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COMPARATIVE EVALUATION OF UNCERTAINTY TRANSFORMATION FOR MEASURING COMPLEX-VALUED QUANTITIES

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Abstract—This paper presents a comparative study on practical evaluation of measurement uncertainty for complex-valued RF and microwave quantities in polar coordinate. The measurement uncertainty is first evaluated in rectangular coordinate to avoid the biased effect, and then transformed into the desired polar coordinate. In this work, uncertainty coordinate transformation from rectangular coordinate to polar coordinate is focused and performed in two ways; the *law of propagation of uncertainty* and the *coordinate rotation*. Their performances are compared through practical evaluations and simulations, and found to be highly consistent when the uncertainty region is distant from the origin of a complex plane.

1. INTRODUCTION

Measurement uncertainty has been recognized to be a key parameter in RF and microwave applications [1–4]. For the often-experienced complex-valued RF and microwave quantities (e.g., the scattering parameters S_{11}, S_{21} etc.), the uncertainty evaluation is sometimes a challenge since there is limited associated information in the *Guide to the Expression of Uncertainty in Measurement* (GUM) [5] which is an internationally recommended guideline and has been widely accepted and followed. A great number of works therefore have been performed in [6–10] for properly evaluating the measurement uncertainty of a complex-valued RF and microwave quantity.

The reported works [7, 10] strongly recommended that for a complex-valued variable S ($S = R + jI$, R and I are the real

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and imaginary components), its uncertainty evaluation should be performed in the real–imaginary rectangular coordinate. However, a polar representation (i.e., magnitude $|S| = \sqrt{R^2 + I^2}$ and phase $\phi = \tan^{-1}(\frac{I}{R})$) is often preferred by the RF and microwave community. For example, it is more natural to characterize an amplifier or an attenuator in terms of its magnitude. Therefore, the uncertainty evaluated in a rectangular coordinate needs to be transformed into the desired polar coordinate, since the direct evaluation of measurement uncertainty for the magnitude ($|S|$) is suggested to be avoided as it is a biased estimator especially when the magnitude is small [7].

The *law of propagation of uncertainty* (LPU) in [5] is then a possible solution to propagate the uncertainty in rectangular coordinate to polar coordinate, and has been preliminarily discussed in [11]. As a continued work of [11], we will compare its performance for the uncertainty coordinate transformation with another method using the *coordinate rotation* [10] in this paper. In the following, both the methods are described in details in Section 2. Their performances are then evaluated with some case studies in Section 3 and numerical simulations in Section 4. Finally, conclusions of this paper are given in Section 5.

2. PARAMETER REPRESENTATION AND TRANSFORMATION METHODOLOGIES

A 2×2 variance–covariance matrix V_{rec} is used to describe the complex-valued variable S in terms of its standard uncertainties $u(R)$ and $u(I)$ for R and I and the sample correlation coefficient r between R and I in a rectangular coordinate,

$$V_{rec} = \begin{bmatrix} u^2(R) & ru(R)u(I) \\ ru(R)u(I) & u^2(I) \end{bmatrix}. \tag{1}$$

It is noted that for evaluations of the standard uncertainties $u(R)$ and $u(I)$ associated with the real quantities R and I, the GUM [5] has been followed.

2.1. Uncertainty Transformation Based on the *Law of Propagation of Uncertainty*

For the uncertainty transformation from rectangular coordinate (R, I) to polar coordinate $(|S|, \phi)$, Jacobian matrix J is used in our study,

$$J = \begin{bmatrix} J_{|S|} \\ J_{\phi} \end{bmatrix} = \begin{bmatrix} \frac{\partial |S|}{\partial R} & \frac{\partial |S|}{\partial I} \\ \frac{\partial \phi}{\partial R} & \frac{\partial \phi}{\partial I} \end{bmatrix}. \tag{2}$$

Following the LPU in matrix form, the covariance matrix V_{pol} for the variable S in polar coordinate can be derived through [7, 9, 10],

$$V_{pol} = JV_{rec}J^T. \tag{3}$$

The standard uncertainties $u(|S|)$ and $u(\phi)$ and the sample correlation coefficient r_{pol} between $|S|$ and ϕ are then obtained from the elements of V_{pol} as

$$u(|S|) = \sqrt{V_{pol}(1,1)}, \tag{4}$$

$$u(\phi) = \sqrt{V_{pol}(2,2)}, \tag{5}$$

$$r_{pol} = \frac{V_{pol}(1,2)}{u(|S|)u(\phi)}. \tag{6}$$

Equations (4) and (5) can be described in terms of $u(R)$, $u(I)$, R, I and r directly as,

$$u(|S|) = \sqrt{\frac{u^2(R)R^2 + u^2(I)I^2}{R^2 + I^2} + r\frac{2u(R)u(I)RI}{R^2 + I^2}}, \tag{7}$$

$$u(\phi) = \sqrt{\frac{u^2(R)I^2 + u^2(I)R^2}{(R^2 + I^2)^2} - r\frac{2u(R)u(I)RI}{(R^2 + I^2)^2}}. \tag{8}$$

Both the equations are the same as the results derived using the separated LPU transformations for $|S|$ and ϕ in [11]. It is also noted that if $u(\phi) > \frac{\pi}{2}$ (happened when the uncertainty region encompasses the origin), $u(\phi) = \frac{\pi}{2}$ [6].

2.2. Uncertainty Transformation Based on the Rectangular Coordinate System Rotation

In [10], the conventional real-imaginary rectangular coordinate system is rotated with an angle of $-\phi$ to align with the radial and tangential directions of ($|S|$, ϕ). A 'radial-tangential' rectangular coordinate system is therefore formed with the covariance matrix V_{rt} which can be obtained through,

$$V_{rt} = M^T V_{rec} M, \tag{9}$$

where M is a matrix for the rotation and defined as,

$$M = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}. \tag{10}$$

The standard uncertainties $u(|S|)$ and $u(\phi)$ and the correlation coefficient r_{pol} between $|S|$ and ϕ can be obtained as

$$u(|S|) = \sqrt{V_{rt}(1,1)}, \tag{11}$$

$$u(\phi) = \tan^{-1}\left(\frac{\sqrt{V_{rt}(2,2)}}{|S|}\right), \tag{12}$$

$$r_{pol} = \frac{V_{rt}(1,2)}{u(|S|)\sqrt{V_{rt}(2,2)}}. \tag{13}$$

In the following, the performance of both the transformation methodologies will be comparatively investigated through some general cases and simulations where $u(R) \neq u(I)$ and $r \neq 0$.

3. PRACTICAL EVALUATIONS AND COMPARISONS

3.1. Case 1: The Uncertainty Region Is Distant from the Origin

For this case, the reported data from 6 repeated measurements of S_{11} of a device in [7] is selected for uncertainty evaluation and performance comparison. The complete information for matrix V_{rec} are evaluated following the GUM [5] and summarized in Table 1. Its corresponding geometric representation for the uncertainty region with 95% level of confidence is shown in Fig. 1.

The evaluated results in polar coordinate using (4), (5) and (6) are summarized together with the results evaluated using (11), (12) and (13) in Table 2. The results estimated through the *direct evaluation* [7], (i.e., $u(|S|) = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(|S_i| - |\bar{S}|)^2}$ and $u(|\phi|) = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(|\phi_i| - |\bar{\phi}|)^2}$, $|S_i|$ and $|\phi_i|$ are the magnitude and phase of ith measurement, and $|\bar{S}|$ and $|\bar{\phi}|$ are their algorithmic mean magnitude and phase.) are also included in Table 2 as a reference. From Table 2,

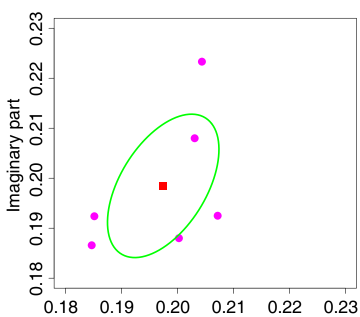

Figure 1. Elliptical region of uncertainty (95% level of confidence) for the repeated measurements given in [7]. Measured values are shown as purple dots and the mean value is shown as a red square.

Table 1. 6 repeated measurements for a complex-valued quantity S_{11} [7].

Repeated No.	Real part	Imaginary part
1	0.1847	0.1866
2	0.1852	0.1924
3	0.2072	0.1925
4	0.2003	0.1880
5	0.2031	0.2080
6	0.2044	0.2233
Mean	0.1975	0.1985
Standard uncertainty	0.0041	0.0059
Correlation coefficient	0.5093	

Table 2. Evaluated standard uncertainties $u(|S|)$ and $u(\phi)$ and the correlation coefficient r_{pol} between $|S|$ and ϕ in polar coordinate for Case 1.

	LPU	*Coordinate rotation* [10]	*Direct evaluation*
$u(\|S\|)$	0.0061	0.0061	0.0062
$u(\phi)$	0.0130	0.0130	0.0127
r_{pol}	0.3960	0.3960	0.3664

it can be observed that both the methods: the *law of propagation of uncertainty* and the *coordinate rotation*, produce highly consistent results when the uncertainty region is distant from the origin of a complex plane. Although the *direct evaluation* is not reliable [7], it still can generate some results very close to the other two methods and therefore it can be used for a 'quick check' purpose in similar application scenarios.

3.2. Case 2: The Uncertainty Region Lies Close to the Origin

This is the case when the uncertainty region encompasses the origin of a complex plane [6]. For simplicity, the measurement data and its associated uncertainty region reported in Table 1 and Fig. 1 are shifted with an offset to make its mean located at the origin. The same as above, the evaluated results using three different methods are summarized in Table 3. From the results shown in Table 3, it is observed that although the uncertainty region lies close to the origin (the uncertainty region is distorted), the *law of propagation of*

Table 3. Evaluated standard uncertainties $u(|S|)$ and $u(\phi)$ and the correlation coefficient r_{pol} between $|S|$ and ϕ in polar coordinate for the offset data in Case 2.

	LPU	*Coordinate rotation* [10]	*Direct evaluation*
$u(\|S\|)$	0.0036	0.0036	0.0024
$u(\phi)$	1.5708	1.5708	0.6876
r_{pol}	−0.3983	−0.3983	0.3091

uncertainty method and the *coordinate rotation* method still produce the same results for this set of data.

However, the results from the *direct evaluation* method is vastly different from the results evaluated by the other two methods. This is because the biased effect for the *direct evaluation* becomes more obvious when the true value becomes close to the origin. Similar as in [7], $u(|S|)$ is taken as an example for explanations of the biased effect.

For the offset data (its mean is located at the origin purposely), $|S|_{true} = 0$ while the estimated mean $|\bar{S}| = \frac{1}{n}\sum_{i=1}^{n}|S_i| > 0$. In such case, the evaluation of $u(|S|)$ directly through

$$u(|S|) = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}\left(|S_i| - |\bar{S}|\right)^2} < u(|S|)_{true}. \tag{14}$$

That is, the evaluated standard uncertainty using the *direct evaluation* method actually underestimates the 'true' underlying standard uncertainty when the uncertainty region lies close to the origin [7]. It should be avoided during the uncertainty evaluation in similar application scenarios (e.g., S_{11} of a well-matched device).

4. EVALUATIONS WITH SIMULATIONS

In Section 3, one set of data with 6 observations (or offset observations) has been used for performance evaluations of three different methods for uncertainty coordinate transformations of a complex-valued quantity. It seems that the *law of propagation of uncertainty* method and the *coordinate rotation* method are mathematically equivalent. To further explore these two methods, numerical simulations are then performed. In this study, R-programming language [12] which is a free software designed for statistical computing is used.

Bivariate normal distribution is used to simulate a complex-valued quantity S measured from an experiment as in [11]. Simulations with $\bar{R} = 0.1975$, $\bar{I} = 0.1985$, $u(R) = 0.0041$, and $u(I) = 0.0059$ have been performed by varying r with a step of 0.2 in the range of $[-1, 1]$. Very consistent uncertainty information has been observed for both the methods which is similar to those in Table 2. This demonstrates the mathematical equivalence between them when the uncertainty region is distant from the origin of a complex plane.

However when simulating the case where the uncertainty region encompasses the origin of a complex plane with $\bar{R} = \bar{I} = 0$, $u(R) = 0.0041$, $u(I) = 0.0059$ by varying r with a step of 0.2 in the range of $[-1, 1]$, the estimated uncertainty information becomes inconsistent/varying. The *law of propagation of uncertainty* method can produce repeatable results, while the *coordinate rotation* method generates some random uncertainty information with the same inputs for the simulations.

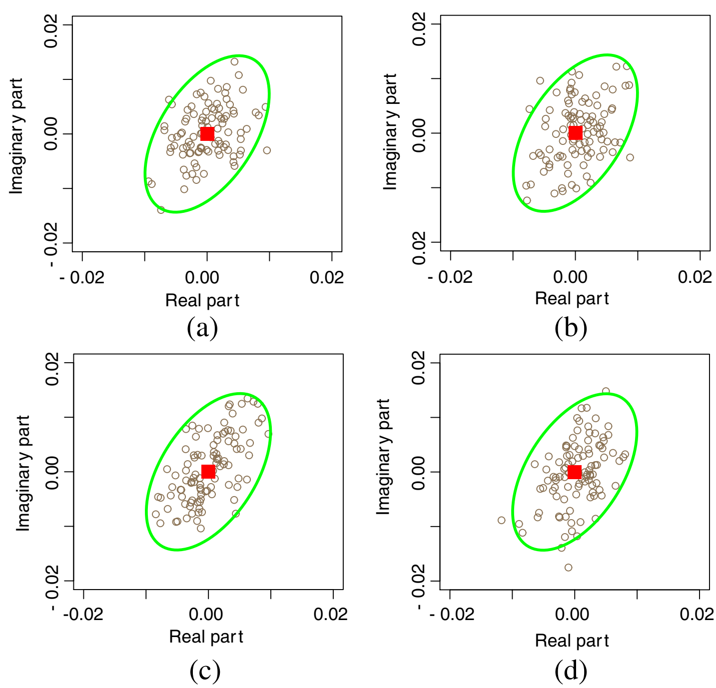

Figure 2. Simulated 100 repeated measurements with $\bar{R} = \bar{I} = 0$, $u(R) = 0.0041$, $u(I) = 0.0059$, and $r = 0.5093$. Elliptical region of uncertainty (95% level of confidence) is shown as green line, and the mean value is plotted as a red square. (a) $u(|S|) = 0.0058$, $r_{pol} = -0.5182$. (b) $u(|S|) = 0.0057$, $r_{pol} = 0.5518$. (c) $u(|S|) = 0.0060$, $r_{pol} = -0.4649$. (d) $u(|S|) = 0.0032$, $r_{pol} = -0.1044$.

In the following, the simulations with $\bar{R} = \bar{I} = 0$, $u(R) = 0.0041$, $u(I) = 0.0059$, and $r = 0.5093$ which are taken from Case 2 are used for a detailed analysis. Fig. 2 shows 4 sets of simulations with 100 repeated measurements together with the uncertainty regions at 95% level of confidence. The calculated uncertainty using the *law of propagation of uncertainty* method is exactly the same as Table 3 for all the simulations. That is, $u(|S|) = 0.0036$, $u(\phi) = 1.5708$, and $r_{pol} = -0.3983$. However, the *coordinate rotation* method produces inconsistent uncertainty information in polar coordinate for the 4 sets of simulations as indicated in Fig. 2 where only $u(\phi) = 1.5708$ is kept.

This is because the *coordinate rotation* method is more sensitive to the rotation angle $-\phi$ for alignment with the radial and tangential directions of $(|S|, \phi)$. From the analysis of the simulated data shown in Fig. 2, it is found that $\bar{\phi}$ for the 4 sets of simulations are slightly different although they are all kept in an identical elliptical uncertainty region. The observations suggest that the *coordinate rotation* method should not be recommended when the uncertainty region encompasses the origin of a complex plane.

5. CONCLUSION

In this paper, we presented a comparative study on practical evaluation of measurement uncertainty for complex-valued RF and microwave quantities in polar coordinate.

It is found that, when transforming the evaluated uncertainty in rectangular coordinate to the desired polar coordinate, the *direct evaluation* method is not reliable and should be avoided. Moreover, the results from the *law of propagation of uncertainty* method and the *coordinate rotation* method are found to be highly consistent when the uncertainty region is distant from the origin of a complex plane. This indicates that both the methods may be suitable for uncertainty coordinate transformation in similar application scenarios.

However when the uncertainty region lies close to the origin, the *law of propagation of uncertainty* method is recommended as it is more robust to the distorted uncertainty region and can produce repeatable results.

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