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RECURSIVE IMPLEMENTATION OF NATURAL FREQUENCY-BASED RADAR DETECTION USING THE LRT SCHEME

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Abstract—We address the performance analysis of the natural frequency-based radar target detection in this paper. We show how to calculate the detection performance recursively by making a polynomial approximation of the probability density function (PDF) of the standard normal distribution. Why we make a polynomial approximation of the PDF of the standard normal distribution is that the PDF of the standard normal distribution is not analytically integrable but that the polynomial is definitely analytically integrable, which makes it possible to calculate the detection performance without look-up table. The Taylor polynomial is used for an approximation of the PDF of the standard normal distribution. We derive the error of the approximation, the bound of the error of approximation, and the optimal polynomial approximation in the sense that the bound of the error of the approximation is minimized. We validate the derived expressions via numerical simulation.

1. INTRODUCTION

There have been many studies on radar target recognition [1–9] and detection [10–23]. In natural frequency-based radar signal processing, accurate estimation of natural frequency is quite important [24–26]. Radar has been one of the classical applications of array signal processing [27–32].

The performance of the natural-frequency-based radar target detection scheme has been analyzed in time domain [10] and in frequency domain [20]. In this paper, we consider recursive implementation of the LRT-based detection scheme in [10].

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2. LATE TIME RESPONSE IN TERMS OF THE NATURAL FREQUENCY

The late time response with sampling interval Δt can be written as (1)

$$y_i = u_i + g_i = \sum_{m=1}^{p} a_m z_m^i + g_i \quad i = 1, 2, \ldots, q \tag{1}$$

where g_i is zero-mean Gaussian-distributed with variance of σ^2 [9].

If we define

$$\mathbf{y}_q = \begin{bmatrix} y_1 & y_2 & \ldots & y_q \end{bmatrix}^{\mathrm{T}} \tag{2}$$

$$\mathbf{a} = \begin{bmatrix} \mathrm{a}_1 & \mathrm{a}_2 & \ldots & \mathrm{a}_p \end{bmatrix}^{\mathrm{T}}, \tag{3}$$

(1) can be written as

$$\mathbf{y}_q = \mathbf{Ba} + \mathbf{g}_q = \mathbf{u}_q + \mathbf{g}_q \tag{4}$$

where $\{\mathbf{B}\}_{im}$ is defined as

$$\{\mathbf{B}\}_{im} = z_m^i, \tag{5}$$

and $\mathbf{g}_q$ and $\mathbf{u}_q$ are similarly defined as $\mathbf{y}_q$.

3. LIKELIHOOD RATIO TEST

Detection problem can be formulated as

$$H_1 : y_i = u_i + g_i = \sum_{m=1}^{p} a_m z_m^i + g_i \tag{6}$$

$$H_0 : y_i = g_i. \tag{7}$$

The ratio of two likelihood functions is calculated:

$$L(\mathbf{y}_q) = \frac{p(\mathbf{y}_q | \mathbf{a}, H_1)}{p(\mathbf{y}_q | H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta. \tag{8}$$

The probabilities of detection and false alarm are defined as the probabilities that $\mathbf{y}_q^T \mathbf{u}_q$ is greater than γ_q under H_1 and H_0, respectively [10]:

$$P_{D,q} = \mathrm{Prob}(\mathbf{y}_q^T \mathbf{u}_q > \gamma_q) \text{ under } H_1 \tag{9}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{\gamma_q - \|\mathbf{u}_q\|^2}{\sigma \|\mathbf{u}_q\|}}^{\infty} e^{-\frac{t^2}{2}} dt \tag{10}$$

$$P_{F,q} = \mathrm{Prob}(\mathbf{y}_q^T \mathbf{u}_q > \gamma_q) \text{ under } H_0 \tag{11}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{\gamma_q}{\sigma \|\mathbf{u}_q\|}}^{\infty} e^{-\frac{t^2}{2}} dt, \tag{12}$$

where γ_q is defined as

$$\gamma_q \equiv \sigma^2 \ln \eta + \frac{1}{2} \|\mathbf{u}_q\|^2. \tag{13}$$

4. RECURSIVE IMPLEMENTATION OF THE LRT DETECTION SCHEME

Let γ_q and γ_{q+N} be defined as

$$\gamma_q = \sigma^2 \ln \eta + \frac{1}{2} \|\mathbf{u}_q\|^2 \tag{14}$$

$$\gamma_{q+N} = \sigma^2 \ln \eta + \frac{1}{2} \|\mathbf{u}_{q+N}\|^2 = \sigma^2 \ln \eta + \frac{1}{2} \left(\|\mathbf{u}_q\|^2 + \sum_{i=q+1}^{q+N} u_i^2 \right). \tag{15}$$

For calculation of $P_{D,q}$, $\beta_{D,q}$ and $\beta_{D,q+N}$ are defined as

$$\beta_{D,q} \equiv \frac{\gamma_q - \|\mathbf{u}_q\|^2}{\sigma\sqrt{\|\mathbf{u}_q\|^2}} = \frac{\sigma^2 \ln \eta - \frac{1}{2} \|\mathbf{u}_q\|^2}{\sigma\sqrt{\|\mathbf{u}_q\|^2}} \tag{16}$$

$$\beta_{D,q+N} \equiv \frac{\gamma_{q+N} - \|\mathbf{u}_{q+N}\|^2}{\sigma\sqrt{\|\mathbf{u}_{q+N}\|^2}} = \frac{\sigma^2 \ln \eta - \frac{1}{2} \left(\|\mathbf{u}_q\|^2 + \sum_{i=q+1}^{q+N} u_i^2 \right)}{\sigma\sqrt{\left(\|\mathbf{u}_q\|^2 + \sum_{i=q+1}^{q+N} u_i^2 \right)}}. \tag{17}$$

P_D for q data is given by

$$P_{D,q} = \frac{1}{\sqrt{2\pi}} \int_{\beta_{D,q}}^{\infty} e^{-\frac{x^2}{2}} dx. \tag{18}$$

For recursive evaluation of $P_{D,q+N}$ from $P_{D,q}$, $P_{D,q+N}$ can be written as, for $\beta_{D,q+N} > \beta_{D,q}$,

$$\begin{aligned} P_{D,q+N} &= \frac{1}{\sqrt{2\pi}} \int_{\beta_{D,q+N}}^{\infty} e^{-\frac{x^2}{2}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\beta_{D,q}}^{\infty} e^{-\frac{x^2}{2}} dx - \frac{1}{\sqrt{2\pi}} \int_{\beta_{D,q}}^{\beta_{D,q+N}} e^{-\frac{x^2}{2}} dx. \end{aligned} \tag{19}$$

Similarly, for $\beta_{D,q+N} < \beta_{D,q}$, the recursive expression is

$$P_{D,q+N} = \frac{1}{\sqrt{2\pi}} \int_{\beta_{D,q+N}}^{\infty} e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\beta_{D,q}}^{\infty} e^{-\frac{x^2}{2}} dx + \frac{1}{\sqrt{2\pi}} \int_{\beta_{D,q+N}}^{\beta_{D,q}} e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\beta_{D,q}}^{\infty} e^{-\frac{x^2}{2}} dx - \frac{1}{\sqrt{2\pi}} \int_{\beta_{D,q}}^{\beta_{D,q+N}} e^{-\frac{x^2}{2}} dx. \tag{20}$$

Note that, from (19) and (20), whether $\beta_{D,q+N}$ is greater then $\beta_{D,q}$ dose not matter in recursive expression of $P_{D,q+N}$ from $P_{D,q}$.

$\beta_{F,q}$ and $\beta_{F,q+N}$ are defined as

$$\beta_{F,q} \equiv \frac{\gamma_q}{\sigma\sqrt{\|\mathbf{u}_q\|^2}} = \frac{\sigma^2 \ln \eta + \frac{1}{2}\|\mathbf{u}_q\|^2}{\sigma\sqrt{\|\mathbf{u}_q\|^2}} \tag{21}$$

$$\beta_{F,q+N} \equiv \frac{\gamma_{q+N}}{\sigma\sqrt{\|\mathbf{u}_{q+N}\|^2}} = \frac{\sigma^2 \ln \eta + \frac{1}{2}\left(\|\mathbf{u}_q\|^2 + \sum_{i=q+1}^{q+N} u_i^2\right)}{\sigma\sqrt{\left(\|\mathbf{u}_q\|^2 + \sum_{i=q+1}^{q+N} u_i^2\right)}}. \tag{22}$$

$P_{F,q}$ is given by

$$P_{F,q} = \frac{1}{\sqrt{2\pi}} \int_{\beta_{F,q}}^{\infty} e^{-\frac{x^2}{2}} dx. \tag{23}$$

Following the same strategy to get (19) and (20) from (18), we can show that, from (23), $P_{F,q+N}$ can be written as

$$P_{F,q+N} = \frac{1}{\sqrt{2\pi}} \int_{\beta_{F,q}}^{\infty} e^{-\frac{x^2}{2}} dx - \frac{1}{\sqrt{2\pi}} \int_{\beta_{F,q}}^{\beta_{F,q+N}} e^{-\frac{x^2}{2}} dx \tag{24}$$

both for $\beta_{F,q+N} > \beta_{F,q}$ and $\beta_{F,q+N} < \beta_{F,q}$.

5. BOUND OF ABSOLUTE VALUES OF THE DERIVATIVES

In Fig. 1, we specify the expressions of the derivative functions of

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \tag{25}$$

L_M and L'_M are defined as

$$\left| f^{(M)}(\xi(x)) \right| \leq L_M \quad \min(\beta_{D,\,q+N}, \beta_{D,q}) \leq x \leq \max(\beta_{D,\,q+N}, \beta_{D,q}) \tag{26}$$

$$\left| f^{(M)}(\xi(x)) \right| \leq L'_M \quad -\infty < x < \infty. \tag{27}$$

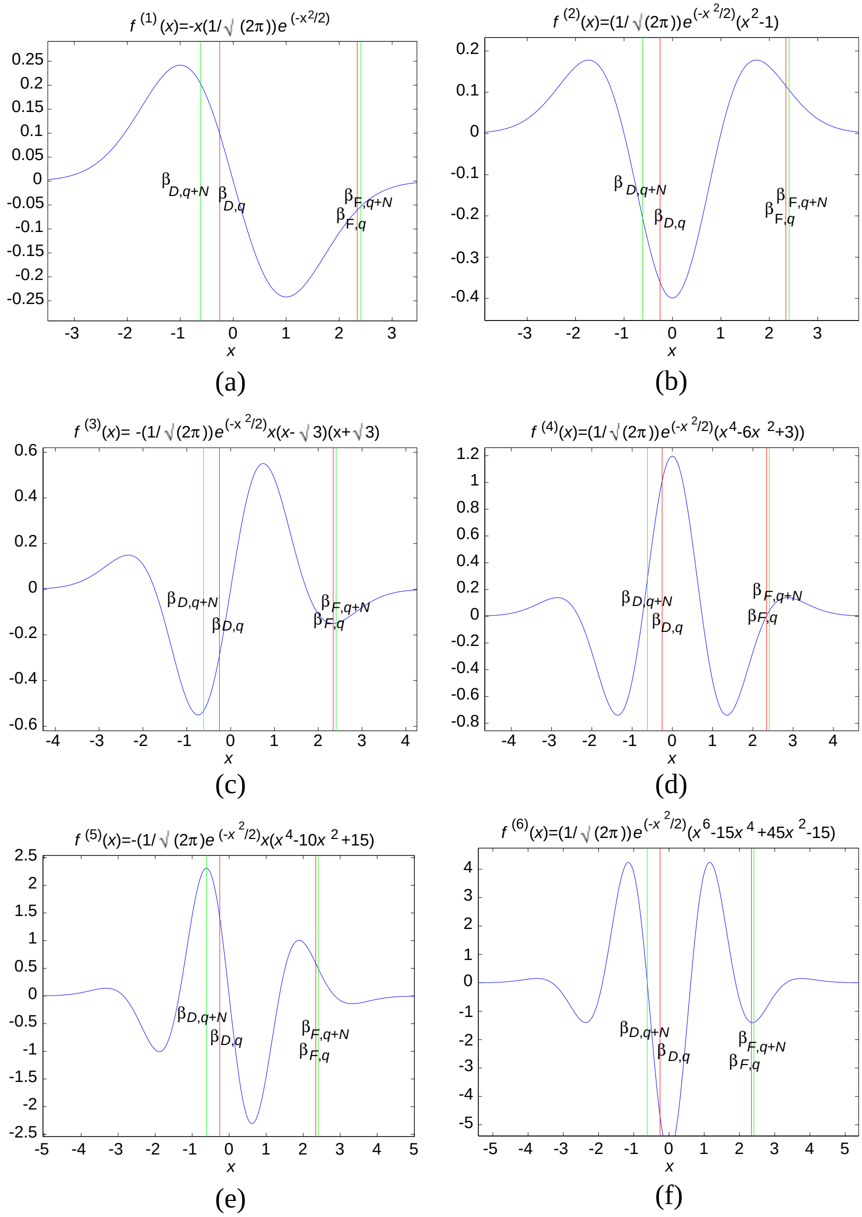

Figure 1. The derivatives of $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $q = 10$, $q + N = 20$, $\beta_{D,q} = -0.2552$, $\beta_{D,q+N} = -0.6173$, $\beta_{F,q} = 2.3412$, $\beta_{F,q+N} = 2.4077$. (a) $f'(x)$. (b) $f''(x)$. (c) $f^{(3)}(x)$. (d) $f^{(4)}(x)$. (e) $f^{(5)}(x)$. (f) $f^{(6)}(x)$.

Let $x_{M+1,i}$ denote the i-th x value satisfying $f^{(M+1)}(x) = 0$. Note that L_M and L'_M in (26) and (27) can also be written as

$$L'_M = \max_i \left| f^{(M)}(x_{M+1,i}) \right| \tag{28}$$

$$L_M = \max\left(\left(\max_{\substack{i \\ b_{M+1,i}=1}} \left| f^{(M)}(x_{M+1,i}) \right| \right), \left| f^{(M)}(\beta_q) \right|, \left| f^{(M)}(\beta_{q+N}) \right| \right) \tag{29}$$

where $b_{M+1,i}$ is defined as

$$b_{M+1,i} \equiv \begin{cases} 1 & \min(\beta_q, \beta_{q+N}) \le x_{M+1,i} \le \max(\beta_q, \beta_{q+N}) \\ 0 & \text{otherwise} \end{cases} . \tag{30}$$

In Fig. 1, we illustrate how the derivative functions of $f(x)$ behave. $\beta_{D,q}$, $\beta_{D,q+N}$, $\beta_{F,q}$ and $\beta_{F,q+N}$, which are associated with L_M, correspond to the specific example in Section 9.

From Fig. 1, it is easy to see that L'_1, L'_2, L'_3 and L'_4 are given by

$$L'_1 = \left| f'(x = -1) \right| = \left| f'(x = 1) \right| = \left| \frac{1}{\sqrt{2\pi}} e^{-\frac{(1)^2}{2}} \right| \tag{31}$$

$$L'_2 = \left| f''(x = 0) \right| = \left| -\frac{1}{\sqrt{2\pi}} \right| \tag{32}$$

$$L'_3 = \left| f^{(3)}\left(x = \sqrt{3 - \sqrt{6}} \right) \right| = \left| \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(3-\sqrt{6})} \sqrt{\left(18 - 6\sqrt{6}\right)} \right| \tag{33}$$

$$L'_4 = \left| f^{(4)}(x) \right| \le \left| f^{(4)}\left(x = \sqrt{5 - \sqrt{10}} \right) \right| = \left| \frac{3}{\sqrt{2\pi}} \right| \tag{34}$$

6. RECURSIVE UPDATE USING POLYNOMIAL APPROXIMATION

There exists $\xi(x)$ between x and $a_{q,N}$ with [33]

$$f(x) = f_{Ta,M}(x; a_{q,N}) + R_{Ta,M}(x; a_{q,N}) \simeq f_{Ta,M}(x; a_{q,N}) \tag{35}$$

where

$$f_{Ta,M}(x; a_{q,N}) = \sum_{k=0}^{M} \frac{f^{(k)}(a_{q,N})}{k!} (x - a_{q,N})^k \tag{36}$$

and

$$R_{Ta,M}(x; a_{q,N}) = \frac{f^{(M+1)}(\xi(x))}{(M+1)!} (x - a_{q,N})^{(M+1)} . \tag{37}$$

Let $a_{D,q,N}$ and $a_{F,q,N}$ denote $a_{q,N}$ values for P_D and P_F, respectively. The recursive expression of $P_{D,q+N}$ from $P_{D,q}$ based on the Taylor series approximation of $f(x)$ with $f_{Ta,M}(x; a_{D,q,N})$ is

$$P_{D,q+N} = P_{D,q} - \int_{\beta_{D,q}}^{\beta_{D,q+N}} f(x)\, dx \tag{38}$$

$$\approx P_{D,q} - \int_{\beta_{D,q}}^{\beta_{D,q+N}} f_{Ta,M}(x, a_{D,q,N})(x)\, dx \tag{39}$$

$$= P_{D,q} - \sum_{k=0}^{M} \frac{f^{(k)}(a_{D,q,N})}{(k+1)!} \left((\beta_{D,q+N} - a_{D,q,N})^{k+1} - (\beta_{D,q} - a_{D,q,N})^{k+1} \right). \tag{40}$$

Note that the error in approximating (38) with (39) is

$$\int_{\beta_{D,q}}^{\beta_{D,q+N}} R_{Ta,M}(x; a_{D,q,N})\, dx. \tag{41}$$

Similarly, the recursive evaluation of the probability of false alarm is

$$P_{F,q+N} = P_{F,q} - \int_{\beta_{F,q}}^{\beta_{F,q+N}} f(x)\, dx \approx P_{F,q} - \int_{\beta_{F,q}}^{\beta_{F,q+N}} f_{Ta,M}(x; a_{F,q,N})\, dx$$

$$= P_{F,q} - \sum_{k=0}^{M} \frac{f^{(k)}(a_{F,q,N})}{(k+1)!} \left((\beta_{F,q+N} - a_{F,q,N})^{k+1} - (\beta_{F,q} - a_{F,q,N})^{k+1} \right). \tag{42}$$

7. ERROR BOUNDS

The Cauchy-Schwarz inequality can be written as

$$\left| \int f(x)\, g(x)\, dx \right| \leq \left[\int (f(x))^2\, dx \int (g(x))^2\, dx \right]^{\frac{1}{2}} \tag{43}$$

where $f(x)$ and $g(x)$ are real-valued functions.

The upper bound of the polynomial approximation is

$$\begin{aligned} &\left| \int_{\min(\beta_q, \beta_{q+N})}^{\max(\beta_q, \beta_{q+N})} R_{Ta,M}(x; a_{q,N})\, dx \right| \\ &= \left| \int_{\min(\beta_q, \beta_{q+N})}^{\max(\beta_q, \beta_{q+N})} \frac{f^{(M+1)}(\xi(x))}{(M+1)!} (x - a_{q,N})^{M+1} \right| \\ &\leq \frac{1}{(M+1)!} \left[\int_{\min(\beta_q, \beta_{q+N})}^{\max(\beta_q, \beta_{q+N})} \left(f^{(M+1)}(\xi(x)) \right)^2 dx \right. \\ &\qquad \left. \int_{\min(\beta_q, \beta_{q+N})}^{\max(\beta_q, \beta_{q+N})} \left((x - a_{q,N})^{M+1} \right)^2 dx \right]^{\frac{1}{2}} \end{aligned} \tag{44}$$

where the last inequality follows from the Cauchy-Schwarz inequality.

Due to $|f^{(M+1)}(\xi(x))dx| \le L_{M+1}$ for $\min(\beta_q, \beta_{q+N}) \le x \le \max(\beta_q, \beta_{q+N})$, we get the inequality :

$$\int_{\min\left(\beta_q,\beta_{q+N}\right)}^{\max\left(\beta_q,\beta_{q+N}\right)} \left(f^{(M+1)}\left(\xi\left(x\right)\right)\right)^2 dx \le {L_{M+1}}^2 \left|\beta_{q+1} - \beta_q\right|. \tag{45}$$

The last integral in (44) can be easily evaluated:

$$\begin{aligned} &\int_{\min\left(\beta_q,\beta_{q+N}\right)}^{\max\left(\beta_q,\beta_{q+N}\right)} \left(\left(x - a_{q,N}\right)^{M+1}\right)^2 dx \\ &= \frac{1}{2M+3}\Big(\left(\max\left(\beta_q, \beta_{q+N}\right) - a_{q,N}\right)^{2M+3} \\ &\quad - \left(\min\left(\beta_q, \beta_{q+N}\right) - a_{q,N}\right)^{2M+3}\Big). \end{aligned} \tag{46}$$

From (44), (45) and (46), we get

$$\begin{aligned} &\left|\int_{\min\left(\beta_q,\beta_{q+N}\right)}^{\max\left(\beta_q,\beta_{q+N}\right)} R_{Ta,M}\left(x; a_{q,N}\right) dx\right| \\ &\le \Big[\frac{1}{(M+1)!}{L_{M+1}}^2 \left|\beta_{q+1} - \beta_q\right| \\ &\quad \times \frac{1}{2M+3}\Big(\left(\max\left(\beta_q, \beta_{q+N}\right) - a_{q,N}\right)^{2M+3} \\ &\quad - \left(\min\left(\beta_q, \beta_{q+N}\right) - a_{q,N}\right)^{2M+3}\Big)\Big] \qquad (47) \\ &\le \Big[\frac{1}{(M+1)!}{L'_{M+1}}^2 \left|\beta_{q+1} - \beta_q\right| \\ &\quad \times \frac{1}{2M+3}\Big(\left(\max\left(\beta_q, \beta_{q+N}\right) - a_{q,N}\right)^{2M+3} \\ &\quad - \left(\min\left(\beta_q, \beta_{q+N}\right) - a_{q,N}\right)^{2M+3}\Big)\Big]. \qquad (48) \end{aligned}$$

8. OPTIMIZATION OF ERROR BOUND

Since $\int_{\min(\beta_{D,q},\beta_{D,q+N})}^{\max(\beta_{D,q},\beta_{D,q+N})} [f^{(M+1)}(\xi(x))]^2 dx$ is not a function of $a_{q,N}$, the minimization of (44) with respect to $a_{q,N}$ reduces to the minimization of the following function:

$$f\left(a_{q,N}\right) \equiv \int_{\min\left(\beta_q,\beta_{q+N}\right)}^{\max\left(\beta_q,\beta_{q+N}\right)} \left(\left(x - a_{q,N}\right)^{M+1}\right)^2 dx$$

$$=\frac{1}{2M+3}\left(\left(\max\left(\beta_q,\beta_{q+N}\right)-a_{q,N}\right)^{2M+3}\right.$$
$$\left.-\left(\min\left(\beta_q,\beta_{q+N}\right)-a_{q,N}\right)^{2M+3}\right). \tag{49}$$

For $\beta_{q+N}>\beta_q$, $f(a_{q,N})$ in (49) becomes

$$f\left(a_{q,N}\right)\equiv\int_{\beta_q}^{\beta_{q+N}}\left(\left(x-a_{q,N}\right)^{M+1}\right)^2dx=\int_{\beta_q}^{\beta_{q+N}}\left(x-a_{q,N}\right)^{2M+2}dx$$
$$=\frac{1}{2M+3}\left(\left(\beta_{q+N}-a_{q,N}\right)^{2M+3}-\left(\beta_q-a_{q,N}\right)^{2M+3}\right). \tag{50}$$

The bound of error is minimized when the derivative of the cost function $f(a_{q,N})$ in (50) with respect to $a_{q,N}$ is equal to zero:

$$\frac{df\left(a_{q,N}\right)}{da_{q,N}}=-\left[\left(\beta_{q+N}-a_{q,N}\right)^{2M+2}-\left(\beta_q-a_{q,N}\right)^{2M+2}\right]=0 \tag{51}$$

The real roots of (51) are given by

$$\beta_{q+N}-a_{q,N}=\pm\left(\beta_q-a_{q,N}\right). \tag{52}$$

Since $\beta_{q+N}-a_{q,N}=\beta_q-a_{q,N}$ implies $\beta_{q+N}=\beta_q$, which is not possible, $\beta_{q+N}-a_{q,N}$ should be equal to $-(\beta_q-a_{q,N})$, from which $a_{q,N}$ is given by

$$a_{q,N}=\frac{1}{2}\left(\beta_{q+N}+\beta_q\right). \tag{53}$$

To confirm that the cost function is actually minimized, not maximized, at the value of $a_{q,N}$ specified in (53), we check the sign of the second derivative function of $f(a_{q,N})$.

$$\frac{d^2f\left(a_{q,N}\right)}{da_{q,N}{}^2}=(2M+2)\left[\left(\beta_{q+N}-a_{q,N}\right)^{2M+1}-\left(\beta_q-a_{q,N}\right)^{2M+1}\right] \tag{54}$$

By substituting (53) in (54), it can be shown that the second derivative of (50) is positive, for $\beta_{q+N}>\beta_q$

$$\left.\frac{d^2f\left(a_{q,N}\right)}{da_{q,N}{}^2}\right|_{a_{q,N}=\frac{1}{2}\left(\beta_{q+N}+\beta_q\right)}=(2M+2)\frac{1}{2^{2M}}\left(\beta_{q+N}-\beta_q\right)^{2M+1}>0 \tag{55}$$

Using the same strategy to get (55) we have, for $\beta_{q+N}<\beta_q$,

$$\frac{df(a_{q,N})}{da_{q,N}}=0\quad\text{at}\quad a_{q,N}=\frac{1}{2}\left(\beta_{q+N}+\beta_q\right), \tag{56}$$

$$\left.\frac{d^2f(a_{q,N})}{da_{q,N}{}^2}\right|_{a_{q,N}=\frac{1}{2}\left(\beta_{q+N}+\beta_q\right)}=(2M+2)\frac{1}{2^{2M}}\left(\beta_q-\beta_{q+N}\right)^{2M+1}>0. \tag{57}$$

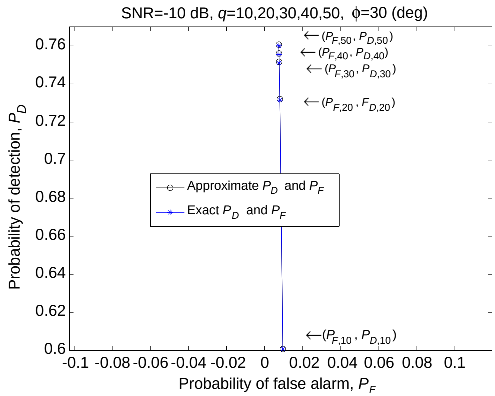

Figure 2. $(P_{F,20}, P_{D,20})$, $(P_{F,30}, P_{D,30})$, $(P_{F,40}, P_{D,40})$ and $(P_{F,50}, P_{D,50})$ based on repeated use of (40) and (42) with $M = 1$ and $a_{q,N} = \frac{\beta_q+\beta_{q,N}}{2}$.

9. NUMERICAL RESULTS

The radius of the wire of the target is equal to 0.5 cm, and the length of the wire is 1 meter. The noiseless frequency response is obtained via the method of moments (MoM). We calculate the back-scattered field. The frequency response is obtained in increments of 7.8 MHz.

The frequency response up to 500 MHz is used, and the number of the natural frequencies of 1 meter long wire for that bandwidth is six [2]. Therefore, the total bandwidth used in the numerical results is 500 MHz.

To get the late time response for the target, the frequency response is inverse Fourier-transformed.

q and N values are set to $q = 10$ and $N = 10$. $\beta_{D,q}$, $\beta_{D,q+N}$, $\beta_{F,q}$ and $\beta_{F,q+N}$ values are calculated using (16) and (17), and given by $\beta_{D,q} = -0.2552$, $\beta_{D,q+N} = -0.6273$, $\beta_{F,q} = 2.3412$ and $\beta_{F,q+N} = 2.4077$. L_M and L'_M values can be obtained from Fig. 1.

In Fig. 2, approximate $(P_{F,20}, P_{D,20})$ is obtained from $(P_{F,10}, P_{D,10})$ using (40) and (42) with $q = 10$ and $N = 10$. Note that $(P_{F,10}, P_{D,10})$ are exact values, rather than approximate values. In evaluating approximate $(P_{F,30}, P_{D,30})$, (40) and (42) are also used with $q = 20$ and $N = 10$. Note that $P_{D,q}$ in the right hand side (RHS) of (40) and $P_{F,q}$ in the RHS of (42) is the approximate $(P_{F,20}, P_{D,20})$ what we have obtained in the previous step. The same procedure is repeated to obtain approximate $(P_{D,40}, P_{F,40})$ and $(P_{D,50}, P_{F,50})$.

That is, in recursive implementation, $P_{D,q}$ and $P_{F,q}$ on the RHS's

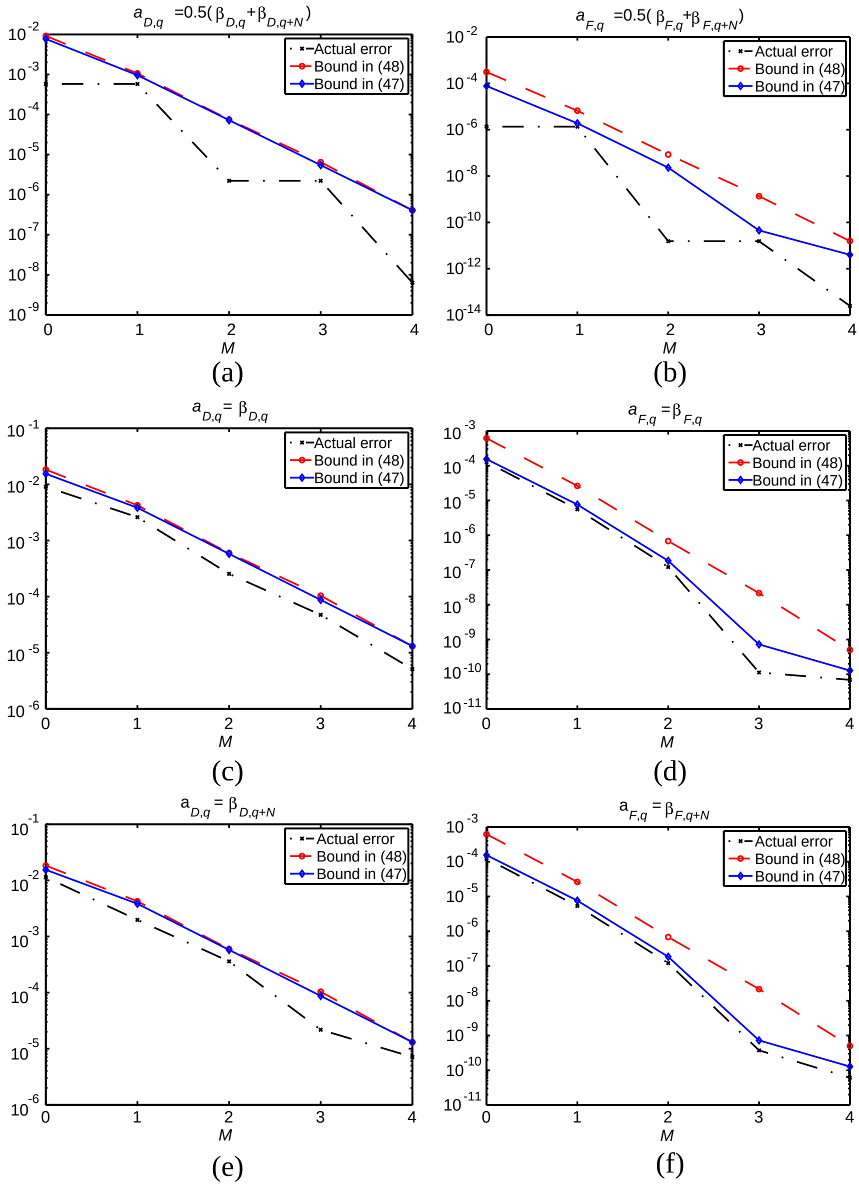

Figure 3. Actual error and two error bounds of Taylor series approximation for $M = 0, \ldots, 4$ (SNR $= -10$ dB). (a) Detection, $a_{D,q} = \frac{\beta_{D,q}+\beta_{D,q+N}}{2}$. (b) False alarm, $a_{F,q} = \frac{\beta_{F,q}+\beta_{F,q+N}}{2}$. (c) Detection, $a_{D,q} = \beta_{D,q}$. (d) False alarm, $a_{F,q} = \beta_{F,q}$. (e) Detection, $a_{D,q} = \beta_{D,q+N}$. (f) False alarm, $a_{F,q} = \beta_{F,q+N}$.

of (40) and (42) are the exact values in the first step. From the second step, $P_{D,q}$ and $P_{F,q}$ on the RHS's of (40) and (42) are the approximate values obtained in the previous step. Therefore, as we recursively evaluate $(P_{F,q}, P_{D,q})$ using (40) and (42), the error due to an polynomial approximation of the standard normal distribution accumulates.

But, the amount of error accumulation is negligible, which is quite clear in Fig. 2. Since the results in Fig. 2 is for $M = 1$, the error accumulation for larger M values gets smaller than the error accumulation for $M = 1$ shown in Fig. 2.

In Fig. 3, the actual error itself and the bound of the error are shown for various $a_{q,N}$ values both for the probability of detection and the probability of false alarm. For detection, the error for $a_{D,q,N} = \frac{1}{2}(\beta_{D,q} + \beta_{D,q+N})$ is obtained from the difference of (20) and (40) with $a_{D,q,N} = \frac{1}{2}(\beta_{D,q} + \beta_{D,q+N})$. The error for $a_{D,q,N} = \beta_{D,q}$ is the obtained from the difference of (20) and (40) with $a_{D,q,N} = \beta_{D,q}$, and the error for $a_{D,q,N} = \beta_{D,q+N}$ is calculated from the difference of (20) and (40) with $a_{D,q,N} = \beta_{D,q+N}$.

For false alarm, the error for $a_{F,q,N} = \frac{1}{2}(\beta_{F,q} + \beta_{F,q+N})$ is obtained from the difference of (24) and (42) with $a_{F,q,N} = \frac{1}{2}(\beta_{F,q} + \beta_{F,q+N})$. The error for $a_{F,q,N} = \beta_{F,q}$ is obtained from the difference of (24) and (42) with $a_{F,q,N} = \beta_{F,q}$, and the error for $a_{F,q,N} = \beta_{F,q+N}$ is calculated from the difference of (24) and (42) with $a_{F,q,N} = \beta_{F,q+N}$.

The error bounds in Fig. 3 are obtained from (47) and (48). For detection, β_q and β_{q+N} are $\beta_q = \beta_{D,q}$ and $\beta_{q+N} = \beta_{D,q+N}$ in (47) and (48), and for false alarm, β_q and β_{q+N} are $\beta_q = \beta_{F,q}$ and $\beta_{q+N} = \beta_{F,q+N}$ in (47) and (48). $a_{q,N}$ in (47) and (48) can be $a_{q,N} = \frac{1}{2}(\beta_q + \beta_{q+N})$, $a_{q,N} = \beta_{q+N}$ or $a_{q,N} = \beta_q$.

From all the results in Fig. 3, for SNR $= -10$ dB, it is quite clear that the error itself is actually less than the bound of the error. It is quite clear that the error and the bound of the error at $a_{D,q,N} = \frac{1}{2}(\beta_{D,q} + \beta_{D,q+N})$ are smaller than those at $a_{D,q,N} = \beta_{D,q}$ or those at $a_{D,q,N} = \beta_{D,q+N}$. To more clearly illustrate the effect of noise on the detection performance, the results for SNR $= 0$ dB are shown in Fig. 4.

Errors and their bounds of the Taylor polynomial approximation are shown in Fig. 5. In Fig. 5, actual error is calculated using (40) for $\beta_q \leq a_{q,N} \leq \beta_{q+N}$, and the error bound is obtained using (47) and (48). From (53), (55), (56) and (57), the error bounds should be minimized at $a_{D,q,N} = \frac{1}{2}(\beta_{D,q} + \beta_{D,q+N})$ for detection, and at $a_{F,q,N} = \frac{1}{2}(\beta_{F,q} + \beta_{F,q+N})$ for false alarm, which is consistent with the results in Fig. 5.

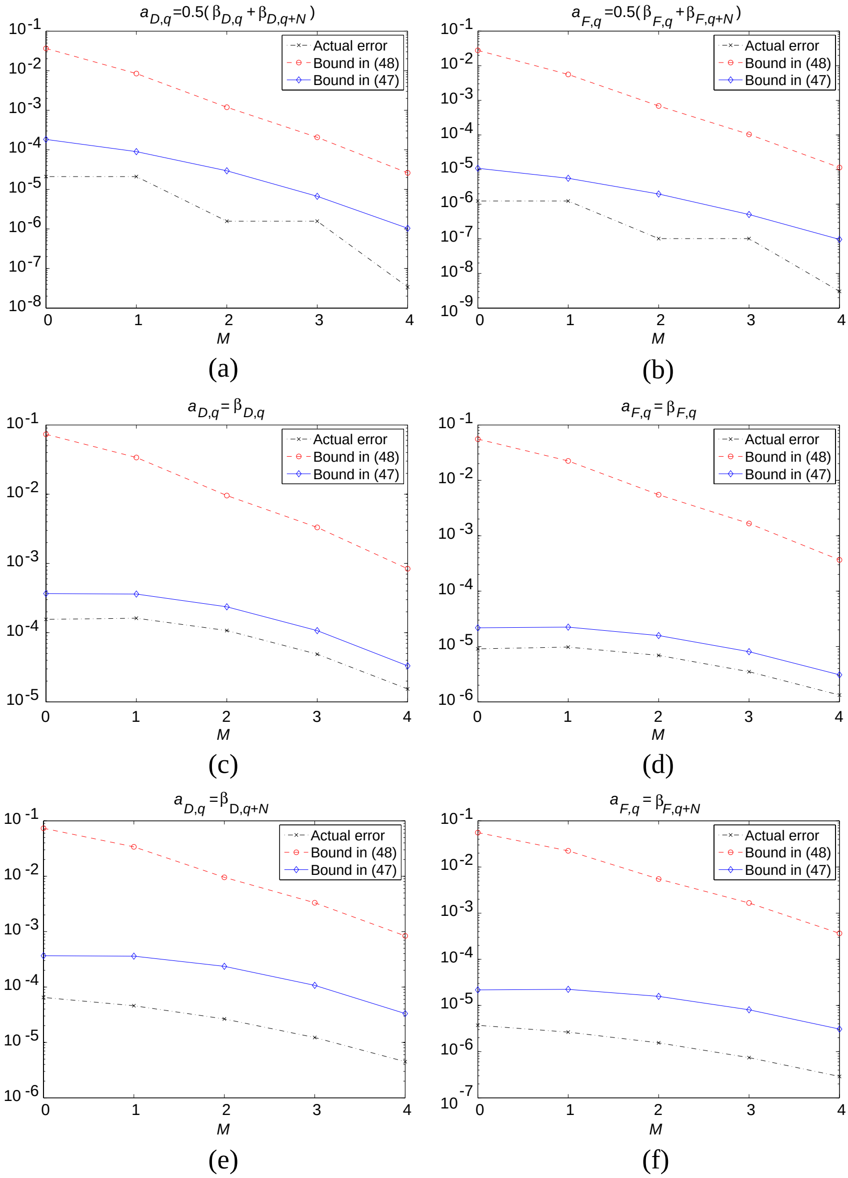

Figure 4. Actual error and two error bounds of Taylor series approximation for $M = 0, \ldots, 4$ (SNR $= 0$ dB). (a) Detection, $a_{D,q} = \frac{\beta_{D,q}+\beta_{D,q+N}}{2}$. (b) False alarm, $a_{F,q} = \frac{\beta_{F,q}+\beta_{F,q+N}}{2}$. (c) Detection, $a_{D,q} = \beta_{D,q}$. (d) False alarm, $a_{F,q} = \beta_{F,q}$. (e) Detection, $a_{D,q} = \beta_{D,q+N}$. (f) False alarm, $a_{F,q} = \beta_{F,q+N}$.

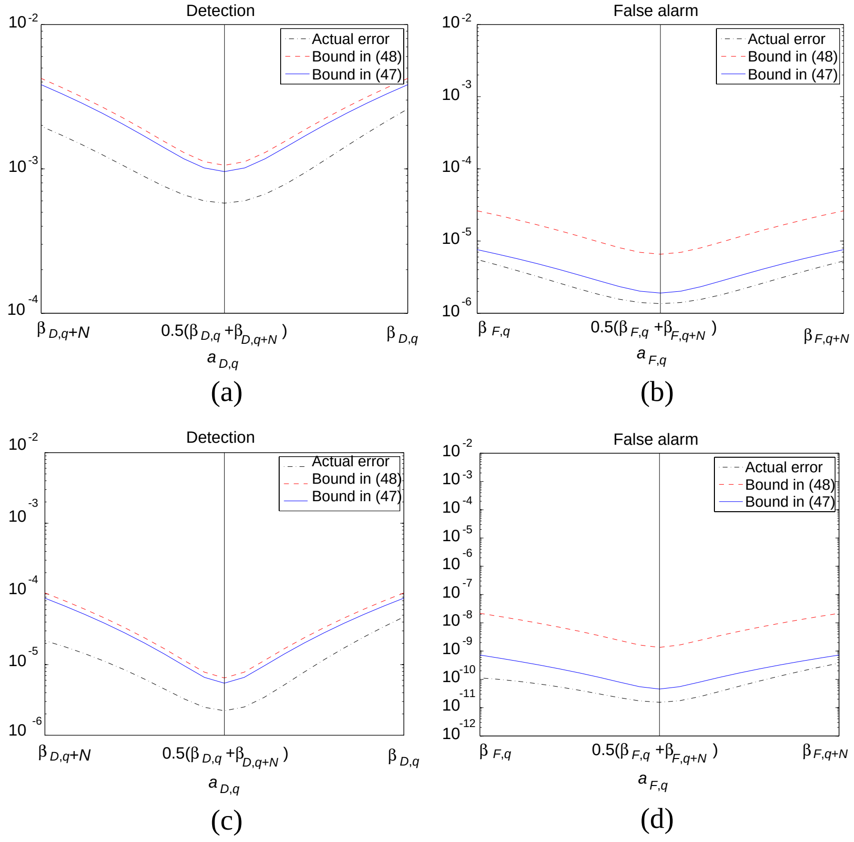

Figure 5. Actual error and two error bounds of Taylor series approximation at various $a_{D,q}$ values and $a_{F,q}$ values (SNR $= -10$ dB). (a) $M = 1$, detection. (b) $M = 1$, false alarm. (c) $M = 3$, detection. (d) $M = 3$, false alarm.

Note that we cannot derive analytically the value of $a_{q,N}$ at which the error itself is minimized, but that we can determine the value of $a_{q,N}$ at which the error bound is minimized. From Fig. 5, it is shown that these two values turn out to be coincident.

In Section 8, it is shown that, in the case of the Tayor series approximation, the error bound is minimized for $a_{q,N} = \frac{1}{2}(\beta_q + \beta_{q+N})$ for all M values.

We illustrate the error values and the bound of the error in Fig. 5, for various $a_{q,N}$ values, between β_q and β_{q+N}. It is easily checked that both the error value and the bound of error are minimized for

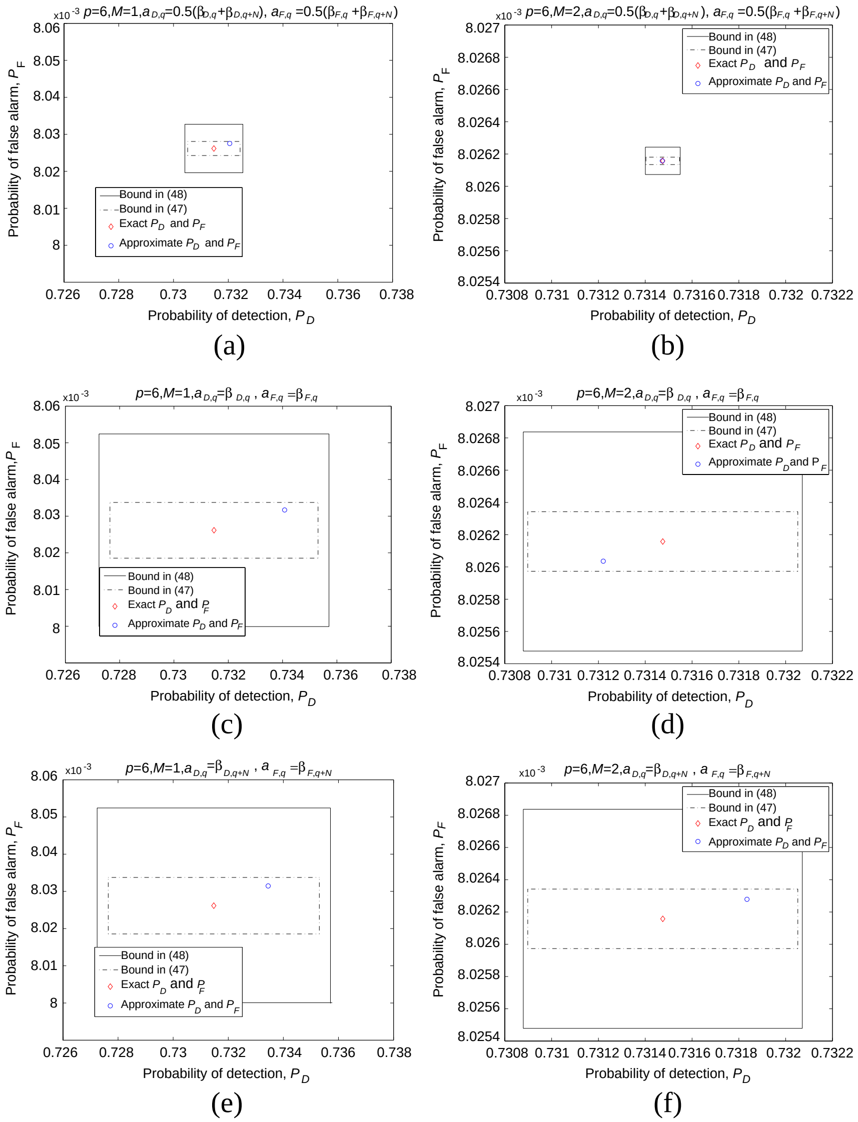

Figure 6. P_D vs P_F of the Taylor polynomial approximation (SNR $= -10$ dB). (a) $M = 1$, $a_{D,q} = \frac{\beta_{D,q}+\beta_{D,q+N}}{2}$, $a_{F,q} = \frac{\beta_{F,q}+\beta_{q+N}}{2}$. (b) $M = 2$, $a_{D,q} = \frac{\beta_{D,q}+\beta_{D,q+N}}{2}$, $a_{F,q} = \frac{\beta_{F,q}+\beta_{F,q+N}}{2}$. (c) $M = 1$, $a_{D,q} = \beta_{D,q}$, $a_{F,q} = \beta_{F,q}$. (d) $M = 2$, $a_{D,q} = \beta_{D,q}$, $a_{F,q} = \beta_{F,q}$. (e) $M = 1$, $a_{D,q} = \beta_{D,q+N}$, $a_{F,q} = \beta_{F,q+N}$. (f) $M = 2$, $a_{D,q} = \beta_{D,q+N}$, $a_{F,q} = \beta_{F,q+N}$.

$a_{q,N} = \frac{1}{2}(\beta_q + \beta_{q+N})$. That is, the actual error as well as two error bounds is minimized at $a_{q,N} = \frac{1}{2}(\beta_q + \beta_{q+N})$ over $\min(\beta_q, \beta_{q+N}) < a_{q,N} < \max(\beta_q, \beta_{q+N})$. Remind that $a_{q,N}$ is the value at which the Taylor series expansion is defined.

It turns out, from Fig. 5, that the value of $a_{q,N}$ at which the error itself is minimized can be found from the value of $a_{q,N}$ at which the error bound is minimized.

Although we cannot derive analytically at what $a_{q,N}$ value the actual error itself is minimized, we can verify empirically that $a_{q,N}$ value, at which the actual error is minimized, actually coincides with $a_{q,N}$ value at which two error bounds are minimized. Note that $a_{q,N}$ value at which two error bounds are minimized is derived analytically and is equal to $a_{q,N} = \frac{1}{2}(\beta_q + \beta_{q+N})$.

In Fig. 6, P_D vs P_F of Taylor series approximation is shown. Two rectangles specify the error bounds in (47) and (48). It is clearly shown that (P_D, P_F) using the Taylor series approximation is actually within two rectangles.

10. CONCLUSIONS

We consider recursive implementation of the performance analysis of the natural frequency-based radar target detection using the likelihood ratio test (LRT) scheme. By making the Taylor polynomial approximation of the PDF of the standard normal distribution, we can make a recursive implementation of the performance analysis. We derive expressions of the error of the integration and the bound of the error of the integration. It is shown that, by adopting higher order polynomial, we can reduce the error of the integration due to a polynomial approximation. We also show how to determine the optimal polynomials. It is analytically proved that the bound of the error for the Taylor polynomial approximation is minimized when the Taylor series expansion at the center of the integration interval is adopted, regardless of the order of the polynomial.

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REFERENCES

1. Chang, Y.-L., C.-Y. Chiang, and K.-S. Chen, "SAR image simulation with application to target recognition," *Progress In Electromagnetics Research*, Vol. 119, 35–57, 2011.
2. Mooney, J. E., Z. Ding, and L. S. Riggs, "Performance analysis of a GLRT automated target discrimination scheme," *IEEE Trans. Antennas and Propagation*, Vol. 49, No. 12, 1827–1835, Dec. 2001.
3. Cho, S.-W. and J.-H. Lee, "Effect of threshold value on the performance of natural frequency-based radar target recognition," *Progress In Electromagnetics Research*, Vol. 135, 527–562, 2013.
4. Lee, J.-H., S.-W. Cho, S.-H. Park, and K.-T. Kim, "Performance analysis of radar target recognition using natural frequency: Frequency domain approch," *Progress In Electromagnetics Research*, Vol. 132, 315–345, 2012.
5. Lee, J.-H. and H.-T. Kim, "Comments on extraction of the natural frequencies of a radar target from a measured response using E-pulse techniques," *IEEE Trans. Antennas and Propagation*, Vol. 53, No. 11, 3853–3855, Nov. 2005.
6. Lee, J.-H., I.-S. Choi, and H.-T. Kim, "Natural frequency based neural network approach to radar target recognition," *IEEE Trans. Signal Processing*, Vol. 51, No. 12, 3191–3197, Dec. 2003.
7. Lee, J.-H. and H.-T. Kim, "Radar target discrimination using transient response reconstruction," *Journal of Electromagnetic Waves and Applications*, Vol. 19, No. 5, 655–669, May 2005.
8. Lee, J.-H. and H.-T. Kim, "Radar target recognition based on late time representation: Closed-form expression for criterion," *IEEE Trans. Antennas and Propagation*, Vol. 54, 2455–2462, Sep. 2006.
9. Lee, J.-H. and H.-T. Kim, "Radar target recognition using least squares estimate," *Microwave and Optical Technology Letters*, Vol. 30, 427–434, Sep. 2001.
10. Mooney, J. E., Z. Ding, and L. Riggs, "Performance analysis of a GLRT in late-time radar target detection," *Progress In Electromagnetics Research*, Vol. 24, 77–96, 1999.
11. Chen, C. C. and L. Peters, "Buried unexploded ordnance identification via complex natural resonances," *IEEE Trans. Antennas Propagat.*, Vol. 45, No. 11, 1645–1654, 1997.
12. Lee, K.-H., C.-C. Chen, F. L. Teixeira, and R. Lee, "Modeling and characterization of geometrically complex UWB antennas using FDTD," *IEEE Trans. Antennas Propagat.*, Vol. 52, No. 8, 1983–1991, 2004.

13. Chen, C. C., B. Higgins, K. O'Neil, and R. Detsch, "Ultrawide-bandwidth fully-polarimetric ground penetrating radar classification of subsurface unexploded ordnance," *IEEE Trans. Geosci. Remote Sens.*, Vol. 39, Vol. 6, 1221–1320, 2001.
14. Magaz, B., A. Belouchrani, and M. Hamadouche, "Automatic threshold selection in OS-CFAR radar detection using information theoretic criteria," *Progress In Electromagnetics Research B*, Vol. 28, 75–94, 2011.
15. Le Marshall, N. W. D. and A. Z. Tirkel, "MIMO radar array for termite detection and imaging," *Progress In Electromagnetics Research B*, Vol. 28, 75–94, 2011.
16. Narayanan, R. M., M. C. Shastry, P.-H. Chen, and M. Levi, "Through-the-wall detection of stationary human targets using doppler radar," *Progress In Electromagnetics Research B*, Vol. 20, 147–166, 2010.
17. Conceicao, R. C., M. O'Halloran, M. Glavin, and E. Jones, "Numerical modeling for ultra wideband radar breast cancer detection and classification," *Progress In Electromagnetics Research B*, Vol. 34, 145–171, 2011.
18. Wang, F.-F. and Y.-R. Zhang, "The support vector machine for dielectric target detection through a wall," *Progress In Electromagnetics Research Letters*, Vol. 23, 119–128, 2011.
19. Liu, B. and W. Chang, "A novel range-spread target detection approach for frequency stepped chirp radar," *Progress In Electromagnetics Research*, Vol. 131, 275–292, 2012.
20. Lee, J.-H. and S.-H. Jeong, "Performance of natural frequency-based target detection in frequency domain," *Journal of Electromagnetic Waves and Applications* Vol. 26, Nos. 17–18, 2426–2437, 2012.
21. Zhou, W., J.-T. Wang, H. W. Chen, and X. Li, "Signal model and moving target detection based on MIMO synthetic aperture radar," *Progress In Electromagnetics Research*, Vol. 131, 311–329, 2012.
22. Hatam, M., A. Sheikhi, and M. A. Masnadi-Shirazi, "Target detection in pulse-train MIMO radars applying ica algorithms," *Progress In Electromagnetics Research*, Vol. 122, 413–435, 2012.
23. Tian, B., D.-Y. Zhu, and Z.-D. Zhu, "A novel moving target detection approach for dual-channel SAR system," *Progress In Electromagnetics Research*, Vol. 115, 191–206, 2011.
24. Lee, J.-H. and H.-T. Kim, "Selecting sampling interval of transient response for the improved Prony method," *IEEE Trans. Antennas*

and Propagation, Vol. 51, No. 1, 74–77, Jan. 2003.

25. Lee, J.-H., I.-S. Choi, and H.-T. Kim, "Natural frequency based neural network approach to radar target recognition," *IEEE Trans. Signal Processing*, Vol. 51, No. 12, 3191–3197, Dec. 2003.
26. Lee, J.-H. and H.-T. Kim, "Hybrid method for natural frequency extraction: Performance improvement using Newton-Raphson method," *Journal of Electromagnetic Waves and Applications*, Vol. 19, No. 8, 1043–1055, Aug. 2005.
27. Cho, S.-W. and J.-H. Lee, "Efficient implementation of the Capon beamforming using the Levenberg-Marquardt scheme for two dimensional AOA estimation," *Progress In Electromagnetics Research*, Vol. 137, 19–34, 2013
28. Lee, J.-H., S.-W. Cho, and I.-S. Choi, "Simple expressions of CEP and covariance matrix for localization using LOB measurements for circular trajectory," *Electronics Express*, Vol. 9, No. 14, 1221–1229, 2012.
29. Lee, J.-H., S.-W. Cho, and H.-S. Kim, "Newton-type method in spectrum estimation-based AOA estimation," *Electronics Express*, Vol. 9, No. 12, 1036–1043, 2012.
30. Lee, J.-H., Y.-S. Jeong, S.-W. Cho, W.-Y. Yeo, and K. Pister, "Application of the Newton method to improve the accuracy of TOA estimation with the beamforming algorithm and the MUSIC algorithm," *Progress In Electromagnetics Research*, Vol. 116, 475–515, 2011.
31. Lee, J.-H., H.-J. Kwon, and Y.-K. Jin, "Numerically efficient implementation of JADE ML algorithm," *Journal of Electromagnetic Waves and Applications*, Vol. 22, Nos. 11–12, 1693–1704, 2008
32. Jeong, Y.-S. and J.-H. Lee, "Estimation of time delay using conventional beamforming based algorithm for UWB systems," *Electromagnetic Waves and Applications*, Vol. 21, No. 15, 2413–2402, 2007
33. Burden, R. L. and J. D. Faires, *Numerical Analysis*, Cengage Learning, Boston, 2010.