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# THE CURRENTS INDUCED BY A HIGH-FREQUENCY WAVE INCIDENT AT A SMALL ANGLE TO THE AXIS OF STRONGLY ELONGATED SPHEROID

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**Abstract**—Previously derived asymptotics for diffraction by strongly elongated body is generalized to the case of nonaxial incidence. By applying "parabolic" equation method the asymptotics of the field in the boundary layer near the surface is constructed. This asymptotics takes into account the rate of elongation of the body and is applicable both to not too much elongated objects, where it reduces to Fock formulae, and to very elongated bodies.

## 1. INTRODUCTION

Many advances in the theory of diffraction were due to the asymptotic methods. Besides giving relatively simple formulae for the fields, they provide considerable physical insight and understanding of diffraction mechanisms. However, the usual asymptotic approach fails in the case of a body with large transverse curvature $1/\rho_t$. For an ordinary body $1/\rho_t$ is of order of curvature $1/\rho$ of the geodesics, i.e., $\rho_t/\rho = O(1)$, and $\rho_t$ does not manifest itself in the leading order approximation. Correction terms [1] show that on surfaces with larger transverse curvature TM creeping waves become less attenuated. To appear in the principal order term of the asymptotics, the transverse curvature should be large in such a way that $\rho/\rho_t = O((k\rho)^{1/3})$, where $k$ is the wave number. This case was called in [2] the case of moderately elongated body. In that case the effect of transverse curvature can be described by introducing effective impedance in the boundary condition. If the transverse curvature is so large that $\rho/\rho_t = O((k\rho)^{2/3})$, which is the case of strongly elongated body in the terminology of [2], the approach with effective impedance does not work and the asymptotics of creeping waves completely changes.

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Recent progress in the analysis of wave effects in presence of elongated bodies is reported in [3–5]. In these papers the new asymptotic procedure is developed for the description of wave processes in Fock domain on a strongly elongated surface. This procedure follows the usual "parabolic" equation approach of [6], however, the supposition that $\rho/\rho_t = O((k\rho)^{2/3})$ modifies the results. In particular the surface is approximated not by a circular cylinder as in the usual case of Fock asymptotics, but by the spheroid with appropriately chosen semiaxes. This results in the replacement of Airy functions in Fock asymptotics with Whittaker functions. The leading order term is expressed in the form of inverse Mellin transform. The transformant crucially depends on the parameter $\chi = k\rho_t^{3/2}\rho^{-1/2}$ which characterizes the rate of elongation of the body. When $\chi \to +\infty$ the asymptotics reduces to Fock formula. Careful checking with test examples computed numerically by M. Duruflé shows [5] that the new asymptotics gives a very good approximation for the induced current on the surface in a wide range of values of parameter $\chi$.

It is worth noting that diffraction and scattering by prolate spheroids are of continuous interest. Most work relies on the use of prolate spheroidal coordinates (summary of results can be found in [7]). Spheroidal functions are not easy to compute especially in the case of high frequencies when they contain the large parameter $kp$, where $p$ is the focal distance. Another difficulty of high-frequency problems is in the large number of terms that should be summed up. Kleshchev [8] and Sammelman et al. [9] have examined high-frequency acoustic scattering from prolate spheroids and reduced the number of terms from $O((kp)^2)$ to $O(kp)$. Elongated spheroids with large aspect ratio of semiaxes $b/a$ posed another difficulty and did not allow to take $b/a$ more than 10 until Voshchinnikov and Farafonov [10] used Jaffé expansions.

Our asymptotic solution does not require extensive computations and is most effective when both the frequency and the aspect ratio $b/a$ are large. However the results of [3–5] are restricted to axially symmetric problems, that is to the case when the incident plane wave runs along the axis of the spheroid. In this paper we generalize the approach to the case of skew incidence. The resulting formulae are relatively simple and may be used as a simple solution. We emphasize that efforts for the construction of simple solutions are undertaken by many authors, mention already cited Ref. [10] where very elongated spheroids are substituted with infinite cylinders and Ref. [11] where perturbation technique is used to deal with spheroids having aspect ratio close to one.

The paper is organized as follows. In Section 1 we present main

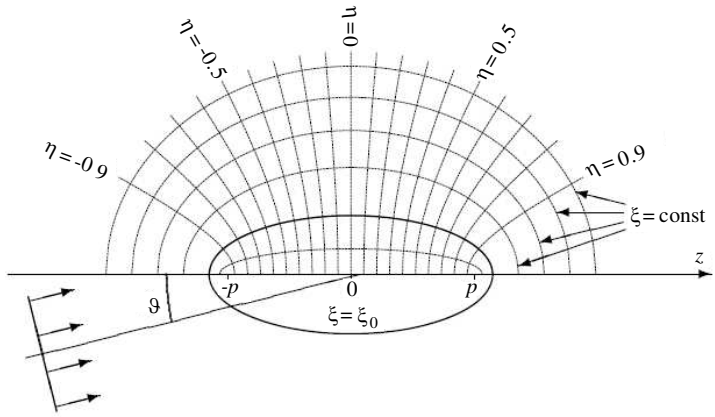


**Figure 1.** Geometry of the problem.

steps of the asymptotic procedure and introduce the general form of the expression for the electromagnetic field in the boundary layer near the surface of strongly elongated spheroid. This general form contains unknown amplitudes related to the incident and reflected waves. In Section 2 we consider plane wave incident on the spheroid at an angle to its axis. Here we formulate assumptions on the smallness of the incident angle. We find amplitudes corresponding to the incident wave in the general formula for the field in the boundary layer. Since the representation of the incident wave is constructed, amplitudes of reflected waves can be easily found. This is done in Section 3. Finally by setting the observation point on surface we find the induced current. Results of computations are presented in Section 4 where we discuss some effects that are specific for diffraction by strongly elongated bodies.

## 2. PROBLEM FORMULATION AND MAIN EQUATIONS

Consider a strongly elongated perfectly conducting convex body of revolution (see Fig. 1). Let a stationary of frequency $\omega$ plane electromagnetic wave be incident at some angle $\vartheta$ to the axis of the body. This wave induces some current on the surface. Our goal is to find such a high-frequency asymptotic formula for the current that takes into account the rate of elongation of the body.

Like in [4, 5] let us approximate the surface with the spheroid having the same radii of curvature at the light-shadow boundary as the body under consideration. The semiaxes of this spheroid are defined

from the formulae

$$\rho = \frac{b^2}{a}, \qquad \rho_t = a.$$

Let the Cartesian coordinate system be chosen such that $z$ axis coincides with the axis of spheroid, $z$ be equal to 0 in its center, and the plane of incidence coincides with the plane $y = 0$. We shall also use cylindrical coordinates $(r, \varphi, z)$, which we introduce by the formulae

$$x = r\cos\varphi, \qquad y = r\sin\varphi,$$

and spheroidal coordinates $(\xi, \eta, \varphi)$

$$r = p\sqrt{\xi^2 - 1}\sqrt{1 - \eta^2}, \qquad z = p\xi\eta.$$

The focal distance $p$ is determined by the semiaxes $a$ and $b$ and for strongly elongated spheroid can be represented by the series in inverse powers of large parameter $kb$, which is our main asymptotic parameter. We have

$$p = \sqrt{b^2 - a^2} = b - \frac{a^2}{2b^2} + \ldots = b - \frac{\chi}{2}\frac{1}{kb} + \ldots$$

In order to symmetrize Maxwell equations we introduce vectors $\mathbf{E}$ and $\mathbf{H}$, where $\mathbf{E}$ is electric vector divided by characteristic impedance of the space $\sqrt{\mu/\varepsilon}$ and $\mathbf{H}$ is magnetic vector. Under the assumption of time dependence in the form $e^{-i\omega t}$ Maxwell equations reduce to

$$\operatorname{curl}\mathbf{E} = ik\mathbf{H}, \qquad \operatorname{curl}\mathbf{H} = -ik\mathbf{E},$$

where $k = \omega\sqrt{\varepsilon\mu}$ is the wave number.

We shall search for the solution of Maxwell equations in the form of Fourier series by $\varphi$. For the part of electromagnetic field which depends on the angle $\varphi$ by means of the multiplier $e^{i\ell\varphi}$ one can express components of $\mathbf{E}$ and $\mathbf{H}$ via angular components $E_\varphi$ and $H_\varphi$. Functions $E_\varphi(\xi, \eta)$ and $H_\varphi(\xi, \eta)$ are solutions of a system of second order differential equations. (These are the usual equations in spheroidal coordinates [12], and they allow variables separation only in the case $\ell = 0$.)

In the standard way of "parabolic" equation technique we search for the solution in the form of a quickly oscillating multiplier and more slowly varying attenuation function. Taking into account that the spheroid is strongly elongated we choose the quick multiplier as $e^{ikb\eta}$. This choice is good, however, only in the middle part of the spheroid and we expect that the asymptotic formula may be not valid near the ends of spheroid. Further we scale the radial coordinate $\xi$. That is we introduce new variable $\tau$ by the formula

$$\xi = 1 + \frac{\chi}{2kb}\tau.$$

The $\tau$ coordinate plays the role of stretched normal to the surface. It can be checked that the choice of the scale is consistent with the usual scaling of normal coordinate by $(k\rho)^{2/3}$. On the axis of the spheroid $\tau = 0$ and on the surface $\tau = 1$.

We represent the attenuation functions in the form of asymptotic series by inverse semi-integer powers of the large parameter $kb$. At the leading order attenuation functions satisfy the system of parabolic equations. Introducing new unknowns $P_\ell$ and $Q_\ell$, such that

$$E_\varphi = \exp\left(ikb\eta + i\ell\varphi\right)\Big\{P_\ell(\tau,\eta) + Q_\ell(\tau,\eta)\Big\}, \tag{1}$$

$$H_\varphi = \exp\left(ikb\eta + i\ell\varphi - \frac{i\pi}{2}\right)\Big\{P_\ell(\tau,\eta) - Q_\ell(\tau,\eta)\Big\}, \tag{2}$$

we reduce this system to two independent parabolic equations

$$L_{\ell-1}P_\ell = 0, \qquad L_{\ell+1}Q_\ell = 0, \tag{3}$$

where

$$L_n = \tau\frac{\partial^2}{\partial\tau^2} + \frac{\partial}{\partial\tau} + \frac{i\chi}{2}\left(1-\eta^2\right)\frac{\partial}{\partial\eta} + \frac{\chi^2\tau}{4} - \frac{n^2}{4\tau} - \frac{\chi^2}{4}\left(1-\eta^2\right) - \frac{i\chi\eta}{2}. \tag{4}$$

The boundary conditions on the surface of perfectly conducting spheroid reduce to

$$P_\ell(1,\eta) + Q_\ell(1,\eta) = 0, \qquad \left(\frac{\partial}{\partial\tau} + \frac{1}{2}\right)\left(P_\ell(1,\eta) - Q_\ell(1,\eta)\right) = 0. \tag{5}$$

If we subtract the incident wave from the total field, the remainder, which we call the secondary field, is subject to the radiation conditions. These conditions mean that for large $\tau$ the secondary field represents waves running to infinity.

So, for every Fourier harmonics $P_\ell$, $Q_\ell$ at the leading order by $kb$ we have got the boundary value problem (3), (5). For smaller order corrections the boundary value problems will be with the same operators in the left-hand sides, but in the right-hand sides instead of zeros, there will be some expressions depending on the principal order terms. The "parabolic" equation methods allows the asymptotics to be constructed up to any order, but we restrict our analysis to the principal order approximation only. That is, our goal is to find such a solution of the above boundary-value problem which is the sum of the given incident plane wave and the secondary field. And the secondary field should satisfy radiation conditions.

First we derive the general representation for the solution. Equation (3) allows variables separation. General solution of the

equation $L_n U_n = 0$ can be written as the integral by the parameter of variables separation

$$U_n = \frac{\exp(-i\chi\eta/2)}{\sqrt{\tau}\sqrt{1-\eta^2}} \int \Omega(s) \left(\frac{1-\eta}{1+\eta}\right)^s F_{s,\,n/2}(-i\chi\tau) ds. \qquad (6)$$

The path of integration and function $\Omega(s)$ are arbitrary, except that we require convergence of the integral. Function $F_{s,\,n/2}(x)$ in (6) is a solution of Whittaker equation [13]

$$\frac{d^2F}{dx^2} + \left(-\frac{1}{4} + \frac{s}{x} + \frac{1-n^2}{4x^2}\right) F = 0. \qquad (7)$$

Standard solutions of Equation (7) are Whittaker functions $M_{s,n/2}(x)$ and $W_{s,n/2}(x)$. The first is regular at $x = 0$ and we choose this function in the expression (6) for the incident field, because the incident field can be also considered in the interior of spheroid and it is regular there up to the axis. Asymptotics

$$W_{s,n/2}(t) \sim t^s e^{-t/2}, \qquad |t| \to +\infty$$

of another solution shows that $U_n$ satisfies the radiation condition if we choose $F = W$ in (6). Therefore we set $F = W$ in the representation of the secondary field.

## 3. REPRESENTATION OF THE INCIDENT FIELD IN THE BOUNDARY LAYER

Arbitrarily polarized plane wave can be represented as the sum of TE and TM waves. We shall consider these waves separately. In TE wave we have

$$\begin{aligned} \vec{E}^{\mathrm{TE}} &= \exp((ikz\cos\vartheta + ikx\sin\vartheta)\,\vec{e}_y, \\ \vec{H}^{\mathrm{TE}} &= \exp((ikz\cos\vartheta + ikx\sin\vartheta)\,\{-\cos\vartheta\vec{e}_x + \sin\vartheta\vec{e}_z\}\,, \end{aligned}$$

where $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ are unit vectors in Cartesian coordinates.

In cylindrical coordinates $(r,\, z,\, \varphi)$ we get

$$\begin{aligned} E_\varphi^{\mathrm{TE}} &= e^{ikz\cos\vartheta}\Big\{ iJ_1(kr\sin\vartheta) \\ &\quad + \sum_{n=1}^{\infty} i^{n-1}\Big(J_{n-1}(kr\sin\vartheta) - J_{n+1}(kr\sin\vartheta)\Big)\cos(n\varphi)\Big\}, \\ H_\varphi^{\mathrm{TE}} &= e^{ikz\cos\vartheta}\cos\vartheta \sum_{n=1}^{\infty} i^{n-1}\Big(J_{n-1}(kr\sin\vartheta) + J_{n+1}(kr\sin\vartheta)\Big)\sin(n\varphi). \end{aligned}$$

In order to rewrite these formulae in the form of representation (6) we decompose the argument of the exponential by inverse powers of the large parameter $\sqrt{kb}$ and neglect vanishing at $kb \to \infty$ terms,

$$kz\cos\vartheta \approx kb\eta + \frac{\eta}{2}\left(\chi(\tau-1) - \beta^2\right), \qquad \beta = \sqrt{kb}\,\vartheta.$$

Further, we assume the incident angle $\vartheta$ to be small, so that $\beta = O(1)$.

Each Fourier component of the incident field is a solution of the Equation (3) and, therefore, can be represented in the form (6). In order to find the amplitudes $\Omega_\ell$ in these representations one can equate both formulae for the solution at some fixed $\tau$. This gives a system of integral equations. The left-hand sides of these equations can be identified with inverse Mellin transform. Therefore solution of these equations can be written explicitly. These calculations were done in [14], and we can use the result

$$\begin{aligned}
&\exp\left(i\frac{\eta}{2}\left(\nu-\beta^2\right)\right) J_\ell\left(\sqrt{1-\eta^2}\sqrt{\nu}\beta\right) \\
&= \frac{1}{\pi}\frac{1}{\sqrt{1-\eta^2}\sqrt{\nu}\beta}\int\limits_{-\infty}^{+\infty}\left(\frac{1-\eta}{1+\eta}\right)^{i\lambda} M_{i\lambda,\ell/2}(i\beta^2)\Omega_\ell(\lambda)M_{i\lambda,\ell/2}(-i\nu)d\lambda,
\end{aligned}$$

where

$$\Omega_\ell = \frac{\Gamma(\ell/2+1/2+i\lambda)\Gamma(\ell/2+1/2-i\lambda)}{\Gamma^2(\ell+1)}.$$

Because of this formula we choose the path of integration in (6) to go along the imaginary axis, and change the integration variable $s = i\lambda$. To write the formulae in a more compact form we introduce the notation

$$\begin{aligned}
U_\ell[R,F] \;=\; & \frac{e^{-i\chi\eta/2}}{\pi\sqrt{1-\eta^2}\sqrt{\chi\tau}\beta}\int\limits_{-\infty}^{+\infty}\left(\frac{1-\eta}{1+\eta}\right)^{i\lambda} \\
& \times\Omega_\ell(\lambda)R(\lambda)M_{i\lambda,\ell/2}\left(i\beta^2\right)F_{i\lambda,\ell/2}(-i\chi\tau)d\lambda
\end{aligned}$$

where dummy parameters $R$ and $F$ will be substituted with particular functions. In the formulae for the incident wave we set $R = 1$ and $F = M$. So, for TE incident wave we can write the representation

$$E_\varphi^{(i)} = e^{ikb\eta}\Big\{iU_1[1,M] + \sum_{n=1}^{\infty} i^{n-1}\big(U_{n-1}[1,M] - U_{n+1}[1,M]\big)\cos(n\varphi)\Big\},$$

$$H_\varphi^{(i)} = e^{ikb\eta}\sum_{n=1}^{\infty} i^{n-1}\big(U_{n-1}[1,M] + U_{n+1}[1,M]\big)\sin(n\varphi).$$

Analogous derivations for TM wave result in a similar formula with $E_\varphi$ replaced with $H_\varphi$ and $H_\varphi$ replaced with $-E_\varphi$ which is the consequence of the natural symmetry of Maxwell equations.

## 4. THE TOTAL FIELD AND INDUCED CURRENTS

For the representation of the secondary field we choose the same path of integration and, as explained above, use Whittaker function $W$. That is, we search the asymptotics of the secondary TE field in the form

$$E_\varphi^{(s)} = e^{ikb\eta}\Big\{iU_1[R_0,W]+\sum_{n=1}^{\infty} i^{n-1}\big(U_{n-1}[R_n,W]-U_{n+1}[T_n,W]\big)\cos(n\varphi)\Big\},$$

$$H_\varphi^{(s)} = e^{ikb\eta}\sum_{n=1}^{\infty} i^{n-1}\big(U_{n-1}[R_n,W]+U_{n+1}[T_n,W]\big)\sin(n\varphi)$$

with additionally introduced subintegral multipliers $R_\ell(\lambda)$ and $T_\ell(\lambda)$. These multipliers play the role of reflection coefficients and can be found when satisfying the boundary conditions (5).

Trigonometric functions are linear independent, therefore each Fourier harmonics should satisfy the boundary conditions separately. When substituting representations for the incident and secondary fields in the boundary conditions we get integral equations. For $n = 0$ we get one equation

$$\Big(U_1[1,M]+U_1[R_0,M]\Big)\Big|_{\tau=1} = 0,$$

and for $n > 0$ we get systems, consisting of two equations

$$\begin{cases}\Big(U_{n-1}[1,M]-U_{n+1}[1,M]+U_{n-1}[R_n,W]-U_{n+1}[T_n,W]\Big)\Big|_{\tau=1}=0,\\ \Big(U_{n-1}[1,\dot{M}]+U_{n+1}[1,\dot{M}]+U_{n-1}[R_n,\dot{W}]+U_{n+1}[T_n,\dot{W}]\Big)\Big|_{\tau=1}=0.\end{cases}$$

Here and below dot denotes derivative of a function.

The integral operators in these equations can be reduced to Mellin transform. Due to uniqueness of the inversion we can equate to zero the subintegral expressions. For $R_0$ we find

$$R_0 = -\frac{M_{i\lambda,1/2}(-i\chi)}{W_{i\lambda,1/2}(-i\chi)}.$$

For other reflection coefficients we get the following system of equations

$$\begin{cases} R_nW_{i\lambda,(n-1)/2}(-i\chi)-T_nC_n(\lambda,\chi)W_{i\lambda,(n+1)/2}(-i\chi)\\ \quad=-M_{i\lambda,(n-1)/2}(-i\chi)+C_n(\lambda,\chi)M_{i\lambda,(n+1)/2}(-i\chi),\\ R_n\dot{W}_{i\lambda,(n-1)/2}(-i\chi)+T_nC_n(\lambda,\chi)\dot{W}_{i\lambda,(n+1)/2}(-i\chi)\\ \quad=-\dot{M}_{i\lambda,(n-1)/2}(-i\chi)-C_n(\lambda,\chi)\dot{M}_{i\lambda,(n+1)/2}(-i\chi),\end{cases}$$

where

$$C_n = \frac{n^2+4\lambda}{4n^2(n+1)^2}\frac{M_{i\lambda,(n+1)/2}(-i\chi)}{M_{i\lambda,(n-1)/2}(-i\chi)}.$$

Solving this system in view of formula

$$\dot{M}_{i\lambda,\ell}W_{i\lambda,\ell} - M_{i\lambda,\ell}\dot{W}_{i\lambda,\ell} = \frac{\Gamma(1+2\ell)}{\Gamma(1/2+\ell-i\lambda)}$$

for the wronskian of Whittaker functions, we get

$$\begin{aligned} R_n = &-\frac{1}{Z_n}\Big(M_{i\lambda,(n-1)/2}(-i\chi)\dot{W}_{i\lambda,(n+1)/2}(-i\chi) \\ &+\dot{M}_{i\lambda,(n-1)/2}(-i\chi)W_{i\lambda,(n+1)/2}(-i\chi) + \frac{C_n\Gamma(n+2)}{\Gamma(n/2+1-i\lambda)}\Big), \quad (8) \\ T_n = &-\frac{1}{Z_n}\Big(W_{i\lambda,(n-1)/2}(-i\chi)\dot{M}_{i\lambda,(n+1)/2}(-i\chi) \\ &+\dot{W}_{i\lambda,(n-1)/2}(-i\chi)M_{i\lambda,(n+1)/2}(-i\chi) + \frac{\Gamma(n)}{C_n\Gamma(n/2-i\lambda)}\Big), \quad (9) \end{aligned}$$

where

$$\begin{aligned} Z_n = &W_{i\lambda,(n-1)/2}(-i\chi)\dot{W}_{i\lambda,(n+1)/2}(-i\chi) \\ &+\dot{W}_{i\lambda,(n-1)/2}(-i\chi)W_{i\lambda,(n+1)/2}(-i\chi). \end{aligned}$$

Substituting these expressions to the formula for $H_\varphi$ and letting $\tau = 1$, we find the induced current. It can be expressed as follows

$$J = e^{ikb\eta}A^{\mathrm{TE}}(\eta,\chi,\varphi), \quad (10)$$

where

$$\begin{aligned} A^{\mathrm{TE}}(\eta,\chi,\varphi) = &-\frac{2}{\pi}\frac{e^{-i\chi\eta/2}}{\sqrt{1-\eta^2}\sqrt{\chi}\beta}\int\limits_{-\infty}^{+\infty}\left(\frac{1-\eta}{1+\eta}\right)^{i\lambda}\sum_{n=1}^{\infty}\frac{i^{n-1}\sin(n\varphi)}{(n+1)!Z_n} \\ &\times\Big(\Gamma(n/2+1+i\lambda)M_{i\lambda,(n+1)/2}(i\beta^2)W_{i\lambda,(n-1)/2}(-i\chi) \\ &+n(n+1)\Gamma(n/2+i\lambda)M_{i\lambda,(n-1)/2}(i\beta^2)W_{i\lambda,(n+1)/2}(-i\chi)\Big)d\lambda \end{aligned}$$

It is also convenient to represent the special function $A^{\mathrm{TE}}$ in terms of Coulomb wave functions $F$ and $H^+$ [13]. Using formulae

$$\begin{aligned} M_{i\lambda,(n+1)/2}(i\beta^2) = &\frac{2ie^{i\pi n/4+\pi\lambda/2}\Gamma(n+2)}{\sqrt{\Gamma(n/2+1+i\lambda)\Gamma(n/2+1-i\lambda)}}F_{\frac{n}{2}}\left(\lambda,\frac{\beta^2}{2}\right), \\ W_{i\lambda,(n-1)/2}(-i\chi) = &-ie^{i\pi n/4+\pi\lambda/2}\sqrt{\frac{\Gamma(n/2+i\lambda)}{\Gamma(n/2-i\lambda)}}H^+_{\frac{n}{2}-1}\left(-\lambda,\frac{\chi}{2}\right) \end{aligned}$$

we get

$$A^{\mathrm{TE}} = -\frac{8}{\pi}\frac{e^{-i\chi\eta/2}}{\sqrt{1-\eta^2}\sqrt{\chi}\beta}\int\limits_{-\infty}^{+\infty}\left(\frac{1-\eta}{1+\eta}\right)^{i\lambda}\sum_{n=1}^{\infty} i^n \sin(n\varphi)$$
$$\times\frac{F_{\frac{n}{2}}\left(\lambda,\frac{\beta^2}{2}\right)H^+_{\frac{n}{2}-1}\left(-\lambda,\frac{\chi}{2}\right)+F_{\frac{n}{2}-1}\left(\lambda,\frac{\beta^2}{2}\right)H^+_{\frac{n}{2}}\left(-\lambda,\frac{\chi}{2}\right)}{H^+_{\frac{n}{2}-1}\left(-\lambda,\frac{\chi}{2}\right)\dot{H}^+_{\frac{n}{2}}\left(-\lambda,\frac{\chi}{2}\right)+\dot{H}^+_{\frac{n}{2}-1}\left(-\lambda,\frac{\chi}{2}\right)H^+_{\frac{n}{2}}\left(-\lambda,\frac{\chi}{2}\right)}d\lambda. \quad (11)$$

The derivations for the case of TM polarization are similar. Representation of the incident wave as was already mentioned can be obtained according to the symmetry principle by interchanging $(E_\varphi,\, H_\varphi)$ components of TE wave with $(H_\varphi,\, -E_\varphi)$ components of TM wave. The same symmetry is inherited in the representation of the secondary field. The reflection coefficient $R_0^{\mathrm{TM}}$ is given by the formula

$$R_0^{\mathrm{TM}} = -\frac{\dot{M}_{i\lambda,1/2}(-i\chi)}{\dot{W}_{i\lambda,1/2}(-i\chi)}.$$

For the other reflection coefficients one gets formulae (8) and (9), where $C_n$ should be taken with the minus sign. In the resulting formula (10) for the induced current the special function $A^{\mathrm{TE}}$ is replaced with

$$A^{\mathrm{TM}}(\eta,\chi,\varphi)=\frac{8}{\pi}\frac{e^{-i\chi\eta/2}}{\sqrt{1-\eta^2}\sqrt{\chi}\beta}\int\limits_{-\infty}^{+\infty}\left(\frac{1-\eta}{1+\eta}\right)^{i\lambda}\left\{\frac{1}{2}\frac{F_0\left(\lambda,\frac{\beta^2}{2}\right)}{\dot{H}_0^+\left(-\lambda,\frac{\chi}{2}\right)}+\sum_{n=1}^{\infty} i^n\cos(n\varphi)\right.$$
$$\left.\times\frac{F_{\frac{n}{2}}\left(\lambda,\frac{\beta^2}{2}\right)H^+_{\frac{n}{2}-1}\left(-\lambda,\frac{\chi}{2}\right)-F_{\frac{n}{2}-1}\left(\lambda,\frac{\beta^2}{2}\right)H^+_{\frac{n}{2}}\left(-\lambda,\frac{\chi}{2}\right)}{H^+_{\frac{n}{2}-1}\left(-\lambda,\frac{\chi}{2}\right)\dot{H}^+_{\frac{n}{2}}\left(-\lambda,\frac{\chi}{2}\right)+\dot{H}^+_{\frac{n}{2}-1}\left(-\lambda,\frac{\chi}{2}\right)H^+_{\frac{n}{2}}\left(-\lambda,\frac{\chi}{2}\right)}\right\}d\lambda. \quad (12)$$

It is worth noting that in the case of axial incidence one should consider the limit as $\beta \to 0$. Only Coulomb wave function $F_{\frac{-1}{2}}$ gives nonzero contribution and special functions $A^{\mathrm{TE}}$ and $A^{\mathrm{TM}}$ simplify to

$$A^{\mathrm{TE}}(\eta,\chi,0) = A(\eta,\chi)\sin\varphi, \qquad A^{\mathrm{TM}}(\eta,\chi,0) = A(\eta,\chi)\cos\varphi, \quad (13)$$

where $A(\eta,\chi)$ is the special function from Refs. [3–5]

$$A(\eta,\chi) = -\frac{4}{\sqrt{\pi}}\frac{e^{-i\chi\eta/2}}{\sqrt{\chi}\sqrt{1-\eta^2}}\int\limits_{-\infty}^{+\infty}\left(\frac{1-\eta}{1+\eta}\right)^{i\lambda}\frac{e^{-\pi\lambda/2}}{\sqrt{\cosh(\pi\lambda)}}$$
$$\times\frac{H^+_{1/2}\left(-\lambda,\frac{\chi}{2}\right)}{H^+_{-\frac{1}{2}}\left(-\lambda,\frac{\chi}{2}\right)\dot{H}^+_{\frac{1}{2}}\left(-\lambda,\frac{\chi}{2}\right)+\dot{H}^+_{-\frac{1}{2}}\left(-\lambda,\frac{\chi}{2}\right)H^+_{\frac{1}{2}}\left(-\lambda,\frac{\chi}{2}\right)}d\lambda.$$

As shown in [5] when $\chi \to +\infty$ function $A$ reduces to Fock function.

## 5. EFFECT OF ELONGATION AND ANGLE OF INCIDENCE

Formulae (11), (12) are not difficult for computations due to the program developed in [15] for Coulomb wave functions. The subintegral expression rapidly decreases at infinity and only a finite interval of $\lambda$ contributes to the integrals. However when the elongation parameter $\chi$ or the scaled angle $\beta$ increase the interval becomes larger. For all the results presented below we choose it as $[-5-\beta^2/2,\ 5+\chi/2]$.

As it was already mentioned for axially incident wave its polarization $\varphi_0$ manifests itself only by the multiplier $\cos(\varphi-\varphi_0)$. For skew incidence distributions of induced current differ more essentially. Consider first the case of TE polarization. Fig. 2 presents current distributions on the bodies characterized by $\chi = 10$ and $\chi = 1$. Solid line corresponds to axial incidence, doted line is for the incidence at $\beta = 0.5$ and dashed line for $\beta = 1$. We present currents in sections at angles $\varphi = 90^\circ$ (curves No. 1), where the current is maximal and almost do not depend on the angle of incidence both for not so much elongated and for very much elongated bodies. This independence of the current on the angle of incidence is an expected result because the position of the geometric shadow boundary in this section remains at $\eta = 0$ for any angle of incidence. The maximal shift of the light-shadow boundary takes place in the section corresponding to $\varphi = 0^\circ$ and $\varphi = 180^\circ$, however the current is equal to zero at this section. Curves No. 2 on Fig. 2 present currents at the angle $\varphi = 45^\circ$, while curves No. 3 correspond to $\varphi = 15^\circ$. These sections lie on the more shadowed side of the body and the light-shadow boundary in these sections shifts to negative values of $\eta$. We see that when $\beta$ increases the current decreases on the body with $\chi = 10$, except for a small domain near the shadowed end of spheroid. This domain is reached by creeping waves that are excited at the opposite side which appears more illuminated. On ordinary bodies creeping waves attenuate much faster when in the case of strongly elongated body, and this effect is not seen. For very much elongated body characterized by $\chi = 1$, the current increases along the whole spheroid (see curves No. 2 and No. 3 on Fig. 2(b)).

For TM wave incidence the current distributions are different. Fig. 3 presents currents on the spheroid with $\chi = 10$. Fig. 4 presents analogous results for the body with $\chi = 1$. Compared to the case of TE polarization, the influence of the incidence angle is more noticeable. So, in the section $\varphi = 90^\circ$ where the light-shadow boundary remains at $\eta = 0$ for any $\beta$ the current essentially depends on $\beta$. For axial incidence the current is zero in this section, but already for $\beta = 0.1$

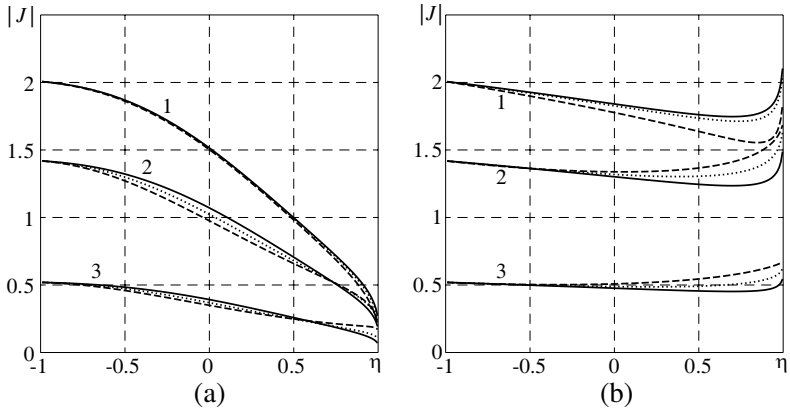


**Figure 2.** The currents on spheroid with (a) $\chi = 10$ and (b) $\chi = 1$ for TE wave incident at $\beta = 0$ (solid lines), $\beta = 0.5$ (dotted lines) and $\beta = 1$ (dashed lines) in sections $\varphi = 90^\circ$ (curves 1), $\varphi = 45^\circ$ (curves 2) and $\varphi = 15^\circ$ (curves 3).

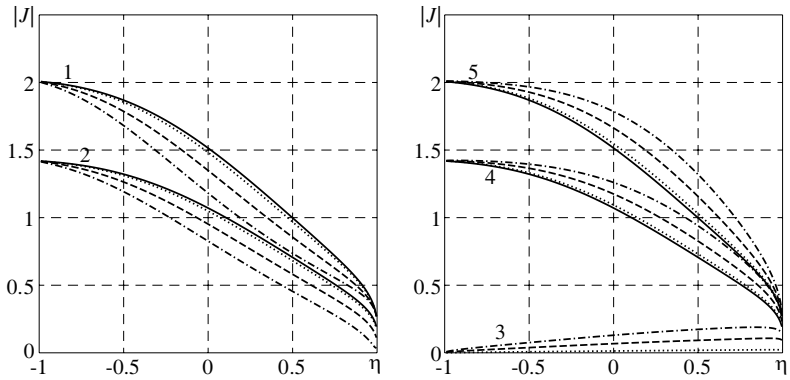


**Figure 3.** The currents on the surface of strongly elongated body with $\chi = 10$ for TM wave incidence at angles $\beta = 0$ (solid lines), $\beta = 0.1$ (doted lines), $\beta = 0.5$ (dashed lines) and $\beta = 1$ (dash-dotted lines) at sections $\varphi = 0^\circ$ (curves 1) and $\varphi = 45^\circ$ (curves 2), $\varphi = 90^\circ$ (curves 3), $\varphi = 135^\circ$ (curves 4) and $\varphi = 180^\circ$ (curves 5).

the current has a significant amplitude. The effect is greater, the more elongated the body is. In other sections on the less illuminated side, the current decreases and in the sections on the more illuminated side it increases.

As can be seen from Figs. 2(b) and 4, for small values of $\chi$ special functions $A^{\mathrm{TE}}$ and $A^{\mathrm{TM}}$ infinitely grow when $\eta \to +1$. The current

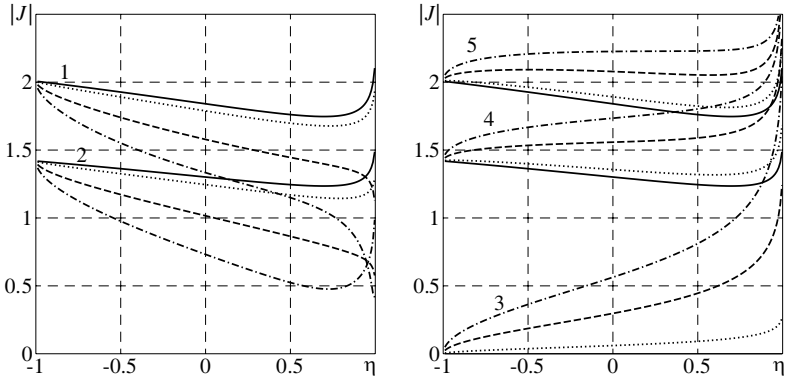


**Figure 4.** Same as on Fig. 3, but for $\chi = 1$.

increases near the shadowed end of the body due to focussing, however it remains finite. That means that the asymptotic formula (10) is not valid in the domain where $\eta$ is close to one. Generally speaking it can give wrong results in that domain (as well in a vicinity of illuminated ending) not only for bodies with small $\chi$, but this is not so noticeable.

## 6. CONCLUSION

We have derived a simple asymptotic formula for the currents on the surface of strongly elongated body induced by a high frequency plane wave incident at a small angle to the axis. The currents are given by special functions that depend on the point on the surface and two parameters. One parameter $\chi = ka^2/b$ characterizes the rate of elongation. The other parameter $\beta = \sqrt{kb}\vartheta$ is the scaled angle of incidence. For $\beta = 0$ the formulae contain ambiguity and computations are not possible. However reducing this ambiguity yields previously derived formulae for the axial incidence.

Numerical results computed according to these new asymptotics are in agreement with previous computations for axial incidence. They also show that the induced currents are more dependent on the angle of incidence for TM polarization and for more elongated bodies.

It is worth noting that the approach allows other types of incident wave to be considered as well. For example it is possible to study spherical waves diffraction, or very popular diffraction of Gaussian beams as in [16].

We note also that by matching with exact solution of diffraction by a paraboloid [17] in the same way as in [18] one can obtain the currents of backward wave that is formed of creeping waves that encircle the

shadowed ending of the body.

We have studied the diffraction field only in a boundary layer in a small vicinity of the surface. However if one knows the currents on the surface, the field in an arbitrary point can be expressed by Green's formula in the form of the integral over the surface.

## REFERENCES


1. Hong, S., "Asymptotic theory of electromagnetic and acoustic diffraction by smooth convex surfaces of variable curvature," *J. Math. Physics*, Vol. 8, No. 6, 1223, 1967.
2. Andronov, I. V. and D. Bouche, "Asymptotic of creeping waves on a strongly prolate body," *Ann. Telecommunications*, Vol. 49, Nos. 3–4, 205–210, 1994.
3. Andronov, I. V., "High-frequency asymptotics for diffraction by a strongly elongated body," *Antennas and Wireless Propagation Letters*, Vol. 8, 872, 2009.
4. Andronov, I. V., "High frequency asymptotics of electromagnetic field on a strongly elongated spheroid," *PIERS Online*, Vol. 5, No. 6, 536–540, 2009.
5. Andronov, I. V., D. P. Bouche, and M. Duruflé, "High-frequency diffraction of plane electromagnetic wave by an elongated spheroid," *IEEE Transactions on Antennas and Propag.*, Vol. 60, No. 11, 5286–5295, 2012.
6. Fock, V. A., "The distribution of currents induced by a plane wave on the surface of a conductor," *Journ. of Phys. of the U.S.S.R.*, Vol. 10, No. 2, 130, 1946.
7. Bowman, J. J., T. B. A. Senior, and P. L. E. Uslenghi, *Electromagnetic and Acoustic Scattering by Simple Shapes*, North-Holland, Amsterdam, 1969.
8. Kleshchev, A. A., "Scattering of sound by perfect spheroids in the limiting case of high frequencies," *Akust. Zhurn.*, Vol. 19, No. 5, 699–704, 1973 (in Russian).
9. Sammelman, G. S., D. H. Trivett, and R. H. Hackman, "High-frequency scattering from rigid prolate spheroids," *J. Acoust. Soc. Am.*, Vol. 83, 46–54, 1988.
10. Voshchinnikov, N. V. and V. G. Farafonov, "Light scattering by an elongated particle: Spheroid versus infinite cylinder," *Meas. Sci. Technol.*, Vol. 13, 249–255, 2002.
11. Kotsis, A. D. and J. A. Roumeliotis, "Electromagnetic scattering

by a metallic spheroid using shape perturbation method," *Progress In Electromagnetics Research*, Vol. 67, 113–134, 2007.

12. Komarov, I. V., L. I. Ponomarev, and S. Y. Slavyanov, *Spheroidal and Coulomb Spheroidal Functions*, Science, Moscow, 1976 (in Russian).
13. Abramowitz, M. and I. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, New York, 1964.
14. Andronov, I. V., "Diffraction of a plane wave incident at a small angle to the axis of a strongly elongated spheroid," *Acoustical Physics*, Vol. 58, No. 5, 521–529, 2012.
15. Thompson, I. J. and A. R. Barnett, "COULCC: A continued-fraction algorithm for Coulomb functions of complex order with complex arguments," *Computer Physics Communications*, Vol. 36, 363–372, 1985.
16. Sun, X., H. W. Wang, and H. Zhang, "Scattering of Gaussian beam by a spheroidal particle," *Progress In Electromagnetics Research*, Vol. 128, 539–555, 2012.
17. Fock, V. A., "Theory of diffraction by a paraboloid of revolution," *Diffraction of Electromagnetic Waves by Some Bodies of Revolution*, 5–56, Soviet Radio, Moscow, 1957.
18. Andronov, I. V. and D. Bouche, "Forward and backward waves in high-frequency diffraction by an elongated spheroid," *Progress In Electromagnetics Research B*, Vol. 29, 209–231, 2011.