BEHAVIOR OF ELECTROMAGNETIC WAVES AT DIELECTRIC FRACTAL-FRACTAL INTERFACE IN FRACTIONAL SPACES

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Abstract—In this paper, reflection and transmission coefficients at dielectric fractal-fractal interface are discussed. The ratio of permittivity of the two dielectric fractal media is kept constant, while the dimension is varied in order to get the desired results. Conventional results are recovered for the integer dimensions. The proposed expressions are useful to study the behavior of electromagnetic waves for non-integer dimensions, multiple fractal interfaces and waveguides. Moreover, it is also helpful to understand the variation in the magnitudes of reflection and transmission coefficients with the difference in dimensionality at interface of the two fractal media.

1. INTRODUCTION

The concept of fractional space is effectively used in many areas of physics to describe the effective parameters of physical systems. Mandelbrot introduced the concept of "Fractal" to describe complex structures [1]. Highly complex structures can be modeled using fractional space concept which cannot be described by Euclidean geometry. Sometime they represent natural occurring phenomena and geometries that are highly complex like roughness of ocean floor, dust particles, snow and mountains etc., better than conventional geometry. We can also find fractals in the field of integrated circuits, weather prediction, image compression algorithms and antenna design. An important property of fractals is that complex geometries can be defined with very less number of parameters because all fractals are self-similar and repeat themselves at different scales. These

complex structures can be characterized by using fractional dimension D, at both macroscopic and microscopic levels [2, 3]. Knowing the importance of fractional space, it has been the subject of interest for researchers for past two decades [4–9].

In order to get maximum benefit of these fractal models, it is vital to simplify the theory of electromagnetics in fractional spaces. In this respect, Martin et al. derived Ampere's and Faraday's laws for non-integer dimensional space [10]. Similarly, solutions of Poisson's and Laplace equation for fractional space are presented in [11, 12]. Solution of electromagnetic wave propagation in fractional space was presented by Zubair et al. and has given solutions for plane waves in D-dimensional fractional space [13–18]. The antenna radiation in fractional space is also investigated and presented by Mughal et al. [19]. Recently, Asad et al. have worked on interaction of electromagnetic waves at dielectric-fractal interface [20], which gives basis to analyze the behavior of these waves for multiple interfaces filled with fractal media. Therefore, one is motivated to work on the behavior of electromagnetic waves at dielectric interface of two fractal media. This work will facilitate us to analyze the behavior of electromagnetic waves for non-integer dimensions and the multiple fractal interfaces, waveguides and also help to find out the desired magnitudes of transmission and reflection coefficients with the difference in the dimensionality of two media.

In this paper, the expressions of transmission and reflection coefficients are derived for dielectric fractal-fractal interface. Classical results are formulated from fractional results when integer dimension space is considered. The transmission and reflection coefficients for parallel polarization are discussed in Section 2. Section 3 covers the transmission and reflection coefficients for perpendicular polarization. Finally in Section 4, it is shown that exact classical results can be recovered from proposed expressions when integer dimension is considered. This approach is useful for other multiple interfaces filled with fractal media as well.

2. PARALLEL POLARIZATION

To study the behavior of electromagnetic waves at an interface of two dielectric-fractal media, consider two dielectric fractal media with constitutive parameters (ϵ_1, μ_1, $1 < D_1 \leq 2$) and (ϵ_2, μ_2, $1 < D_2 \leq 2$), both media exhibits quasi fractional space properties. The fractionality exists only in z-axis of the fractional space and boundary is assumed to be infinite. The time dependency $\exp(i\omega t)$ has been omitted throughout the paper. For simplicity, we also assumed that the
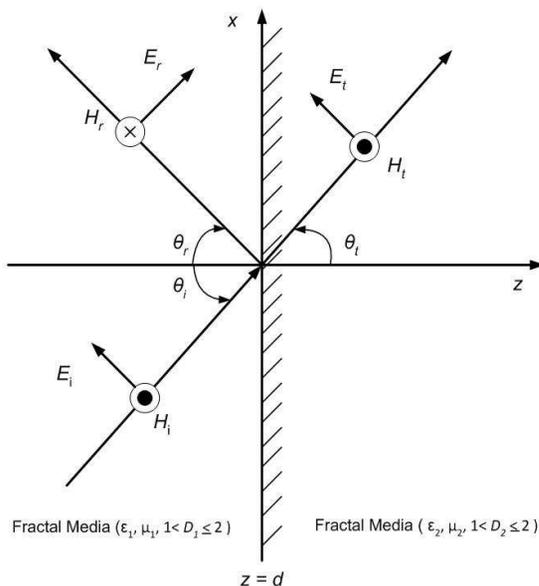
Figure 1. Parallel polarized wave at dielectric fractal-fractal interface.

permeability of the both media is same, i.e., $\mu_1 = \mu_2 = \mu_0$, where μ_0 is the permeability of free space. The geometry of the incident, reflected and transmitted waves for parallel polarization is shown in Figure 1, where the dielectric fractal-fractal interface is located at $z = d$. The equations for incident, reflected and transmitted waves are as follows:

$$\mathbf{E}_i = (\hat{\mathbf{a}_x} \cos\theta_i - \hat{\mathbf{a}_z} \sin\theta_i) \, E_0 \exp\left(-i\beta_1 \sin\theta_i x\right)$$
$$(\beta_1 \cos\theta_i z)^{n_1} H_{n_1}^2 (\beta_1 \cos\theta_i z) \tag{1}$$

$$\mathbf{E}_r = (\hat{\mathbf{a}_x} \cos\theta_r + \hat{\mathbf{a}_z} \sin\theta_r) \, E_0\Gamma \exp\left(-i\beta_1 \sin\theta_r x\right)$$
$$(\beta_1 \cos\theta_r z)^{n_1} H_{n_1}^1 (\beta_1 \cos\theta_r z) \tag{2}$$

$$\mathbf{E}_t = (\hat{\mathbf{a}_x} \cos\theta_t - \hat{\mathbf{a}_z} \sin\theta_t) \, E_0 \mathrm{T} \exp\left(-i\beta_2 \sin\theta_t x\right)$$
$$(\beta_2 \cos\theta_t z)^{n_2} H_{n_2}^2 (\beta_2 \cos\theta_t z) \tag{3}$$

$$\mathbf{H}_i = \hat{\mathbf{a}_y} \frac{E_0}{\eta_1} \exp\left(-i\beta_1 \sin\theta_i x\right)(\beta_1 \cos\theta_i z)^{n_{h1}} H_{n_{h1}}^2 (\beta_1 \cos\theta_i z) \tag{4}$$

$$\mathbf{H}_r = -\hat{\mathbf{a}_y} \frac{E_0\Gamma}{\eta_1} \exp\left(-i\beta_1 \sin\theta_r x\right)(\beta_1 \cos\theta_r z)^{n_{h1}} H_{n_{h1}}^1 (\beta_1 \cos\theta_r z) \tag{5}$$

$$\mathbf{H}_t = \hat{\mathbf{a}_y} \frac{E_0\mathrm{T}}{\eta_2} \exp\left(-i\beta_2 \sin\theta_t x\right)(\beta_2 \cos\theta_t z)^{n_{h2}} H_{n_{h2}}^2 (\beta_2 \cos\theta_t z) \tag{6}$$

where $\beta_1 = \omega\sqrt{\mu_1\epsilon_1}$ and $\beta_2 = \omega\sqrt{\mu_2\epsilon_2}$ are wave numbers. The wave impedance of both media are $\eta_1 = \sqrt{\mu_1/\epsilon_1}$ and $\eta_2 = \sqrt{\mu_2/\epsilon_2}$. Subscript 1, 2 are used to represent first and second media. The exponential function is used to describe wave propagation in x direction and Hankel function of order n is used to represent wave propagation in z direction. Backward traveling waves are represented by Hankel function of first kind of order n and forward traveling waves are represented by Hankel function of second kind of order n [15, 20]. The value of $n_i = |3 - D_i|/2$ and $n_{hi} = |D_i - 1|/2$ for $i = 1, 2$, and D is the dimension. Γ and T are unknown reflection and transmission coefficients. The general solution of electromagnetic wave in fractional space is valid only for far field. Hence value of d should be very large. At the interface $(z = d)$, tangential components of magnetic and electric fields are continues, i.e.,

$$E_{ix}(z=d) + E_{rx}(z=d) = E_{tx}(z=d) \tag{7}$$

$$H_{iy}(z=d) + H_{ry}(z=d) = H_{ty}(z=d) \tag{8}$$

The unknown coefficients can be find out by putting the equations of electric and magnetic fields in above boundary conditions. By putting (1)–(6) in (7)–(8), We get

$$E_0 \cos\theta_i \exp\left(-i\beta_1\sin\theta_i x\right)(\beta_1\cos\theta_i d)^{n_1} H_{n_1}^2(\beta_1\cos\theta_i d)$$

$$+\Gamma E_0 \cos\theta_r \exp\left(-i\beta_1\sin\theta_r x\right)(\beta_1\cos\theta_r d)^{n_1} H_{n_1}^1(\beta_1\cos\theta_r d)$$

$$= T E_0 \cos\theta_t \exp\left(-i\beta_2\sin\theta_t x\right)(\beta_2\cos\theta_t d)^{n_2} H_{n_2}^2(\beta_2\cos\theta_t d) \tag{9}$$

and

$$\frac{E_0}{\eta_1} \exp\left(-i\beta_1\sin\theta_i x\right)(\beta_1\cos\theta_i d)^{n_{h1}} H_{n_{h1}}^2(\beta_1\cos\theta_i d)$$

$$+\frac{\Gamma E_0}{\eta_1} \exp\left(-i\beta_1\sin\theta_r x\right)(\beta_1\cos\theta_r d)^{n_{h1}} H_{n_{h1}}^1(\beta_1\cos\theta_r d)$$

$$= \frac{T E_0}{\eta_2} \exp\left(-i\beta_2\sin\theta_t x\right)(\beta_2\cos\theta_t d)^{n_{h2}} H_{n_{h2}}^2(\beta_2\cos\theta_t d) \tag{10}$$

Equations (9) and (10) are the functions of x and z. The condition of continuity must hold at the interface, for all x. The variation of functions of x must be same on both sides of the interface. Hence,

$$\beta_1\sin\theta_i = \beta_1\sin\theta_r = \beta_2\sin\theta_t \tag{11}$$

and we have two important relations as follows:

$$\theta_i = \theta_r \tag{12}$$

$$\beta_1\sin\theta_i = \beta_2\sin\theta_t \tag{13}$$

by inserting (12) and (13) in (9) and (10), we have the following two relations,

$$\cos\theta_i(\beta_1\cos\theta_i d)^{n_1}H_{n_1}^2(\beta_1\cos\theta_i d)$$
$$+\Gamma\cos\theta_i(\beta_1\cos\theta_i d)^{n_1}H_{n_1}^1(\beta_1\cos\theta_i d)$$
$$= \mathrm{T}\cos\theta_t(\beta_2\cos\theta_t d)^{n_2}H_{n_2}^2(\beta_2\cos\theta_t d) \tag{14}$$

$$\frac{1}{\eta_1}(\beta_1\cos\theta_i d)^{n_{h1}}H_{n_{h1}}^2(\beta_1\cos\theta_i d)$$
$$+\frac{1}{\eta_1}\Gamma(\beta_1\cos\theta_i d)^{n_{h1}}H_{n_{h1}}^1(\beta_1\cos\theta_i d)$$
$$= \frac{1}{\eta_2}\mathrm{T}(\beta_2\cos\theta_t d)^{n_{h2}}H_{n_{h2}}^2(\beta_2\cos\theta_t d) \tag{15}$$

by solving the (14) and (15) simultaneously, we can determine the required reflection and transmission coefficients as given below,

$$\Gamma_\| = \frac{\eta_2\cos\theta_t CE - \eta_1\cos\theta_i AG}{\eta_1\cos\theta_i BG + \eta_2\cos\theta_t CF} \tag{16}$$

$$\mathrm{T}_\| = \frac{\eta_2\cos\theta_i(BE + AF)}{\eta_1\cos\theta_i BG + \eta_2\cos\theta_t CF} \tag{17}$$

where,

$$A = (\beta_1\cos\theta_i d)^{n_1}H_{n_1}^2(\beta_1\cos\theta_i d) \tag{18a}$$
$$B = (\beta_1\cos\theta_i d)^{n_1}H_{n_1}^1(\beta_1\cos\theta_i d) \tag{18b}$$
$$C = (\beta_2\cos\theta_t d)^{n_2}H_{n_2}^2(\beta_2\cos\theta_t d) \tag{18c}$$
$$E = (\beta_1\cos\theta_i d)^{n_{h1}}H_{n_{h1}}^2(\beta_1\cos\theta_i d) \tag{18d}$$
$$F = (\beta_1\cos\theta_i d)^{n_{h1}}H_{n_{h1}}^1(\beta_1\cos\theta_i d) \tag{18e}$$
$$G = (\beta_2\cos\theta_t d)^{n_{h2}}H_{n_{h2}}^2(\beta_2\cos\theta_t d) \tag{18f}$$

for parallel polarization, Brewster's angel ($\theta_i = \theta_b$) can be find out by setting (16) equal to zero,

$$\Gamma_\| = \frac{\eta_2\cos\theta_t CE - \eta_1\cos\theta_b AG}{\eta_1\cos\theta_b BG + \eta_2\cos\theta_t CF} = 0 \tag{19}$$

or,

$$\cos\theta_b = \frac{\eta_2}{\eta_1}\frac{CE}{AG}\cos\theta_t \tag{20}$$

by using (13), (20) can be written as:

$$\sin\theta_b = \sqrt{\frac{\epsilon_2 AG/\epsilon_1 CE - \mu_2 CE/\mu_1 AG}{\epsilon_2 AG/\epsilon_1 CE - \epsilon_1 CE/\epsilon_2 AG}} \tag{21}$$

since sine function cannot exceed unity, (21) exists only if:

$$\epsilon_2 AG/\epsilon_1 CE - \mu_2 CE/\mu_1 AG \le \epsilon_2 AG/\epsilon_1 CE - \epsilon_1 CE/\epsilon_2 AG \quad (22)$$

for $\mu_1 = \mu_2$, (21) becomes,

$$\sin \theta_b = \sqrt{\frac{\epsilon_2 AG/\epsilon_1 CE - CE/AG}{\epsilon_2 AG/\epsilon_1 CE - \epsilon_1 CE/\epsilon_2 AG}} \quad (23)$$

$$\theta_b = \sin^{-1} \sqrt{\frac{\epsilon_2 AG/\epsilon_1 CE - CE/AG}{\epsilon_2 AG/\epsilon_1 CE - \epsilon_1 CE/\epsilon_2 AG}} \quad (24)$$

for parallel polarization, the reflection coefficient reduces to zero at Brewster's angle θ_b given by (21) or (24).

3. PERPENDICULAR POLARIZATION

The geometry of the incident, reflected and transmitted waves for perpendicular polarization is shown in Figure 2. The corresponding equations are as follows:

$$\mathbf{E}_i = \hat{\mathbf{a}_y} E_0 \exp\left(-i\beta_1 \sin \theta_i x\right)(\beta_1 \cos \theta_i z)^{n_1} H_{n_1}^2 (\beta_1 \cos \theta_i z) \quad (25)$$

$$\mathbf{E}_r = \hat{\mathbf{a}_y} E_0 \Gamma \exp\left(-i\beta_1 \sin \theta_r x\right)(\beta_1 \cos \theta_r z)^{n_1} H_{n_1}^1 (\beta_1 \cos \theta_r z) \quad (26)$$
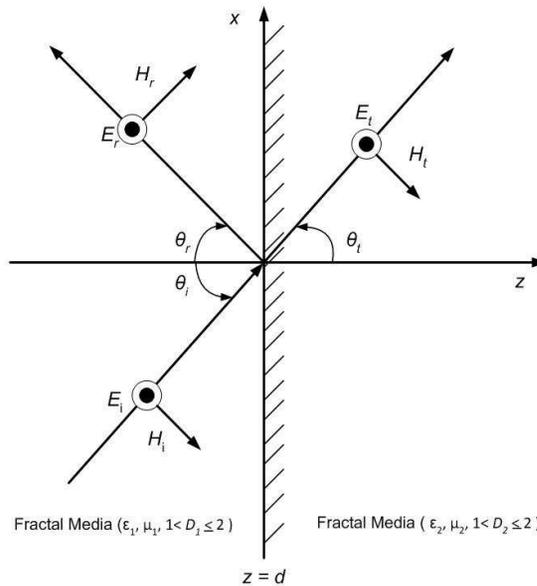

Figure 2. Perpendicular polarized wave at dielectric fractal-fractal interface.

$$\mathbf{E}_t = \hat{\mathbf{a}_y} E_0 T \exp\left(-i\beta_2 \sin\theta_t x\right)(\beta_2 \cos\theta_t z)^{n_2} H_{n_2}^2(\beta_2 \cos\theta_t z) \quad (27)$$

$$\mathbf{H}_i = (-\hat{\mathbf{a}_x}\cos\theta_i + \hat{\mathbf{a}_z}\sin\theta_i)\frac{E_0}{\eta_1}\exp\left(-i\beta_1\sin\theta_i x\right)$$
$$(\beta_1 \cos\theta_i z)^{n_{h1}} H_{n_{h1}}^2(\beta_1 \cos\theta_i z) \quad (28)$$

$$\mathbf{H}_r = (\hat{\mathbf{a}_x}\cos\theta_r + \hat{\mathbf{a}_z}\sin\theta_r)\frac{E_0\Gamma}{\eta_1}\exp\left(-i\beta_1\sin\theta_r x\right)$$
$$(\beta_1 \cos\theta_r z)^{n_{h1}} H_{n_{h1}}^1(\beta_1 \cos\theta_r z) \quad (29)$$

$$\mathbf{H}_t = (-\hat{\mathbf{a}_x}\cos\theta_t + \hat{\mathbf{a}_z}\sin\theta_t)\frac{E_0 T}{\eta_2}\exp\left(-i\beta_2\sin\theta_t x\right)$$
$$(\beta_2 \cos\theta_t z)^{n_{h2}} H_{n_{h2}}^2(\beta_2 \cos\theta_t z) \quad (30)$$

by following same procedure of previous case of parallel polarization, we can determine the required reflection and transmission coefficients as given below,

$$\Gamma_\perp = \frac{\eta_2 \cos\theta_i CE - \eta_1 \cos\theta_t AG}{\eta_1 \cos\theta_t BG + \eta_2 \cos\theta_i CF} \quad (31)$$

$$T_\perp = \frac{\eta_2 \cos\theta_i (BE + AF)}{\eta_1 \cos\theta_t BG + \eta_2 \cos\theta_i CF} \quad (32)$$

4. RESULTS AND SIMULATIONS

The coefficients of reflection and transmission derived in previous sections are for non-integer dimension space. The classical results
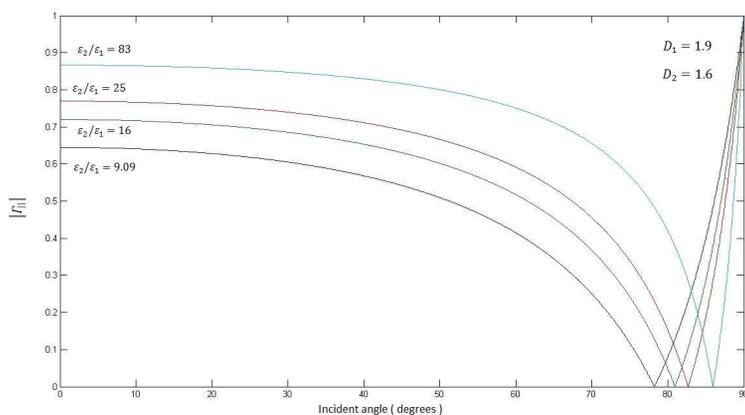

Figure 3. Magnitude of reflection coefficients for parallel polarization for non-integer dimension space.

can be found by taking integer values of dimension i.e., $D = 2$. For $D = 2$, the order of Hankel function will become $n = 1/2$. For far-field approximation, the expression for Hankel function of first kind is, as follow:

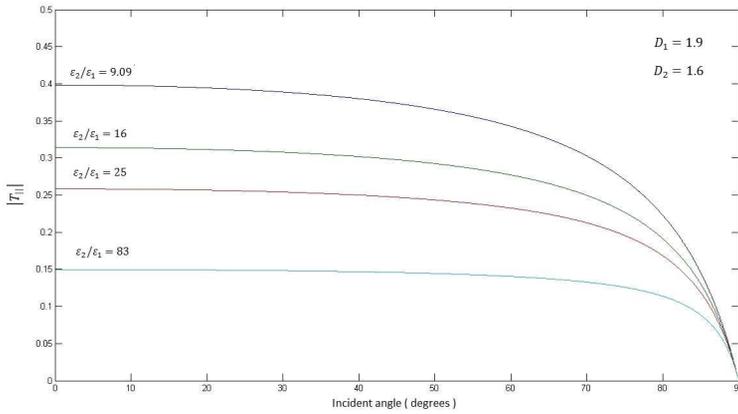$$H_{1/2}^1(z) = \sqrt{\frac{2}{\pi z}} e^{j(z-\pi/2)} \tag{33}$$

Figure 4. Magnitude of transmission coefficients for parallel polarization for non-integer dimension space.
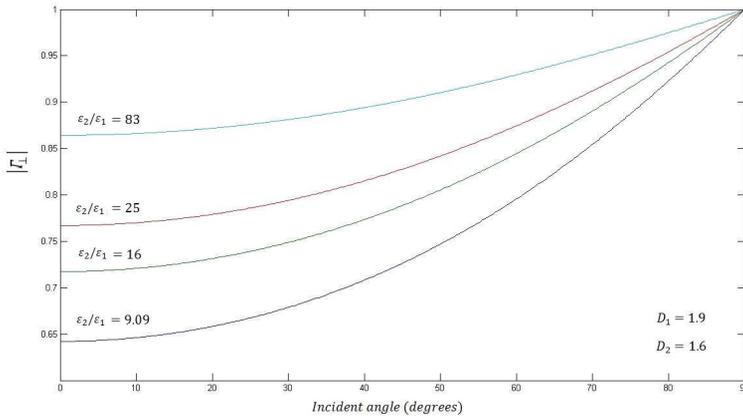

Figure 5. Magnitude of reflection coefficients for perpendicular polarization for non-integer dimension space.

and the expression of the Hankel function of second kind is,

$$H_{1/2}^2(z) = \sqrt{\frac{2}{\pi z}} e^{-j(z-\pi/2)} \tag{34}$$

by inserting (33) and (34) in (16), (17) and (24),

$$\Gamma_\parallel = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t} \tag{35}$$
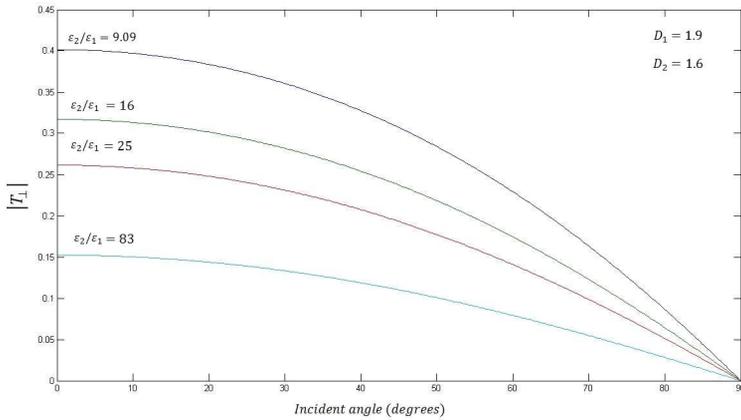

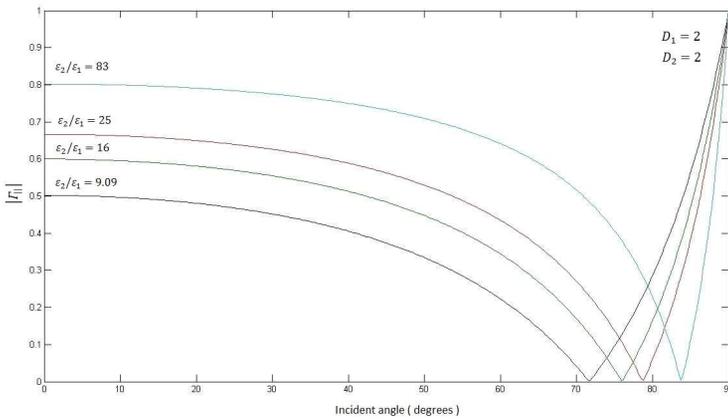**Figure 6.** Magnitude of transmission coefficients for perpendicular polarization for non-integer dimension space.

Figure 7. Magnitude of reflection coefficients for parallel polarization for integer dimension space.

$$T_{\parallel} = \frac{2\eta_2 \cos \theta_i}{\eta_1 \cos \theta_i + \eta_2 \cos \theta_t} \tag{36}$$

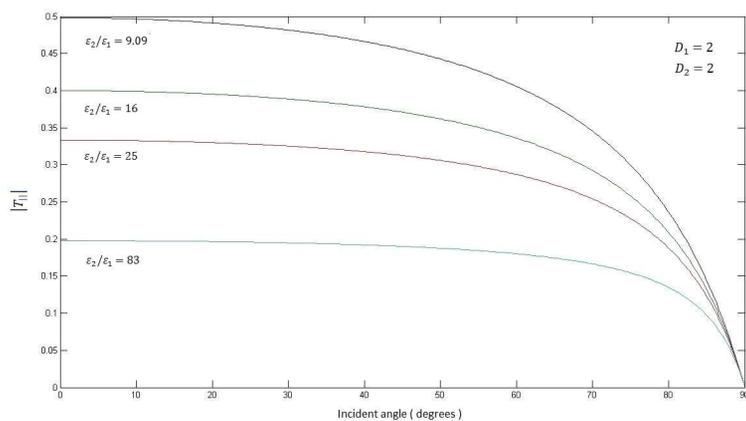$$\theta_b = \sin^{-1} \sqrt{\frac{\epsilon_2}{\epsilon_1 + \epsilon_2}} \tag{37}$$

Figure 8. Magnitude of transmission coefficients for parallel polarization for integer dimension space.
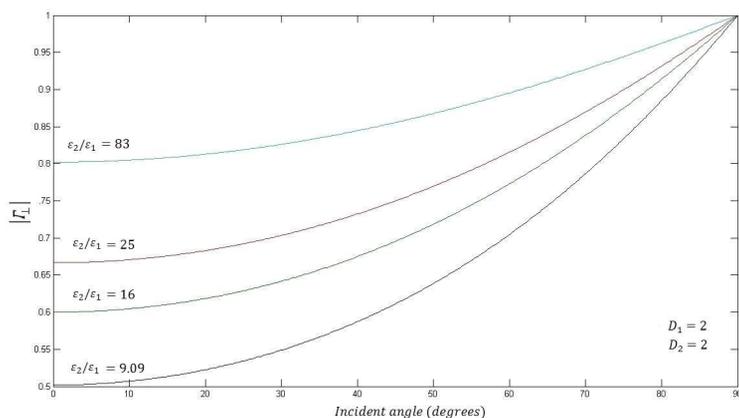

Figure 9. Magnitude of reflection coefficients for perpendicular polarization for integer dimension space.

Similarly, by inserting (33) and (34) in (31) and (32) the reflection and transmission coefficients for perpendicular polarization are as following:

$$\Gamma_\perp = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \tag{38}$$

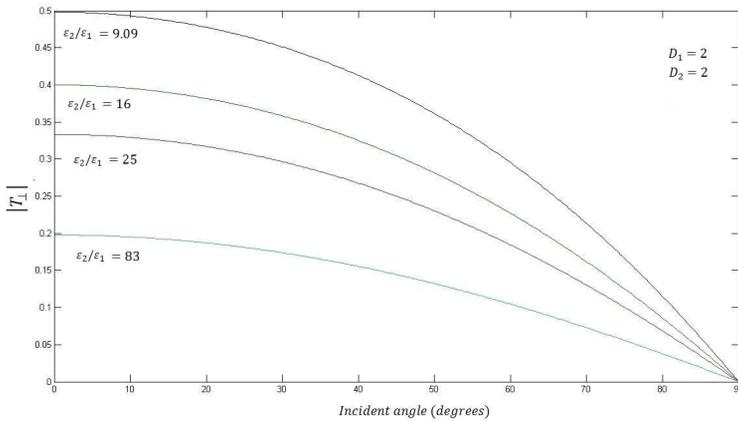$$T_\perp = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \tag{39}$$

Figure 10. Magnitude of transmission coefficients for perpendicular polarization for integer dimension space.
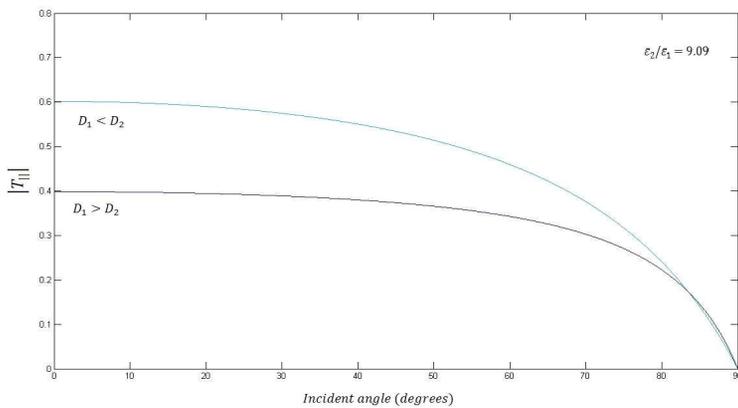

Figure 11. Magnitude of transmission coefficients for parallel polarization for varying dimension D_1 and D_2.

which are the same as that given by Balanis [21]. The transmission and
reflection coefficients for both parallel and perpendicular polarization
are plotted against varying incident angles. Figure 3 is the plot of
reflection coefficients for parallel polarization against varying angle
of incidence with non-integer dimension. The plot of transmission
coefficients for parallel polarization against varying angle of incidence
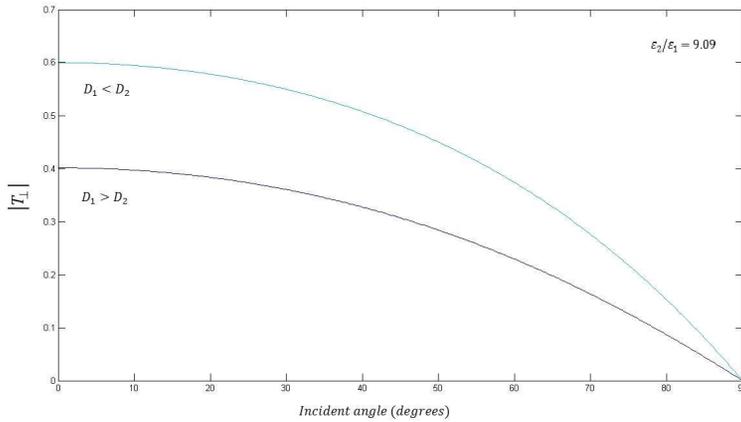with non-integer dimension is shown in Figure 4. Same is done for

Figure 12. Magnitude of transmission coefficients for perpendicular
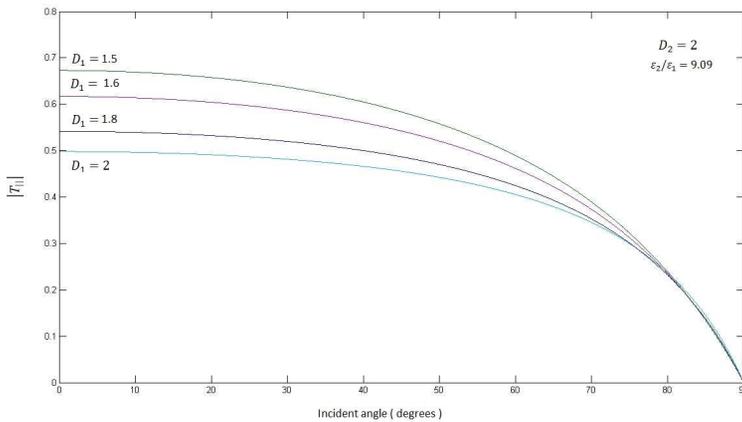polarization for varying dimension D_1 and D_2.

Figure 13. Magnitude of transmission coefficients for parallel
polarization for varying dimension D_1.

perpendicular polarization and their plots are shown in Figure 5 and Figure 6. All plots are calculated for fix ratio of permittivities of both fractal media. Classical results are recovered and plotted for comparison. Plots of transmission and reflection coefficients for both parallel and perpendicular polarization with integer dimension are shown in Figure 7 through Figure 10. These are exactly same as given
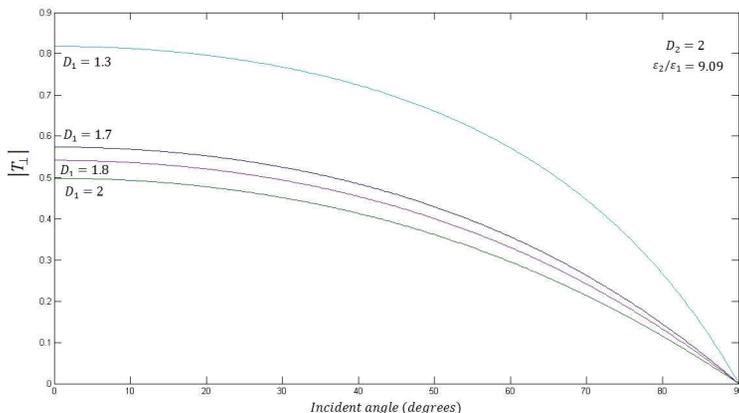

Figure 14. Magnitude of transmission coefficients for perpendicular polarization for varying dimension D_1.
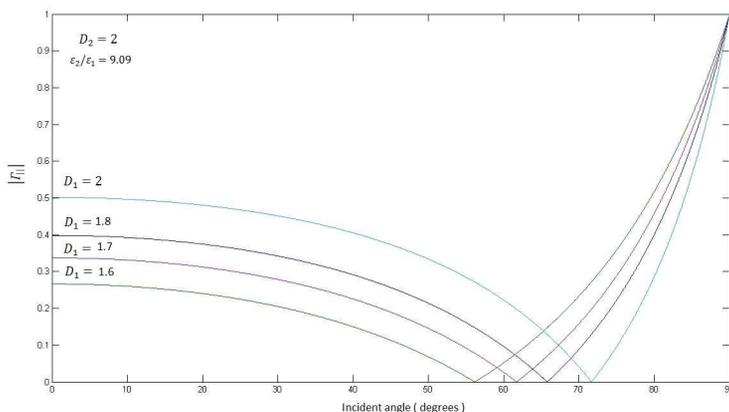

Figure 15. Change in magnitude of reflection coefficient and Brewster's angle for parallel polarization for varying dimension, D_1.
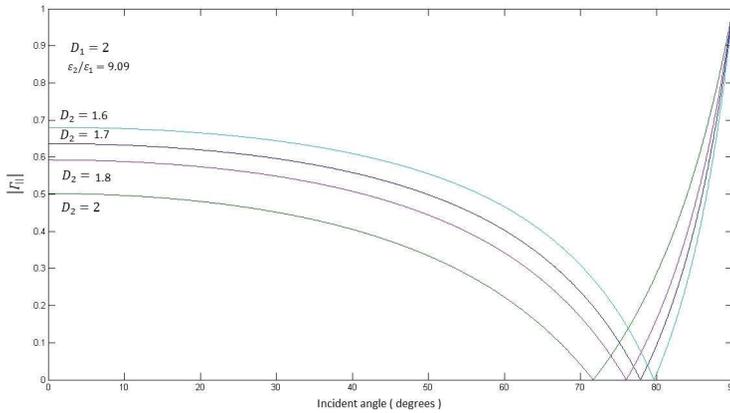
Figure 16. Change in magnitude of reflection coefficient and Brewster's angle for parallel polarization for varying dimension, D_2.

by Balanis for the same fix ratio of permittivities of both media [21].

Furthermore, it was also investigated that the magnitudes of transmission and reflection coefficients changes with the variation of the difference in dimensionality of the two fractal media. Hence using this important feature of fractals, dimension D can be used as a third parameter in order to find out required magnitude of transmission and reflection coefficients shown in Figure 11 through Figure 14. Figure 15 and Figure 16 show the change in magnitude of reflection coefficient and Brewster's angle with change in dimension.

5. CONCLUSION

Reflection and transmission coefficients at dielectric fractal-fractal interface are derived in this paper. The effect of fractional dimension on the magnitude of reflection and transmission coefficients for different ratio of permittivity of the two dielectric fractal media was investigated. Conventional results are recovered for the integer dimensions. The proposed expressions are useful to study the behavior of electromagnetic waves for non-integer dimension, multiple fractal interfaces and waveguides. Moreover, this work is also helpful to understand the variation in the magnitudes of reflection and transmission coefficients with the difference in dimensionality at interface of the two fractal media.

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