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RELATIVISTIC TRANSFORMATION OF MAGNETIC DIPOLE MOMENT

Alexander Kholmetskii[1, *], Oleg Missevitch[2], and Tolga Yarman[3, 4]

[1]Belarus State University, 4 Nezavisimosti Avenue, Minsk 220030, Belarus

[2]Institute for Nuclear Problems, Belarus State University, 11 Bobruiskaya Street, Minsk 220030, Belarus

[3]Okan University, Akfirat, Istanbul, Turkey

[4]Savronik, Eskisehir, Turkey

Abstract—We consider three different definitions of magnetic dipole moment for electrically neutral compact bunches of charged particles and show that, in general, they are not equivalent to each other with respect to their relativistic transformation. In particular, we prove that the "configurational" definition of magnetic dipole moment $\mathbf{m}_c = \frac{1}{2}\int\limits_V (\mathbf{r}\times\mathbf{j})dV$ (in the common designations) and its definition through generated electromagnetic field ("source" definition $\mathbf{m}_s$) or experienced force ("force" definition $\mathbf{m}_f$) lead to different relativistic transformations of $\mathbf{m}_c$ and $\mathbf{m}_s$ ($\mathbf{m}_f$). The results obtained shed light on the available disagreements with respect to relativistic transformation of a magnetic dipole moment, and they can be used in covariant formulation of classical electrodynamics in material media.

1. INTRODUCTION

Nowadays, the problem of relativistic transformation of polarization/magnetization and related transformation of electric/magnetic dipole moment for a compact bunch of charges represents the subject of textbook consideration in macroscopic electrodynamics [1–3]. However, surprisingly enough, a consensus on this subject is not achieved

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* Corresponding author: Alexander Kholmetskii (khol123@yahoo.com).

to the moment. More specifically, all authors agree with the transformation of polarization $\mathbf{P}$ (in Gaussian units)

$$\mathbf{P} = \gamma \mathbf{P}_0 - \frac{(\gamma - 1)}{v^2} (\mathbf{P}_0 \cdot \mathbf{v}) \mathbf{v} + \gamma \frac{\mathbf{v} \times \mathbf{M}_0}{c^2} \tag{1}$$

and the related transformation of electric dipole moment $\mathbf{p} = \int_V \mathbf{P} dV$

$$\mathbf{p} = \mathbf{p}_0 - \frac{(\gamma - 1)}{\gamma v^2} (\mathbf{p}_0 \cdot \mathbf{v}) \mathbf{v} + \frac{\mathbf{v} \times \mathbf{m}_0}{c^2}, \tag{2}$$

where $\mathbf{M}$ is magnetization, $\mathbf{m} = \int_V \mathbf{M} dV$ the magnetic dipole moment, $\mathbf{v}$ the velocity of the bunch of charges in the frame of observation, $\gamma = (1 - v^2/c^2)^{-1/2}$ the Lorentz factor, and V the volume of the bunch. Hereinafter the subscript "0" denotes the given quantity defined in the rest frame of dipole.

However, various books and papers disagree with respect to transformation properties of magnetization and magnetic dipole moment. In particular, Panofsky and Phillips [1] write the transformation for magnetization in the form

$$\mathbf{M}_{//} = (\mathbf{M}_0)_{//}, \tag{3a}$$

$$\mathbf{M}_{\perp} = \gamma \left((\mathbf{M}_0)_{\perp} + \mathbf{v} \times \mathbf{P}_0 \right), \tag{3b}$$

which follows from the fact that the components of $\mathbf{M}$ and $\mathbf{P}$ constitute the anti-symmetric polarization-magnetization tensor used in the covariant formulation of classical electrodynamics in material media [1, 2]. Hereinafter the subscript "//" stands for the component collinear to $\mathbf{v}$, while the subject "$\perp$" denotes the component to be orthogonal to $\mathbf{v}$. Equations (3a)–(3b) imply that the magnetic dipole moment obeys the transformation (see, e.g., Ref. [4])

$$\mathbf{m} = \mathbf{m}_0 - \frac{(\gamma - 1)}{\gamma v^2} (\mathbf{m}_0 \cdot \mathbf{v}) \mathbf{v} + \mathbf{p}_0 \times \mathbf{v}. \tag{4}$$

However, Fisher [5] derives a quite different transformation rule for magnetic dipole moment, proceeding from the formal definition of magnetic dipole moment

$$\mathbf{m} = \frac{1}{2} \int_V (\mathbf{r} \times \mathbf{j})\, dV \tag{5}$$

($\mathbf{j}$ being the current density) and applying the Lorentz transformations for space-time four-vector and for charge density-current density four-vector:

$$\mathbf{m} = \frac{\mathbf{m}_0}{\gamma} - \frac{(\gamma - 1)}{2(\gamma + 1) v^2} \mathbf{v} \times (\mathbf{v} \times \mathbf{m}_0) + \frac{1}{2} (\mathbf{p}_0 \times \mathbf{v}). \tag{6}$$

The factor 1/2 in front of the last term in rhs of Equation (5) is also present in the corresponding equation in the book by Jackson [3].

Here we notice that the contribution to the magnetic dipole moment proportional to the cross product $(\mathbf{p}_0 \times \mathbf{v})$ has the non-relativistic origin, and the disagreement of Equations (4) and (6) with respect to this term has been recently clarified by Hnizdo [6]. He has shown that in first order in (v/c), the transformation

$$\mathbf{m} \approx \mathbf{m}_0 + \mathbf{p}_0 \times \mathbf{v} \tag{7}$$

must be correct, because only with its implementation the sum of polarization $(\mathbf{j}_p)$ and magnetization $(\mathbf{j}_\mu)$ current densities yields the total current density for a moving dipole. Hnizdo also made an important note that the definition of magnetization

$$\mathbf{M} = \frac{1}{2} (\mathbf{r} \times \mathbf{j}) \tag{8}$$

and the definition of magnetic dipole moment (5) is correct, when $\nabla \cdot \mathbf{j} = 0$, and in Section 2 we will essentially use this requirement. Finally, Hnizdo concludes that the transformation

$$\mathbf{m} \approx \mathbf{m}_0 + \frac{1}{2} \mathbf{p}_0 \times \mathbf{v} \tag{9}$$

resulting from Fischer's Equation (5) occurs inappropriately.

Hnizdo's analysis is convincing enough; at the same time, it has been made in first order in (v/c) only.

The goal of the present paper is to derive the exact relativistic transformation for a magnetic dipole moment of a compact bunch of charges. We emphasize that in this paper we do not consider the magnetic dipole moment for point-like charges (in the classical meaning), which is directly related to spin, where the covariant formalism of its description is well developed (see, e.g., [2, 3, 7, 8]). Below we imply that the bunch of charges is characterized by its internal structure, so that the magnetization can be always introduced. Thus, we deal with the magnetic dipole moments, which are defined via the volume integration of magnetization.

For simplicity we further assume that the proper electric dipole moment $\mathbf{p}_0 = 0$, so that the cross product $\mathbf{p}_0 \times \mathbf{v}$ is vanishing. It allows us to avoid a repetition of analysis by Hnizdo [6] with respect to the term $\mathbf{p}_0 \times \mathbf{v}$ (where we completely agree), and to focus our attention to the terms of order $(v/c)^2$ and higher. We stress that in this case the Equations (4) and (6) continue to disagree with each other. Indeed, at $\mathbf{p}_0 = 0$ Equation (4) takes the form

$$\mathbf{m} = \mathbf{m}_0 - \frac{(\gamma - 1)}{\gamma v^2} (\mathbf{m}_0 \cdot \mathbf{v}) \mathbf{v}, \tag{10}$$

while Equation (6) reads:

$$\mathbf{m} = \frac{\mathbf{m}_0}{\gamma} - \frac{(\gamma - 1)}{2(\gamma + 1)v^2}\mathbf{v} \times (\mathbf{v} \times \mathbf{m}_0). \tag{11}$$

We see that only in the case, where the vectors $\mathbf{m}_0$, $\mathbf{v}$ are collinear with each other, both Equations (10) and (11) yield the equality $\mathbf{m} = \mathbf{m}_0/\gamma$. However, for any other orientation of $\mathbf{m}_0$, these equations give different results.

In the present paper, we will show that the determination of correct relativistic transformation for magnetic dipole moment requires to carry out a careful analysis of parameters of compact bunches of charges and the notion of magnetic dipole moment itself, as seen in different inertial reference frames. This way we find the explanation for disagreement of Equations (10), (11) and obtain the general solution of the problem of transformation of magnetic dipole moment. In our general approach, we verify the results obtained with the simplest representative of magnetic dipole: a closed conducting loop of rectangular shape, carrying a steady current (Section 2). In Section 3 we discuss the results obtained.

2. MAGNETIZATION AND MAGNETIC DIPOLE MOMENT OF A MOVING DIPOLE

In what follows, for simplicity, we consider electrically neutral magnetic dipoles with the vanishing proper electric dipole moment, i.e., $\mathbf{p}_0 = 0$. In this case we can test both transformations (10), (11) with the model of magnetic dipole mentioned above: the conducting circuit of rectangular shape with the proper size of the segment l_0, which lies in the plane xy and carries a steady current I_0 in its rest frame (see Fig. 1). The proper magnetic dipole moment of this circuit $\mathbf{m}_0$ is parallel to the axis z, and we assume that the entire circuit moves at the constant velocity v along the axis x. In this case the vectors $\mathbf{m}_0$ and $\mathbf{v}$ are orthogonal to each other, and we respectively derive from Equations (10) and (11):

$$\mathbf{m}_\perp = \mathbf{m}_0, \tag{10a}$$

$$\mathbf{m}_\perp = \frac{\mathbf{m}_0}{\gamma} + \frac{(\gamma - 1)}{2(\gamma + 1)}\mathbf{m}_0 = \frac{\mathbf{m}_0}{2} + \frac{\mathbf{m}_0}{\gamma(\gamma + 1)}. \tag{11a}$$

Now let us directly calculate the magnetic dipole moment of the circuit via Equation (5), which can be presented in the form convenient for further analysis:

$$\mathbf{m} = \frac{1}{2}\int\limits_{V_1+V_3} (\mathbf{r} \times \mathbf{j})\, dV + \frac{1}{2}\int\limits_{V_2+V_4} (\mathbf{r} \times \mathbf{j})\, dV. \tag{12}$$

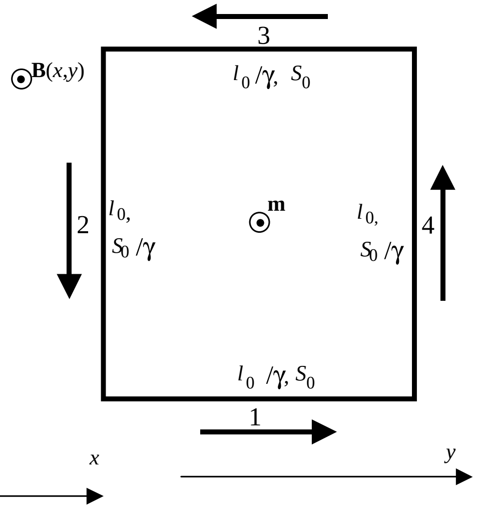

Figure 1. Rectangular conducting loop lying in the xy-plane moves at the constant velocity v along the x-axis of the laboratory frame K. A proper length of each segment of the loop is equal to l_0, the proper area of cross section of conducting wire is S_0. The proper magnetic moment of the loop is equal to $\mathbf{m}_0$ and lies in the positive z-direction. The direction of current in each segment is shown in bold arrows. The lengths and cross-section areas of each segment of moving circuit are also indicated. The total current in segments 1, 3 is equal to $I_1 = I_3 = \gamma I_0$; the total current in segments 2, 4 is equal to $I_2 = I_4 = I_0/\gamma$. The proper current in each segment of moving circuit is equal to I_0/γ, see Equation (35).

Here V_i stands for the volume of segment i, and the numeration of the segments is indicated in Fig. 1.

Using special Lorentz transformation for space-time and charge density-current density four-vectors [3], we derive for the segments 1 and 3:

$$j_{1,3} = \gamma j_0, \tag{13a}$$
$$(\mathbf{r}\times\mathbf{j})_{1,3} = \hat{\mathbf{z}}\gamma l_0 j_0/2, \tag{13b}$$

and also for the segments 2, 4:

$$j_{2,4} = j_0, \tag{14a}$$
$$(\mathbf{r}\times\mathbf{j})_{2,4} = \hat{\mathbf{z}} l_0 j_0/2\gamma, \tag{14b}$$

where $j_0 = I_0/S_0$ in the current density in the rest frame of the circuit. Inserting Equations (13b), (14b) into Equation (12), and taking also into account the relativistic transformation of volume ($V = V_0/\gamma$), we obtain:

$$\mathbf{m}_{\perp} = \frac{\mathbf{m}_0}{2} + \frac{\mathbf{m}_0}{2\gamma^2}, \tag{15}$$

which disagrees with both transformations (10a) and (11a).

In order to understand the origin of the indicated conflict among Equations (10a), (11a) and (15), we have to look closer at the definition of magnetization and magnetic dipole moment. Our analysis of various publications (e.g., Refs. [1–3, 9–13]) shows that, in fact, there are three possible definitions of these quantities:

- definition based on Equations (8), (5), which involves the spatial distribution (configuration) of currents in a considered medium. Thus it can be named as "configurational" definition, and below we supply the related vectors of magnetization and magnetic dipole moment by the subscript "c";

- definition based on determination of EM fields generated by a magnetized medium (magnetic dipole) as the sources of these fields; hereinafter we name it as "source" definition and supply the related vectors of magnetization and magnetic dipole moment by the subscript "s";

- definition based on determination of forces acting on a magnetized medium (magnetic dipole) in an external EM field; hereinafter we name it as "force" definition and supply the related vectors of magnetization and magnetic dipole moment by the subscript "f".

The presented classification, as we are aware, was never mentioned in the scientific literature before, because usually it is tacitly implied that all of these definitions yield the identical result with respect to calculated magnetization and magnetic dipole moment.

However, we stress that the latter statement is correct only in the rest frame of magnetized medium (magnetic dipole), so that we can omit the introduced subscripts c, s and f for the proper magnetization and magnetic dipole moment, i.e., $\mathbf{m}_{c0} = \mathbf{m}_{s0} = \mathbf{m}_{f0} \equiv \mathbf{m}_0$. However, for a moving medium (magnetic dipole), the definitions listed above, in general, yield different values for magnetization (magnetic dipole moment) with their different relativistic transformations[†].

The validity of this assertion has already been seen in the comparison of Equations (10) and (11) (and their particular

[†] Similar definitions can be introduced with respect to polarization and electric dipole moment, too. However, for the case of a moving electric dipole, in contrast to the motion of magnetic dipole, all of these definitions occur equivalent to each other. We omit here the corresponding proof of this statement, and only mention that this circumstance explains the common consensus with respect to relativistic transformations of polarization (Equation (1)) and electric dipole moment (Equation (2)).

forms (10a), (11a) presented above), which obviously differ from each other. According to our classification, Equation (10) stems from the "field" definition of magnetic dipole moment (since the related magnetization is used in covariant formulation of classical electrodynamics in material media, entering into the source term of Maxwell equations [1]), whereas Equation (11) has been derived in Ref. [5] via the Lorentz transformation of the quantities entering into Equation (5) (i.e., with the use of "configurational" definition of magnetic dipole moment).

In these conditions it is important to clarify the transformation rule for the magnetic dipole moment $\mathbf{m}_f$ in its "force" definition. For this purpose we further calculate the force acting on the circuit of Fig. 1 in an external static magnetic field $\mathbf{B}$, which lies in the positive z-direction and depends on x and y coordinates. In our analysis we use the expression for the force, acting on a moving dipole, in the form [14]

$$\mathbf{F} = \nabla\left(\mathbf{p}\cdot\mathbf{E}\right) + \nabla\left(\mathbf{m}_f\cdot\mathbf{B}\right) - \frac{1}{c^2}\frac{d}{dt}\left(\mathbf{m}_f\times\mathbf{E}\right) + \frac{d}{dt}\left(\mathbf{p}\times\mathbf{B}\right), \quad (16)$$

where we take into account that in these equations the "force" definition of magnetic dipole moment is used.

For the circuit in Fig. 1 and time-independent magnetic field, we obtain from Equation (16)

$$\mathbf{F} = \nabla\left(\mathbf{m}_f\cdot\mathbf{B}\right) + \mathbf{p}_{rel}\times\left(\mathbf{v}\cdot\nabla\right)\mathbf{B}, \quad (17)$$

where

$$\mathbf{p}_{rel} = \frac{\mathbf{v}\times\mathbf{m}_0}{c^2} \quad (18)$$

is the electric dipole moment of the moving circuit, which emerges due to its relativistic polarization.

One can see that for the problem in question Equations (17), (18) give two force components:

$$\left(F_m\right)_x = \left(m_f - \frac{v^2}{c^2}m_0\right)\frac{\partial B}{\partial x}, \quad (19a)$$

$$\left(F_m\right)_y = m_f\frac{\partial B}{\partial y}. \quad (19b)$$

Our next problem is to express the forces (19a)–(19b) via the magnetic dipole moment of circuit $\mathbf{m}_0$ measured in its rest frame.

For this purpose we calculate the forces F_x and F_y via the Lorentz force law for a laboratory observer. Hence we derive:

$$F_x = I_4 l_4 B\left(x_0 + l, y_0, 0\right) - I_2 l_2 B\left(x_0, y_0, 0\right), \quad (20)$$

$$F_y = I_3 l_3 B\left(x_0, y_0 + l_0, 0\right) - I_1 l_1 B\left(x_0, y_0, 0\right), \quad (21)$$

where I_i is the current in the segment i, l_i is the length of the segment i, and here we denoted the coordinates of the left bottom corner of the circuit as $\{x_0, y_0, 0\}$.

The values of current I_i can be found from Equations (13a), (14a), complemented by the transformation of area of cross-section of each segment of circuit: $S_1 = S_3 = S_0$, $S_2 = S_4 = S_0/\gamma$. Hence we get

$$I_1 = I_3 = \gamma I_0, \tag{22a}$$

$$I_2 = I_4 = I_0/\gamma. \tag{22b}$$

Taking also into account that $l_1 = l_3 = l_0/\gamma$, $l_2 = l_4 = l_0$, we obtain from Equations (20) and (21):

$$F_x = \frac{m_0}{\gamma^2}\frac{\partial B}{\partial x}, \tag{23}$$

$$F_y = m_0\frac{\partial B}{\partial y}. \tag{24}$$

Comparing now Equations (19a)–(19b) with corresponding Equations (23), (24) and taking into account that for the problem of Fig. 1, $\mathbf{m}_0 \perp \mathbf{v}$, we derive the relationship

$$(\mathbf{m}_f)_\perp = \mathbf{m}_0. \tag{25}$$

Addressing again to Fig. 1 and considering the case, where the velocity $\mathbf{v}$ is collinear with $\mathbf{m}_0$, we derive in a similar way

$$(\mathbf{m}_f)_{//} = \frac{\mathbf{m}_0}{\gamma}. \tag{26}$$

Comparing Equations (25), (26) with Equation (10), we reveal that for the circuit in Fig. 1 the magnetic dipole moment in its "force" definition $\mathbf{m}_f$ obeys the same transformation rule, just like the magnetic dipole moment in "source" definition $\mathbf{m}_s$ does. Since any current-loop model of magnetic dipole can be presented to be filled by a large number of small rectangular closed circuits of Fig. 1 (where the currents in adjacent sections mutually cancel each other, giving rise to a non-compensated boundary current), the validity of transformation (10) for the circuit in Fig. 1 with respect to "force" definitions of the magnetic dipole moment signifies the general validity of Equation (10) for $\mathbf{m}_f$.

Thus we conclude that the "source" definition and "force" definition of magnetic dipole moment occur equivalent to each other for the force law (16), and both $\mathbf{m}_s$, $\mathbf{m}_f$ obey transformation (10).

In order to achieve further progress in the analysis of introduced definitions of magnetization and magnetic dipole moment for a moving dipole, we now emphasize that among these definitions, the "configuration" definition is the most simple with respect to formal analysis and therefore, next we focus our attention to this definition of a moving magnetic dipole $\mathbf{m}_c$ given by Equation (5).

Now we point out an inconsistency in the derivation of its transformation (6) suggested by Fisher [5]. We remind that Fischer obtained Equation (6) via the Lorentz transformations for the distance [15]

$$\mathbf{r} = \mathbf{r}_0 - \frac{(\gamma - 1)}{\gamma v^2} (\mathbf{r}_0 \cdot \mathbf{v}) \mathbf{v} \tag{27}$$

and for current density

$$\mathbf{j} = \mathbf{j}_0 + \frac{(\gamma - 1)}{v^2} (\mathbf{j}_0 \cdot \mathbf{v}) \mathbf{v} \tag{28}$$

(in the case, where the proper charge density of magnetic dipole $\rho_0 = 0$), with further substitution of Equations (27), (28) into Equation (5) and with taking into account of relativistic transformation of volume $dV = dV_0/\gamma$.

We have no objections against Equation (27). However, considering transformation (28), we remind that that the measurement of current density $\mathbf{j}$ in a moving medium is carried out with a cross-section of unit area, passing through the medium, when this area is *at rest* in the laboratory. If so, not only the proper current density of the carriers of current (e.g., the conduction electrons for the circuit in Fig. 1) contributes to $\mathbf{j}$, but also the convective current of charges of opposite sign, moving along with the medium and immovable in its rest frame (i.e., the positive ions for the moving circuit of Fig. 1). At the same time, it is obvious that the convective current of positive ions does not contribute to the magnetic dipole moment of a moving dipole. Moreover, this convective component of current density does not satisfy the equality $\nabla \cdot \mathbf{j} = 0$, which, as shown in [6], is required for correctness of the definitions (8) and (5). Thus, the substitution of Equation (28) into Equation (5) constitutes a physically incorrect operation.

Therefore, before applying Equations (8), (5) to a moving magnetic dipole, one has to re-define properly the current density in these equations, in order to make it divergence-free.

In order to solve this problem we address to our recent paper [10], where we introduced the notion *of proper current density* $\mathbf{j}_{pr}$ for a moving magnetic dipole to be measured with the cross-sectional unit area, *co-moving* with the dipole. In this definition, we exclude the convective currents of positive ions. Thus, $\mathbf{j}_{pr}$ is completely determined by the flow of free carriers of current, and $\nabla \cdot \mathbf{j}_{pr} = 0$ in any inertial reference frame.

The proper current density can be formally defined as a number of charges passing across a unit area S during a unit time interval, when this area co-moves with the magnetic dipole at the velocity $\mathbf{v}$. Designating ρ_{cr} the charge density of carriers of current, as seen in the laboratory frame, the number of such charges (which directly determines the proper current density) is equal to

$$\mathbf{j}_{pr} = \rho_{cr} \left(\mathbf{u} - \mathbf{v} \right), \tag{29}$$

where $\mathbf{u}$ is the velocity of carries of current in the laboratory frame. Applying the Einstein law of velocity composition [15]

$$\mathbf{u} = \left(\frac{\mathbf{u}_0}{\gamma} + \mathbf{v} + \frac{(\gamma - 1)}{\gamma} \frac{(\mathbf{u}_0 \cdot \mathbf{v})\,\mathbf{v}}{v^2} \right) \left(1 + \frac{\mathbf{u}_0 \cdot \mathbf{v}}{c^2} \right)^{-1}, \tag{30}$$

and the relativistic transformation for the charge density

$$\rho_{cr} = \gamma \left(\rho_{cr} \right)_0 \left(1 + \frac{\mathbf{u}_0 \cdot \mathbf{v}}{c^2} \right)^{-1} \tag{31}$$

(where $(\rho_{cr})_0$, $\mathbf{u}_0$ are respectively the charge density and flow velocity of carriers of current in the rest frame of magnetic dipole), we further substitute Equations (30), (31) into Equation (29) and obtain:

$$\mathbf{j}_{pr} = \mathbf{j}_0 - \frac{(\gamma - 1)}{\gamma} \frac{(\mathbf{j}_0 \cdot \mathbf{v})\,\mathbf{v}}{v^2}, \tag{32}$$

where $\mathbf{j}_0$ is the current density of carries of current in the rest frame of dipole.

We point out that due to the equality $\nabla \cdot \mathbf{j}_{pr} = 0$, Equation (32) complies with the continuity requirement for the proper current

$$I_{pr} = \int\limits_S \mathbf{j}_{pr} \cdot d\mathbf{S} \tag{33}$$

for any current-loop model of magnetic dipole. For example, for the circuit in Fig. 1, Equation (32) yields

$$(\mathbf{j}_{pr})_1 = (\mathbf{j}_{pr})_3 = \mathbf{j}_0, \quad (\mathbf{j}_{pr})_2 = (\mathbf{j}_{pr})_4 = \mathbf{j}_0/\gamma.$$

Taking into account the transformation of area

$$S_1 = S_3 = S_0/\gamma, \quad S_2 = S_4 = S_0, \tag{34}$$

we derive from Equation (33) the equality of proper current in all segments of the circuit, i.e.,

$$(I_{pr})_1 = (I_{pr})_2 = (I_{pr})_3 = (I_{pr})_4 = I_0/\gamma. \tag{35}$$

In the view of definition of current as $I = dQ/dt$, Equation (35) has a clear physical meaning and reflects the time dilation effect for a moving circuit.

In contrast, applying the transformation (28) for the *total* current density, used by Fisher [5] in the derivation of transformation (6), we derive for the circuit in Fig. 1

$$\mathbf{j}_1 = \mathbf{j}_3 = \mathbf{j}_0, \quad \mathbf{j}_2 = \mathbf{j}_4 = \gamma \mathbf{j}_0,$$

so that with Equations (33), (34) we get

$$I_1 = I_3 = I_0/\gamma, \quad I_2 = I_4 = \gamma I_0$$

(see also Equations 22(a)–(22b). We observe that $I_1 = I_3 \neq I_2 = I_4$, which simply indicates that the contribution of convective current of positive ions into the total current (measured by means of cross-sectional area resting in a laboratory) is different in the different segments of the circuit (on this subject see also Refs. [4, 10, 16]). As we have mentioned above, such a convective current of immovable charges (in the rest frame of dipole) does not contribute to the magnetization and magnetic dipole moment of moving medium and thus, in physically meaningful "configurational" definition of magnetization/magnetic dipole moment, we have to replace in Equations (8), (5) the total current density $\mathbf{j}$ by the proper charge density $\mathbf{j}_{pr}$, arriving at the new definitions

$$\mathbf{M}_c = \frac{1}{2}\left(\mathbf{r} \times \mathbf{j}_{pr}\right), \tag{36a}$$

$$\mathbf{m}_c = \frac{1}{2}\int\limits_V \left(\mathbf{r} \times \mathbf{j}_{pr}\right) dV, \tag{36b}$$

where $\nabla \cdot \mathbf{j}_{pr} = 0$. Hence, using the Lorentz transformations (27) and (32), we determine the relativistic transformation for the "configurational" magnetization:

$$\begin{aligned} \mathbf{M}_c &= \frac{1}{2}\left(\mathbf{r}_0 - \frac{(\gamma-1)}{\gamma}\frac{(\mathbf{r}_0 \cdot \mathbf{v})\,\mathbf{v}}{v^2}\right) \times \left(\mathbf{j}_0 - \frac{(\gamma-1)}{\gamma}\frac{(\mathbf{j}_0 \cdot \mathbf{v})\,\mathbf{v}}{v^2}\right) \\ &= \mathbf{M}_0 - \frac{1}{2}\frac{(\gamma-1)}{\gamma v^2}\left[(\mathbf{j}_0 \cdot \mathbf{v})\,\mathbf{r}_0 \times \mathbf{v} - (\mathbf{r}_0 \cdot \mathbf{v})\,\mathbf{j}_0 \times \mathbf{v}\right], \end{aligned} \tag{37}$$

where we designated $\mathbf{M}_0 = \frac{1}{2}(\mathbf{r}_0 \times \mathbf{j}_0)$ the magnetization of medium in its rest frame.

Further we use the equality

$$(\mathbf{j}_0 \cdot \mathbf{v})\,\mathbf{r}_0 \times \mathbf{v} - (\mathbf{r}_0 \cdot \mathbf{v})\,\mathbf{j}_0 \times \mathbf{v} = (\mathbf{v} \times (\mathbf{r}_0 \times \mathbf{j}_0)) \times \mathbf{v}, \tag{38}$$

which is straightforwardly proved via the vector identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b}). \tag{39}$$

Combining Equations (37), (38), we arrive at the relativistic transformation of magnetization in its "configurational definition":

$$\mathbf{M}_c = \mathbf{M}_0 - \frac{(\gamma - 1)}{\gamma v^2} (\mathbf{v} \times \mathbf{M}_0) \times \mathbf{v}. \tag{40}$$

Applying again the vector identity (39), we can present Equation (40) in the equivalent form

$$\mathbf{M}_c = \frac{\mathbf{M}_0}{\gamma} + \frac{(\gamma - 1)}{\gamma} \frac{(\mathbf{M}_0 \cdot \mathbf{v}) \mathbf{v}}{v^2}. \tag{41}$$

Taking into account the relativistic transformation of volume, $V = V_0/\gamma$, we further derive the transformation of "configurational" magnetic dipole moment:

$$\mathbf{m}_c = \frac{\mathbf{m}_0}{\gamma^2} + \frac{(\gamma - 1)}{\gamma^2} \frac{(\mathbf{m}_0 \cdot \mathbf{v}) \mathbf{v}}{v^2}. \tag{42}$$

The latter equation replaces the Fisher's transformation (6) for magnetic dipole moment in its configurational definition.

We again emphasize that the proper current density entering into the definitions (36a)–(36b) is the divergence-free in any inertial reference frame. Hence we can conjecture that Equation (42) provides the implementation of the known equality for small dipole

$$\mathbf{m}_c = I_{pr} \mathbf{S} \tag{43}$$

for any inertial observer. Indeed, for such a small dipole, we can adopt that its area is plane, and it can be divided into the elements

$$\Delta \mathbf{S}_0 = \mathbf{a}_0 \times \mathbf{b}_0, \tag{44}$$

in the rest frame of the dipole, where $\mathbf{a}_0$, $\mathbf{b}_0$ are some small vectors. Applying the transformation (27) to the vectors $\mathbf{a}$, $\mathbf{b}$ of a moving area, we obtain:

$$\begin{aligned}
\Delta \mathbf{S} &= \left(\mathbf{a}_0 - \frac{(\gamma - 1)}{\gamma v^2} (\mathbf{a}_0 \cdot \mathbf{v}) \mathbf{v} \right) \times \left(\mathbf{b}_0 - \frac{(\gamma - 1)}{\gamma v^2} (\mathbf{b}_0 \cdot \mathbf{v}) \mathbf{v} \right) \\
&= \Delta \mathbf{S}_0 - \frac{(\gamma - 1)}{\gamma v^2} (\mathbf{b}_0 \cdot \mathbf{v}) \mathbf{a}_0 \times \mathbf{v} - \frac{(\gamma - 1)}{\gamma v^2} (\mathbf{a}_0 \cdot \mathbf{v}) \mathbf{v} \times \mathbf{b}_0 \\
&= \Delta \mathbf{S}_0 - \frac{(\gamma - 1)}{\gamma v^2} \left[(\mathbf{b}_0 \cdot \mathbf{v}) \mathbf{a}_0 - (\mathbf{a}_0 \cdot \mathbf{v}) \mathbf{b}_0 \right] \times \mathbf{v} \\
&= \Delta \mathbf{S}_0 - \frac{(\gamma - 1)}{\gamma v^2} [\mathbf{v} \times (\mathbf{a}_0 \times \mathbf{b}_0)] \times \mathbf{v} = \Delta \mathbf{S}_0 - \frac{(\gamma - 1)}{\gamma v^2} (\mathbf{v} \times \Delta \mathbf{S}_0) \times \mathbf{v} \\
&= \frac{\Delta \mathbf{S}_0}{\gamma} + \frac{(\gamma - 1)}{\gamma} \frac{(\Delta \mathbf{S}_0 \cdot \mathbf{v}) \mathbf{v}}{v^2},
\end{aligned} \tag{45}$$

where we used the identity (39) twice. Substituting $\Delta\mathbf{S}$ in place of $\mathbf{S}$ into Equation (43), and using the relativistic transformation for proper current $I_{pr} = (I_{pr})_0/\gamma$ (see Equation (35)), we arrive at the transformation (42). It proves the validity of Equation (43) for a small dipole in any inertial reference frame.

The results obtained above are straightforwardly extended to the case, where the electric dipole moment $\mathbf{p}_0$ of a compact bunch of charges is not equal to zero. Then Equations (42) and (10) take the form:

$$\mathbf{m}_c = \frac{\mathbf{m}_0}{\gamma^2} + \frac{(\gamma-1)}{\gamma^2}\frac{(\mathbf{m}_0\cdot\mathbf{v})\,\mathbf{v}}{v^2} + \mathbf{p}_0\times\mathbf{v} \tag{46}$$

for "configurational" definition of magnetic dipole moment, and

$$\mathbf{m}_{s,f} = \mathbf{m}_0 - \frac{(\gamma-1)}{\gamma}\frac{(\mathbf{m}_0\cdot\mathbf{v})\,\mathbf{v}}{v^2} + \mathbf{p}_0\times\mathbf{v} \tag{47}$$

for "source" and "field" definitions of magnetic dipole moment. The related transformations for magnetization are directly obtained from Equations (46), (47) via taking into account the relativistic transformation of volume ($V = V_0/\gamma$), i.e.,

$$\mathbf{M}_c = \frac{\mathbf{M}_0}{\gamma} + \frac{(\gamma-1)}{\gamma}\frac{(\mathbf{M}_0\cdot\mathbf{v})\,\mathbf{v}}{v^2} + \gamma\mathbf{P}_0\times\mathbf{v}, \tag{48}$$

$$\mathbf{M}_{s,f} = \gamma\mathbf{M}_0 - \frac{(\gamma-1)}{v^2}(\mathbf{M}_0\cdot\mathbf{v})\,\mathbf{v} + \gamma\mathbf{P}_0\times\mathbf{v}. \tag{49}$$

3. DISCUSSION

Analyzing transformations (46), (47), we have to answer the question: *why* the transformation for $\mathbf{m}_c$ even in its corrected definition (36b) differs from the transformation for $\mathbf{m}_s$ ($\mathbf{m}_f$), as well as to clarify better the conditions, providing the coincidence of transformation for $\mathbf{m}_s$ and $\mathbf{m}_f$ obtained above.

Answering the first question we consider, first of all, a compact bunch of charges $\mathbf{q}_i$ with the position vectors $\mathbf{r}_i$ and velocities $\mathbf{v}_i$ at some fixed time moment. Then at a large distance R from the bunch, the average vector potential is determined by the equation

$$\bar{\mathbf{A}} = \frac{1}{c}\sum_i \frac{\overline{q_i\mathbf{v}_i}}{|\mathbf{R}-\mathbf{r}_i|},$$

and the decomposition of this equation to the first order in $1/R$ gives:

$$\bar{\mathbf{A}} = \frac{1}{cR}\sum_i q_i\bar{\mathbf{v}}_i - \frac{1}{c}\sum_i \overline{q_i\mathbf{v}_i\left(\mathbf{r}_i\cdot\nabla\frac{1}{R}\right)}. \tag{50}$$

Further, it is implied (see, e.g., Ref. [12]) that

$$\sum_i q_i \bar{\mathbf{v}}_i = 0, \tag{51}$$

while the second term in rhs of Equation (50) can be transformed to the form

$$-\frac{1}{c}\sum_i q_i \mathbf{v}_i \overline{\left(\mathbf{r}_i \cdot \nabla \frac{1}{R}\right)} = \frac{(\mathbf{m} \times \mathbf{R})}{R^3},$$

where the magnetic dipole moment is defined by Equation (5). If so, the "configurational" definition of magnetic dipole moment, given by Equation (5), might coincide with its source definition, yielding

$$\mathbf{A} = \frac{(\mathbf{m} \times \mathbf{R})}{R^3}. \tag{52}$$

However, we emphasize that the adopted equality (51) is valid, in general, only in a reference frame, where a bunch of charges rests as the whole, and where indeed $\mathbf{m}_{0c} = \mathbf{m}_{0f} = \mathbf{m}_{0s}$. In any other reference frame, wherein the velocity of the bunch is not zero, the equality (51) is not fulfilled, at least in the order $(v/c)^2$, even for electrically neutral bunch ($\sum\limits_i q_i = 0$). Therefore, in this frame Equation (52) loses its validity, when the magnetic dipole moment is defined by Equation (36b) ("configurational" definition). At the same time, replacing the total current density $\mathbf{j}$, entering into the original Equation (5), by the proper current density $\mathbf{j}_{pr}$ according to our definition (36b), one has to remember that just the total current density (representing the resultant flow of carries of current (conduction electrons) and immovable charges (positive ions)) enters into the source term of corresponding Maxwell equation. In other words, the flow of positive ions moving along with a medium, does contribute to the generated electromagnetic fields, but does not contribute to magnetization/magnetic dipole moment of the medium.

In these conditions, it looks reasonable to abandon the "configurational" definition of magnetic dipole moment and to re-define magnetization and magnetic dipole moment in the way, which directly keeps the Lorentz-invariance of Maxwell equations in material media, as, in fact, is tacitly done in the common approach. However, the fact that this "source" definition of magnetic dipole moment differs, in general, from its "configurational" definition has been missed in the scientific literature. Thus, in the calculation of electromagnetic field generated by a moving magnetic dipole, even its corrected "configurational" definition (36b) is not applicable.

In order to clarify better the physical origin of the difference between $\mathbf{m}_s\,(\mathbf{m}_f)$ and $\mathbf{m}_c$, we subtract Equations (47) and (46):

$$\begin{aligned}\mathbf{m}_{s,f}-\mathbf{m}_c &= \left[\mathbf{m}_0-\frac{(\gamma-1)}{\gamma}\frac{(\mathbf{m}_0\cdot\mathbf{v})\,\mathbf{v}}{v^2}+\mathbf{p}_0\times\mathbf{v}\right]\\ &\quad-\left[\frac{\mathbf{m}_0}{\gamma^2}+\frac{(\gamma-1)}{\gamma^2}\frac{(\mathbf{m}_0\cdot\mathbf{v})\,\mathbf{v}}{v^2}+\mathbf{p}_0\times\mathbf{v}\right]\\ &=\frac{\left(\gamma^2-1\right)\mathbf{m}_0}{\gamma^2}-\frac{(\gamma-1)}{\gamma}\frac{(\mathbf{m}_0\cdot\mathbf{v})\,\mathbf{v}}{v^2}\left(1+\frac{1}{\gamma}\right)\\ &=\frac{\left(\gamma^2-1\right)}{\gamma^2}\left(\mathbf{m}_0-\frac{(\mathbf{m}_0\cdot\mathbf{v})\,\mathbf{v}}{v^2}\right)=\frac{\mathbf{v}\times(\mathbf{m}_0\times\mathbf{v})}{c^2}=-\left(\mathbf{p}_{rel}\times\mathbf{v}\right),\end{aligned}$$

where we have used Equations (39) and (18). Hence

$$\mathbf{m}_c=\mathbf{m}_{s,f}+\left(\mathbf{p}_{rel}\times\mathbf{v}\right),\tag{53}$$

which shows that the "configurational" definition of magnetic dipole moment, in comparison with its "source" definition, contains the additional term $(\mathbf{p}_{rel}\times\mathbf{v})$. This term represents the contribution to the total magnetic dipole moment, which emerges due to a motion of electric dipole, caused by relativistic polarization of the original magnetic dipole.

Similar relationship can be written for the related magnetizations, i.e.,

$$\mathbf{M}_c=\mathbf{M}_{s,f}+\left(\mathbf{P}_{rel}\times\mathbf{v}\right).\tag{54}$$

Using Equations (53), (54), it is possible, in general, to reformulate classical electrodynamics in material media in terms of configurational definitions of magnetization $\mathbf{M}_c$ corrected according to Equation (36a), which, however, occurs less convenient than the common way.

At the same time, one should notice that the "configurational" magnetic moment $\mathbf{m}_c$ for a moving dipole is evaluated via the simple expression (43), and in some particular problems, its direct calculation via Equation (43) with further application of Equation (53) might yield the fastest solutions.

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