

AN FFT-BASED APPROACH IN ACCELERATION OF DISCRETE GREEN'S FUNCTION METHOD FOR ANTENNA ANALYSIS

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Abstract—In this paper, the fast Fourier transform (FFT) to perform spatial convolutions of the time domain discrete Green's functions (DGF) method related to the analysis of the antenna with more than one dimension has been proposed. For this aim, the discrete Green's functions and the currents on the antenna have been appropriately defined periodic so as to use the zero padded fast Fourier transform. The computational complexity of this approach is $O(N_w N_x N_y N_z \log(N_x N_y N_z))$, contrary to $O(N_w N_x^2 N_y^2 N_z^2)$ for direct implementation of the convolutions. Simulation results demonstrate the great efficiency of the FFT-based spatial convolutions in the modeling of planar antennas.

1. INTRODUCTION

The time domain analysis is inevitable in electromagnetic problems. The finite-difference time-domain (FDTD) method is the differential equation based time domain method and has been extensively used for analysis of antennas, especially, wide band antennas [1–5]. However, the FDTD method can be very time consuming, practically, along with optimization techniques, due to the fact that the determining the fields at free space nodes and terminating computational space using absorbing boundary conditions are necessary in the FDTD method. With the aim of avoiding the need for absorbing boundary conditions and calculation of free space nodes around the antenna, the dyadic FDTD compatible Green's function, referred to as the discrete Green's function (DGF), in infinite free space, has been proposed by Vazquez and Parini in 1999 [6]. By the multidimensional z-transform of FDTD

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equations in time and spatial domains, they derived the analytical closed form of the discrete Green's functions (DGFs). Then, they applied those expressions in the modelling of the dipole antenna [7]. Some references have been presented for the other formulation of the DGFs. In [8], Kastner obtained an analytical expression for frequency and time domain of DGFs. Jeng also derived new closed form expressions for the dyadic discrete Green's function in free space using the ordinary z-transform along with the spatial partial differential operators [9]. In [10], a new technique for acceleration of the DGF computation based on the Jeng's formulation on the CPU-GPU heterogeneous parallel processing system has been proposed. It is worthwhile to point out that the discrete version of the Green's function is quite different from the discrete samples of continuous one. The need for the DGF to reflect the dispersion and anisotropy properties of FDTD equations was described by Kastner [8].

The DGF method has a significant potential to solve time domain EM problems. However, this method has been used for the modeling of wire antennas such as log periodic dipole and Yagi Uda arrays and presented significant saving in memory storage and computation time [11, 12]. One reason this method has not yet been considered in the analysis of planar antennas is there are multi dimensional convolutions for calculating current distribution on antennas. The direct implementation of multidimensional convolutions requires a much larger number of operations per time steps than conventional FDTD.

The acceleration of electromagnetic numerical simulations based on the fast Fourier transform (FFT) algorithm has been presented recently [13–15]. In this paper, we have shown that the DGF method can be combined with FFT algorithms to increase its performance in modeling of planar antennas. Due to the discrete nature of convolutions of this method, FFT can be used for improving the computational speed compared to the regular FDTD method. With this aim and without loss of generality, a simple planar bow-tie antenna has been considered and current distributions on it have been calculated using DGF method. Results and computational speed of DGF and FDTD methods have been compared.

2. DYADIC DISCRETE GREEN'S FUNCTION

According to [11], scattered electric field of an antenna at the time step ($n\Delta t$) can be obtained using the convolution of the current induced on the antenna and discrete Green's functions as (we denote a space point in a uniform rectangular lattice as $(i, j, k) = (i\Delta x, j\Delta y, k\Delta z)$, where

Δx , Δy , and Δz are, respectively, the lattice space increments in the x , y , and z coordinate directions):

$$\begin{aligned} \left[\vec{E}_{scat} \right]_{i,j,k}^n &= \sum_{n'=0}^n \sum_{i',j',k'} [\bar{G}]_{i-i',j-j',k-k'}^{n-n'} \left[\vec{J} \right]_{i',j',k'}^{n'} \\ i, j, k \text{ anywhere but } i', j', k' \text{ on the antenna} \end{aligned} \quad (1)$$

where $[\bar{G}]$ is the dyadic discrete Green's function. Due to the fact that the sum of the incident and scattered electric fields on the electric conductor antenna must vanish, we have:

$$\begin{aligned} \left[\vec{E}_{scat} \right]_{i,j,k}^n + \left[\vec{E}_{inc} \right]_{i,j,k}^n &= 0 \quad i, j, k \text{ on the antenna} \\ \sum_{n'=0}^n \sum_{i',j',k'} [\bar{G}]_{i-i',j-j',k-k'}^{n-n'} \left[\vec{J} \right]_{i',j',k'}^{n'} + \left[\vec{E}_{inc} \right]_{i,j,k}^n &= 0 \\ i, j, k, i', j', k' \text{ on the antenna} \end{aligned} \quad (2)$$

The update equation for the electric current on the antenna can be achieved using the property of the zero time step discrete Green's function which occurs at $n = n'$ and is equal to spatial delta Kroneker function. Therefore, the time step $n = n'$ can be considered separately from the rest of the time steps resulting in the following:

$$\begin{aligned} \begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix}_{i,j,k}^n &= \varepsilon \begin{bmatrix} E_{xinc} \\ E_{xiny} \\ E_{xinz} \end{bmatrix}_{i,j,k}^n + \varepsilon \sum_{n'=0}^{n-1} \sum_{i',j',k'} \begin{bmatrix} G_{xx} & G_{xy} & G_{xz} \\ G_{yx} & G_{yy} & G_{yz} \\ G_{zx} & G_{zy} & G_{zz} \end{bmatrix}_{i-i',j-j',k-k'}^{n-n'} \begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix}_{i',j',k'}^{n'} \quad (3) \end{aligned}$$

As we can see, in DGF method, the density of current at a certain time instant is in terms of the incident field at the same time instant and the current densities at previous time steps and the calculations are only performed on the cells in which the scatterer is placed instead of updating the fields in an iterative fashion as occurs in the FDTD method.

The dyadic discrete vector Green's function, $[G]$, can be obtained through the discrete Green's function of the scalar wave equation by applying the relationship between them. For example, the three matrix elements are computed as (it is assumed that $\Delta x = \Delta y = \Delta z$):

$$[G_{ejxx}]_{i,j,k}^n = -\frac{\mu}{\Delta t} \left(g_{i,j,k}^n - g_{i,j,k}^{n-1} - \alpha_x \sum_{l=0}^n g_{i-1,j,k}^{l-1} - 2g_{i,j,k}^{l-1} + g_{i+1,j,k}^{l-1} \right)$$

$$\begin{aligned}
[G_{ejxy}]_{i,j,k}^n &= -\frac{\mu}{\Delta t} \left(\alpha_x \sum_{l=0}^n g_{i,j+1,k}^{l-1} - g_{i-1,j+1,k}^{l-1} + g_{i-1,j,k}^{l-1} - g_{i,j,k}^{l-1} \right) \\
[G_{xz}]_{i,j,k}^n &= -\frac{\mu}{\Delta t} \left(\alpha_x \sum_{l=0}^n g_{i,j,k+1}^{l-1} - g_{i-1,j,k+1}^{l-1} + g_{i-1,j,k}^{l-1} - g_{i,j,k}^{l-1} \right) \quad (4)
\end{aligned}$$

$g_{i,j,k}^n$ is the impulse response of the scalar wave equation (scalar discrete Green's function) and can be achieved using multi dimensional z-transform as [6]:

$$\begin{aligned}
g_{i,j,k}^{n+1} &= \sum_{m=0}^{n/2} (-1)^m \binom{n-m}{m} \cdot \sum (n-2m; p_x, p_y, p_z) \prod_{\substack{t=i,j,k \\ s=x,y,z}} \\
&\alpha_s^{p_s} J_{p_s-t}^{(l,l)}(\xi_s) (R_{2s} - R_{1s})^{p_s-l} \quad (5)
\end{aligned}$$

where

$$\begin{aligned}
\alpha_s &= \frac{c^2 \Delta t^2}{\Delta s^2}, \quad \beta_s = \frac{3\alpha_s - 1}{3\alpha_s}, \quad R_{1,2s} = \beta_s \pm \sqrt{\beta_s^2 - 1} \\
\xi_s &= \frac{R_{1s} + R_{2s}}{R_{1s} - R_{2s}}, \quad (n-2m; p_x, p_y, p_z) = \frac{(n-2m)!}{p_x! p_y! p_z!}
\end{aligned}$$

and $J_n^{(\alpha,\beta)}(\xi)$ is the orthogonal Jacobi polynomial. In other words, $g_{i,j,k}^n$ is the solution of the second order central difference approximation of the scalar wave equation with Kronecker delta excitation expressed as (it has been considered as $i' = j' = k' = n' = 0$ due to the shifting capability of the Green's functions):

$$\begin{aligned}
&\frac{g_{i,j,k}^{n+1} - 2g_{i,j,k}^n + g_{i,j,k}^{n-1}}{c^2 (\Delta t)^2} - \frac{g_{i+1,j,k}^n - 2g_{i,j,k}^n + g_{i-1,j,k}^n}{(\Delta x)^2} - \frac{g_{i,j+1,k}^n - 2g_{i,j,k}^n + g_{i,j-1,k}^n}{(\Delta y)^2} \\
&- \frac{g_{i,j,k+1}^n - 2g_{i,j,k}^n + g_{i,j,k-1}^n}{(\Delta z)^2} = \delta_{i-i', j-j', k-k'}^{n-n'} \quad (6)
\end{aligned}$$

and can also be computed through a time stepping FDTD equation with Kronecker delta excitation as:

$$\begin{aligned}
g_{i,j,k}^{n+1} &= 2g_{i,j,k}^n + \alpha_x (g_{i+1,j,k}^n - 2g_{i,j,k}^n + g_{i-1,j,k}^n) \\
&+ \alpha_y (g_{i,j+1,k}^n - 2g_{i,j,k}^n + g_{i,j-1,k}^n) \\
&+ \alpha_z (g_{i,j,k+1}^n - 2g_{i,j,k}^n + g_{i,j,k-1}^n) - g_{i,j,k}^{n-1} \quad (7)
\end{aligned}$$

We rewrite Eq. (3) as:

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix}_{i,j,k}^n = \varepsilon \begin{bmatrix} E_{xinc} \\ E_{xiny} \\ E_{xinz} \end{bmatrix}_{i,j,k}^n + \varepsilon \begin{bmatrix} S_x \\ S_y \\ S_z \end{bmatrix}_{i,j,k}^{n-1} \quad (8)$$

where

$$\begin{bmatrix} S_x \\ S_y \\ S_z \end{bmatrix}_{i,j,k}^{n-1} = \varepsilon \sum_{n'=0}^{n-1} \sum_{i',j',k'} \begin{bmatrix} G_{xx} & G_{xy} & G_{xz} \\ G_{yx} & G_{yy} & G_{yz} \\ G_{zx} & G_{zy} & G_{zz} \end{bmatrix}_{i-i',j-j',k-k'}^{n-n'} \begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix}_{i',j',k''}^{n'} \quad (9)$$

3. DGF IMPLEMENTATION CHALLENGES AND SOLUTIONS

The implementation of Eq. (3) (or Eq. (8)) for the antenna with more than one dimension has some drawbacks. First, it is necessary that all previous time instant currents to be stored for the calculation of the summation of the convolution. The second problem is that, in addition to the main loop of updating current, another time loop is needed for the calculation of time convolution which, in turn, includes several loops for the spatial convolutions. It is noted that the convolutions cannot be calculated recursively according to the nature of DGFs. For the former, Parini and Vazquez have proposed windowing of DGFs, due to the fact that the amplitude of the DGFs decreases to the level that can be truncated [7]. For the latter, we suppose to use the FFT algorithm due to the discrete nature of the simulation. According to the convolution theorem, the convolution in one domain can be converted to the point-wise multiplication in another domain.

At this stage, the problem is that the currents and DGFs are not periodic. As we know, the Fourier transform of the discrete and non-periodic function is continuous and periodic which we get away from the discrete solution. Hence, we have to define DGFs and the currents periodic appropriately so as to use fast Fourier transform. For the FFT in time domain, the period of DGFs and currents can be assumed to be the width of the window function (N_w). While the period of the DGFs and currents, for the FFT in spatial domain, is supposed the maximum number of DGFs required. If the maximum dimensions of the antenna along x , y , and z axis are D_x , D_y , and D_z , respectively, the maximum number of the DGFs, along every axis, will be $N_x = 2 \times (D_x/\Delta x) + 1$, $N_y = 2 \times (D_y/\Delta y) + 1$, and $N_z = 2 \times (D_z/\Delta z) + 1$ which is related to the relative positions between every two nodes considering the direction of the current element. The period of the DGFs and currents along axis x , y , and z can be assumed N_x , N_y , and N_z , respectively. Note that the period of the currents is larger than the nodes that the currents are calculated; hence, the currents are zero-padded to make the period of current as the period of the DGFs.

Now, the question is for which convolutions FFT must be used, time or spatial or both. To answer this question, we have considered the computational complexity. The maximum computational complexity

of Eq. (9) for direct implementation of convolutions is $O(N_w N_x^2 N_y^2 N_z^2)$ at each time step. (Note that in case $n < N_w$, the computational complexity at each time step is $O(n N_x^2 N_y^2 N_z^2)$ and in case $n \geq N_w$, the computational complexity at each time step is $O(N_w N_x^2 N_y^2 N_z^2)$, so we have considered the maximum complexity). If Fourier transform is used for the computation of Eq. (9), it can be considered as:

$$\vec{S}_{i,j,k}^{n-1} = \text{IMFFT}(\text{MFFT}(\vec{G}(i, j, k, n)) \otimes \text{MFFT}(\vec{J}(i, j, k, n))) \quad (10)$$

where MFFT and IMFFT denote the multi-dimensional FFT and inverse multi-dimensional FFT, respectively. \otimes is the matrices element-by-element product operator. The complexity of a single multidimensional MFFT or IMFFT is $O(N_1 N_2 \dots \log(N_1 N_2 \dots))$, so the overall computational complexity is $O(3N_1 N_2 \dots \log(N_1 N_2 \dots) + N_1 N_2 \dots) = O(N_1 N_2 \dots \log(N_1 N_2 \dots))$, where $O(N_1 N_2 \dots)$ is the complexity of element-by-element product of matrices.

According to the above discussion, the computational complexity of Eq. (9) at each time step for the computation of the spatial convolutions using FFT is $O(N_w N_x N_y N_z \log(N_x N_y N_z))$, for the computation of the time convolution using FFT is $O(N_w N_x^2 N_y^2 N_z^2 \log(N_w))$, and for the computation of the time and spatial convolutions using FFT is $O(N_w N_x N_y N_z \log(N_x N_y N_z N_w))$. The computational complexity of mentioned algorithms is listed in Table 1. FFT-based time convolution algorithm has the highest complexity, while FFT-based spatial convolutions algorithm has the lowest complexity which is appropriate to speed up the DGF method.

Table 1. Comparison of the computational complexity of DGF method and FDTD.

Algorithm	Computational complexity at each time step
DGF with Direct implementation of the multi-dimensional convolutions	$O(N_w N_x^2 N_y^2 N_z^2)$
DGF with MFFT for spatial convolutions	$O(N_w N_x N_y N_z \log(N_x N_y N_z))$
DGF with FFT for time convolution	$O(N_w N_x^2 N_y^2 N_z^2 \log(N_w))$
DGF with MFFT for time and spatial convolutions	$O(N_w N_x N_y N_z \log(N_x N_y N_z N_w))$

To confirm above discussions, we have considered a half wavelength dipole antenna and simulated it with regular FDTD, DGF with direct time and spatial convolutions, DGF with FFT for time domain convolution, DGF with FFT for spatial domain convolutions, and DGF with FFT for time and spatial domain convolutions. In this example, the half wavelength dipole with $\lambda/40$ grid resolution is considered and 21 Green's function is needed in the calculation due to the fact that there are 21 relative positions between every two nodes and the Green's function is the function of the relative positions. Therefore, the period of DGFs and currents is considered as 21 for the FFT in spatial domain. (Note that, in this example, since the dipole antenna is one dimension and only one Green's function is computed along the antenna (for example G_{zz} for the antenna along the z axis), the maximum number of Green's function is dependent on the relative positions, not the direction of the element currents). In addition, the width of the step window function for truncation of the DGF is chosen 100 as a result of test and trail. Therefore, the period of currents and DGFs for FFT in time domain is considered 100.

For a 32-bit PC computer with quad core 3.2 GHz CPU and 4 GB RAM and MATLAB programming, the results are shown in Fig. 1. Each of the implementing algorithms of the DGF is faster than the FDTD for the one dimensional antenna. As expected, applying FFT for the time domain convolution reduce the execution speed as a result of having the maximum computational complexity. In contrast, with the use of FFT for the spatial domain convolution, the execution speed increases remarkably. The execution speed of the two dimensional FFT for the computation of the time and spatial domain convolutions is approximately equal to that of the direct convolution.

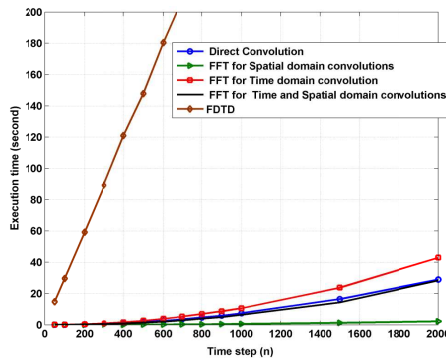


Figure 1. Comparing run time of FDTD method and different algorithms used to implement DGF method for modeling of half wavelength dipole antenna.

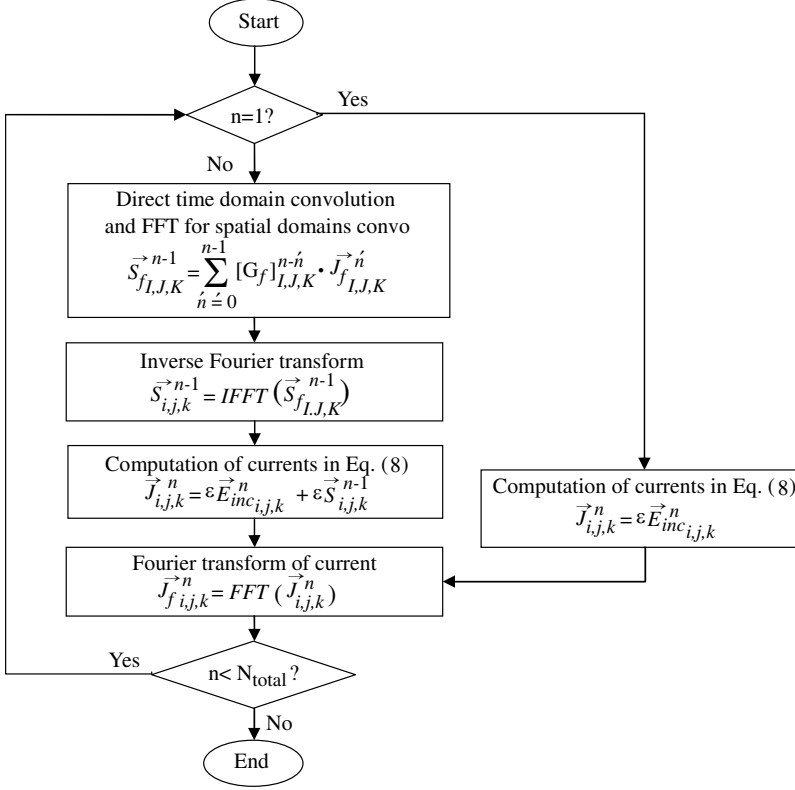


Figure 2. The flow chart of the applying FFT for computation of the spatial convolution in current solving of Eq. (8).

The flow chart of the application of FFT for the computation of the spatial convolutions in current solving of Eq. (8) is represented in Fig. 2. At the start, the variable \vec{S} is zero and the current vector \vec{J} is computed using the first term of the right side of the Eq. (8). In the next steps, the variable \vec{S} is computed using the point wise multiplication of the three dimensional spatial FFT of the previous currents and discrete Green's functions together with the direct convolution of the time domain.

4. PLANAR ANTENNA MODELING USING DGFS

In this section, we have computed time domain current distribution on the bowtie dipole antenna shown in Fig. 3 using DGF method. Since the DGF method is based on the FDTD equations, the discrete representation of the structure follows the same scheme as that in the

Yee algorithm. Fig. 4(a) shows the relative positions of the electric and magnetic currents in the DGF method having the same distribution of the electric and magnetic fields in the Yee grid. Accordingly, the current distribution of the bowtie dipole antenna is shown in Fig. 4(b). As we can see, since the antenna is supposed electric conductor, we have only considered electric currents. In addition, the antenna has two dimension; hence, two component of currents (J_x and J_y) have been calculated.

The spatial and time increments have been selected as $\Delta x = \Delta y = \Delta z = 1\text{ mm}$ and $c\Delta t = 0.5\Delta x$, respectively. In FDTD simulation, a spatial grid of $40 \times 60 \times 20$, with the same spatial and time

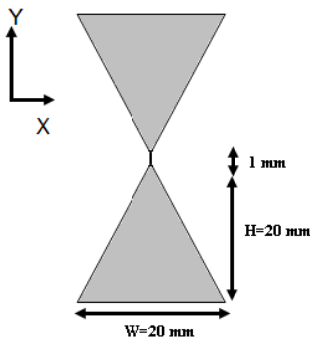


Figure 3. Geometry of the bowtie dipole antenna.

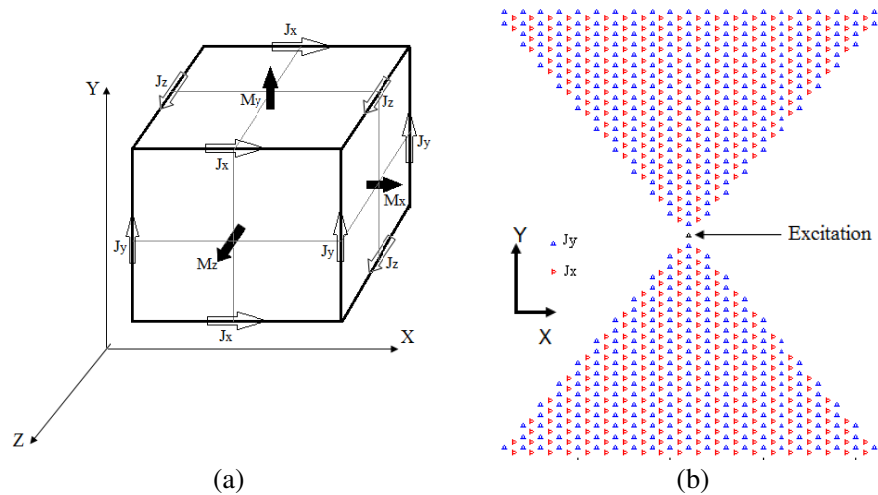


Figure 4. (a) 3D distribution of the electric and magnetic currents in the DGF cells. (b) Current distribution on the bowtie dipole antenna.

increments, and 11 cells of the CPML absorbing condition have been used. Whereas, in the DGFs simulation, computations are performed only on the surface of the antenna with the spatial grid of 21×42 . In the simulation with DGF, because the antenna is two dimensional, the two dimensional spatial FFT has been applied. The maximum number of DGFs along the x -axis is 41 and along the y -axis is 83, which is related to the G_{xy} and G_{yx} . It is worth mentioning that, in determining of these Green's functions, the direction of the current element is important. Considering the above description, the period of the currents and DGFs is assumed to be 41 along the x -axis and 83 along the y -axis in the two dimensional spatial FFT. Since the period of the currents in the x and y direction is larger than the nodes that the currents are calculated, in nodes where there are no currents, their current values are padded with zero. The antenna is excited by the incident Gaussian electric field in y direction at feed point. The temporal response of current at the feed point in comparison with FDTD method is shown in Fig. 5. Fig. 6 shows the return loss of the antenna. Very good agreement is observed between FDTD and DGF methods.

In Fig. 7, the execution speed of the FDTD, DGF with direct convolution and DGF with spatial FFT are compared. As expected, the runtime of the direct convolution of the DGF method for the modeling of the two dimensional antenna is much more than that of the one dimensional antenna. It is also considerably more than the runtime of the FDTD method. However, with the use of the two dimensional spatial FFT, the speed of the runtime of the DGF is approximately 85 percent better than that of the FDTD method.

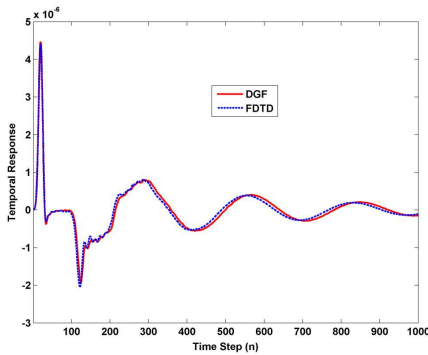


Figure 5. The temporal response of the current at the feed point to the Gaussian incident field.

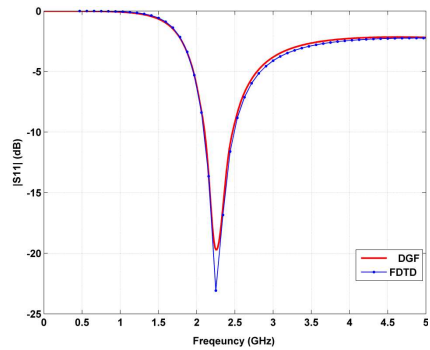


Figure 6. The return loss of the bowtie antenna.

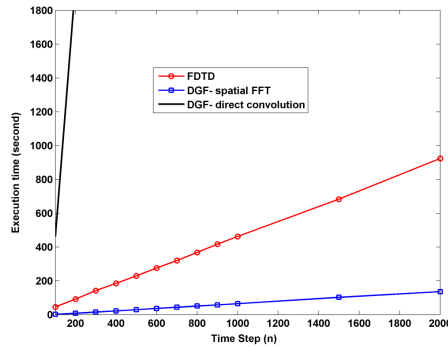


Figure 7. Comparing run time of the FDTD method and the DGF method for the modeling of the bowtie antenna.

5. CONCLUSIONS

In conclusion, we have shown that the implementation of the DGF method has faced some restrictions on the calculation of the multidimensional convolutions in modeling of antennas with more than one dimension. To solve the problem, we have supposed to use spatial discrete Fourier transform. The proposed method demonstrates very favorable runtime speed in the numerical simulations of the bowtie antenna using the DGF method compared to the FDTD method and direct convolution implementation of DGF method.

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