Progress In Electromagnetics Research M, Vol. 31, 29–43, 2013

ON THE SUCCESS OF ELECTROMAGNETIC ANALYTICAL APPROACHES TO FULL TIME-DOMAIN FORMULATION OF SKIN EFFECT PHENOMENA

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Abstract—Maxwell equations can be used to formulate an analytical full time-domain theory of skin effect phenomena in circular cylindrical conductors without any detour into the frequency domain. The paper shows how this can be done and concomitantly provides the means to determine the time-varying per unit length voltage drop along the conductor from a given time-varying conductor current. The developed relationship between voltage and current is not very complicated and led the authors to examine the reasons why it has never been utilized in transient analysis, nor given special emphasis in the literature. Those reasons are thoroughly examined and the conclusion is that the conditions required for the application of a purely time-domain skin effect theory are very restrictive.

1. INTRODUCTION

The skin effect problem is of major concern in high frequency regimes and in transient analysis. The topic is usually connected with transmission-line problems, where both the real and imaginary parts of the per unit length internal impedance of line conductors depend on the frequency, leading to signal attenuation and distortion. The problem has been studied for well over a century, from circular cylindrical conductors to conductors with more complicated cross sections [1–7].

Recently, the interest in circular and tubular geometries has been re-examined, including homogenous and inhomogeneous structures. However, most publications have been focused on frequency-domain developments and associated computational techniques [8–15]. Finite

Received 24 April 2013, Accepted 13 May 2013, Scheduled 23 May 2013

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difference time domain methods (FDTD), numerical inversion of Laplace transform, and circuital approaches, are usually employed to incorporate skin effect phenomena in time-domain transmission-line analysis [16–24]. Full time-domain analyses of skin effect in cylindrical conductors have been attempted in [10, 25]. These approaches do start with time-domain Maxwell equations but, at a certain stage, detours into the s or $j\omega$ domain are employed, involving direct/inverse Laplace/Fourier transforms, and/or convolution integrals. So, those approaches are really not fully time-domain formulated, for intrusive frequency-domain concepts are at play. The magnetic vector potential tool was used by Maxwell [1] to deal with time-domain skin effect phenomena but, since then, little attention has been paid to real full time-domain formulations of the skin effect theory. This paper is aimed at filling in this gap.

2. FREQUENCY-DOMAIN STANDARD APPROACH

The standard approach usually employed to deal with time-domain skin effect problems involves a detour in the frequency-domain. Given the excitation current $i(t)$, its Fourier transform is firstly obtained

$$\tilde{I}(\omega) = \mathfrak{F}\left\{i(t)\right\} = \int_{-\infty}^{\infty} i(t)\, e^{-j\omega t}\, dt \tag{1}$$

Next, by taking into account the frequency dependency of the per unit length (pul) complex impedance of the conductor $\bar{Z}(\omega)$, the time-domain pul voltage drop along the conductor surface is determined using the inverse Fourier transform (if it exists):

$$v(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{V}(\omega)\, e^{j\omega t} d\omega \ ; \ \ \tilde{V}(\omega) = \bar{Z}(\omega)\, \tilde{I}(\omega) = \mathfrak{F}\left\{v(t)\right\} \tag{2}$$

For homogeneous circular cylindrical conductors of radius a characterized by conductivity σ and permeability μ, the pul complex impedance is given by [26]

$$\bar{Z}(\omega) = R_{dc} \frac{J_0(2\bar{\kappa})}{\bar{\kappa}^{-1} J_1(2\bar{\kappa})}; \quad R_{dc} = \frac{1}{\sigma\pi a^2}; \quad \bar{\kappa}^2 = -j\omega\tau; \quad \tau = \sigma\mu\left(\frac{a}{2}\right)^2 \tag{3}$$

where R_{dc} is the pul dc resistance, $\bar{\kappa}$ is a dimensionless complex parameter proportional to $\sqrt{\omega}$, τ is a real constant parameter, with dimensions of time, calculated from the material properties and conductor radius. J_0 and J_1 are respectively zero order and first order Bessel functions of the first kind.

The Bessel functions appearing in the numerator and denominator of (3) can be described by well-known convergent power series:

$$\begin{cases} J_0(2\bar{\kappa}) \simeq \sum\limits_{n=0}^{m} \frac{(-1)^n\left(\bar{\kappa}^2\right)^n}{n!n!} = \sum\limits_{n=0}^{m} (n+1)g_n(j\omega\tau)^n \\ \bar{\kappa}^{-1}J_1(2\bar{\kappa}) \simeq \sum\limits_{n=0}^{m} \frac{(-1)^n\left(\bar{\kappa}^2\right)^n}{n!(n+1)!} = \sum\limits_{n=0}^{m} g_n(j\omega\tau)^n \end{cases} \tag{4}$$

where

$$g_n = \frac{1}{n!(n+1)!} \tag{5}$$

the first g_n coefficients being: $g_0 = 1$, $g_1 = 1/2$, $g_2 = 1/12$, $g_3 = 1/144$.

The expressions in (4) are exact when $m \to \infty$. However, from a computational viewpoint the summations must be truncated at some order m. It should be kept in mind that the truncation order m should be a multiple of 4, since $j^n = j^{n+4}$, by this way it is guaranteed that the same number of positive and negative contributions are accounted in both the real and imaginary parts of $(j\omega\tau)^n$.

Using (2)–(4) leads to

$$\tilde{V}(\omega) = \underbrace{\left(\frac{R_{dc} \sum\limits_{n=0}^{m} (n+1)g_n(j\omega\tau)^n}{\sum\limits_{n=0}^{m} g_n(j\omega\tau)^n} \right)}_{\bar{Z}(\omega)} \tilde{I}(\omega) \tag{6}$$

The power series developments in (4), and in (6), converge very slowly when $|2\bar{\kappa}| > 1$. For high-frequency situations, when $|2\bar{\kappa}| \gg 1$, asymptotic developments for the Bessel functions J_0 and J_1 should be employed [26],

$$J_0(2\bar{\kappa}) \simeq \frac{\cos(2\bar{\kappa} - \pi/4)}{\sqrt{\pi\bar{\kappa}}}; \qquad J_1(2\bar{\kappa}) \simeq \frac{\sin(2\bar{\kappa} - \pi/4)}{\sqrt{\pi\bar{\kappa}}}$$

leading to

$$\bar{Z}(\omega) = R_{dc}\bar{\kappa}\cot(2\bar{\kappa} - \pi/4) \tag{7}$$

Approximate results for the real and imaginary parts of $\bar{Z}(\omega) = R(\omega) + jX(\omega)$ are available for low frequency and high frequency limits. From (6) and (7):

$$\begin{cases} \text{Low frequency} & \begin{cases} R \approx R_{dc} = 1/(\sigma\pi a^2) \\ X \approx \omega L_i = \omega\mu/(8\pi) \end{cases} \\ \text{High frequency} & \{ R \approx X = 1/(\sigma 2\pi a \delta_S) \end{cases} \tag{8}$$

where δ_S is the skin penetration depth, $\delta_S = \sqrt{2/(\omega\mu\sigma)}$.

3. FULL TIME-DOMAIN SKIN EFFECT THEORY

For very good conductors, the displacement current $\partial\mathbf{D}/\partial t$ is negligibly small compared to the conduction current $\mathbf{J} = \sigma\mathbf{E}$ up to the optical spectrum. Therefore, for a homogeneous time-invariant medium with conductivity σ and permeability μ, the Maxwell equations can be written as [26]

$$\begin{cases} \nabla \times \mathbf{H} = \sigma\mathbf{E} \\ \nabla \times \mathbf{E} = -\mu\frac{\partial\mathbf{H}}{\partial t} \end{cases} \tag{9}$$

In circular cylindrical conductors carrying z-oriented currents, the electric and magnetic fields are, respectively, $\mathbf{E} = E(r,t)\hat{z}$ and $\mathbf{H} = H(r,t)\hat{\phi}$. Hence, from (9)

$$\begin{cases} \dfrac{\partial H(r,t)}{\partial r} + \dfrac{H(r,t)}{r} = \sigma E(r,t) \\ \dfrac{\partial E(r,t)}{\partial r} = \mu\dfrac{\partial H(r,t)}{\partial t} \end{cases} \tag{10}$$

3.1. Field Solution

Consider now that in the interval $0 \le r \le a$ both fields H and E can be described by radial expansions with time-dependent coefficients. Noting that $H(r) = -H(-r)$, and $E(r) = E(-r)$, the radial expansion of H will only include odd powers of r, while the radial expansion of E will only include even powers of r

$$\begin{cases} E(r,t) = \sum\limits_{n=0}^{m} e_{2n}(t)\ \left(\frac{r}{a}\right)^{2n} \\ H(r,t) = \sum\limits_{n=0}^{m} h_{2n+1}(t)\ \left(\frac{r}{a}\right)^{2n+1} \end{cases} \tag{11}$$

where m is the truncation order of the summations. Substituting (11) into (10) leads to

$$e_{2n} = \frac{\mu a}{2n}h'_{2n-1} = \frac{2(n+1)}{\sigma a}h_{2n+1} \tag{12}$$

where $h'_k = dh_k/dt$. The function $e_0(t)$ will be determined in Section 3.2, with the help of the boundary condition $H(a,t) = i(t)/(2\pi a)$, at the interface $r = a$.

The functions e_{2n} and h_{2n+1} can be determined recursively with the help of (13)

$$e_0 \to h_1 \to h'_1 \to e_2 \to h_3 \ldots e_{2n} \to h_{2n+1} \to h'_{2n+1} \to e_{2n+2} \to h_{2n+3} \ldots \tag{13}$$

The set of functions e_{2n} and h_{2n+1} (for $n = 0, 1, \ldots, m$) can then be expressed in terms of e_0 and all its time-derivatives $e_0^{(n)} = d^n e_0/dt^n$. For illustration purposes the cases $n = 0$, 1, 2 are offered below

$$n = 0 \begin{cases} e_0 = ? \\ h_1 = \dfrac{\sigma a e_0}{2} \rightarrow h_1' = \dfrac{\sigma a e_0^{(1)}}{2} \end{cases}$$

$$n = 1 \begin{cases} e_2 = \dfrac{\mu a h_1'}{2} = \dfrac{(a^2\sigma\mu)}{2^2} e_0^{(1)} \\ h_3 = \dfrac{\sigma a e_2}{4} = \dfrac{\sigma}{4}\dfrac{\sigma\mu}{2^2} e_0^{(1)} \rightarrow h_3' = \dfrac{\sigma a}{4}\dfrac{(a^2\sigma\mu)}{2^2} e_0^{(2)} \end{cases}$$

$$n = 2 \begin{cases} e_4 = \dfrac{\mu a h_3'}{4} = \dfrac{(a^2\sigma\mu)^2}{4^2 2^2} e_0^{(2)} \\ h_5 = \dfrac{\sigma a e_4}{6} = \dfrac{\sigma a}{6}\dfrac{(a^2\sigma\mu)^2}{4^2 2^2} e_0^{(2)} \rightarrow h_5' = \dfrac{\sigma a}{6}\dfrac{(a^2\sigma\mu)^2}{4^2 2^2} e_0^{(3)} \end{cases}$$

In general, for every integer value of n the following results apply

$$\begin{cases} e_{2n} = \left(\dfrac{a^2\sigma\mu}{4}\right)^n \dfrac{e_0^{(n)}}{n!n!} = (n+1) g_n \tau^n e_0^{(n)} \\ h_{2n+1} = \dfrac{\sigma a}{2}\left(\dfrac{a^2\sigma\mu}{4}\right)^n \dfrac{e_0^{(n)}}{n!(n+1)!} = \dfrac{\sigma a}{2} g_n \tau^n e_0^{(n)} \end{cases} \tag{14}$$

For compactness, the normalized nth time-derivate of $e_0(t)$ is introduced, identified by a tilde mark,

$$\tilde{e}_0^{(n)} = \tau^n e_0^{(n)} = \tau^n \frac{d^n e_0(t)}{dt^n}, \quad \tau = \sigma\mu \left(\frac{a}{2}\right)^2 \tag{15}$$

Since the constant τ has dimensions of time, all the normalized functions $\tilde{e}_0^{(n)}$ turn out to have the same physical dimensions (V/m).

Substituting (14)–(15) into (11), the time-domain field solutions are obtained

$$\begin{cases} E(r,t) = \sum\limits_{n=0}^{m} (n+1) g_n \, \tilde{e}_0^{(n)} \left(\frac{r}{a}\right)^{2n} \\ H(r,t) = \frac{\sigma r}{2} \sum\limits_{n=0}^{m} g_n \, \tilde{e}_0^{(n)} \left(\frac{r}{a}\right)^{2n} \end{cases} \tag{16}$$

3.2. Characterization of the Generating Function $\tilde{e}_0(t)$

The boundary condition ($r = a$) for the magnetic field of a current-carrying cylindrical conductor results from the consideration that the current intensity $i(t)$ can be obtained from the line integral of H along a circumferential path of radius a

$$i(t) = \oint_{r=a} H ds = 2\pi a H(a,t) \tag{17}$$

Then, from (16) and (17),

$$i(t) = \sigma\pi a^2 \sum_{n=0}^{m} g_n\, \tilde{e}_0^{(n)} \;\; \rightarrow \;\; \sum_{n=0}^{m} g_n\, \tilde{e}_0^{(n)} = R_{dc} i(t) \tag{18}$$

Successive time-differentiation of (18), from $k = 0$ to $k = m$, yields

$$\sum_{n=0}^{m-k} g_n \tilde{e}_0^{(n+k)} = R_{dc} \tilde{i}^{(k)} \tag{19}$$

where, from (15), account has been taken of $d^k \tilde{e}_0^{(n)}/dt^k = \tilde{e}_0^{(n+k)}/\tau^k$. The quantity $\tilde{i}^{(k)}$ in (19), with dimensions of ampere, is the normalized kth time-derivative of $i(t)$,

$$\tilde{i}^{(k)} = \tau^k \frac{d^k i(t)}{dt^k} = \tau^k i^{(k)} \tag{20}$$

The result in (19) defines a set of $m+1$ linear equations, which can be put in matrix notation:

$$\begin{bmatrix} \tilde{e}_0^{(0)} \\ \tilde{e}_0^{(1)} \\ \tilde{e}_0^{(2)} \\ \vdots \\ \tilde{e}_0^{(m)} \end{bmatrix} = R_{dc} \begin{bmatrix} 1 & a_1 & a_2 & \dots & a_m \\ & 1 & a_1 & \ddots & \vdots \\ & & 1 & \ddots & a_2 \\ & & & \ddots & a_1 \\ & & & & 1 \end{bmatrix} \begin{bmatrix} \tilde{i}^{(0)} \\ \tilde{i}^{(1)} \\ \tilde{i}^{(2)} \\ \vdots \\ \tilde{i}^{(m)} \end{bmatrix} \tag{21a}$$

$$\begin{bmatrix} 1 & a_1 & a_2 & \dots & a_m \\ & 1 & a_1 & \ddots & \vdots \\ & & 1 & \ddots & a_2 \\ & & & \ddots & a_1 \\ & & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & g_1 & g_2 & \dots & g_m \\ & 1 & g_1 & \ddots & \vdots \\ & & 1 & \ddots & g_2 \\ & & & \ddots & g_1 \\ & & & & 1 \end{bmatrix}^{-1} \tag{21b}$$

The upper-triangular Toeplitz matrix [27] with entries g_n is unimodular (det = 1). Its inverse is also an upper-triangular Toeplitz matrix, whose entries a_k can be determined by matrix inversion, $[\mathbf{a}] = [\mathbf{g}]^{-1}$, or recursively through

$$a_k = -\sum_{n=1}^{k} g_n a_{k-n} = -\frac{a_{k-1}}{1!2!} - \frac{a_{k-2}}{2!3!} - \ldots - \frac{a_1}{(k-1)!k!} - \frac{1}{k!(k+1)!} \tag{22}$$

where the first a_k coefficients are given by: $a_0 = 1$, $a_1 = -1/2$, $a_2 = 1/6$, $a_3 = -7/144$.

From (21a), the generating function $\tilde{e}_0(t)$ and all its derivatives can be evaluated:

$$\tilde{e}_0^{(n)} = R_{dc} \sum_{k=0}^{m-n} a_k \, \tilde{i}^{(n+k)} \tag{23}$$

3.3. Consistency Checks

3.3.1. Evaluation of the Current Intensity Based on the E-field

Based on magnetic field considerations the following result was obtained in (17)–(18)

$$i(t) = \frac{1}{R_{dc}} \sum_{n=0}^{m} g_n \, \tilde{e}_0^{(n)} \tag{24}$$

The current intensity $i(t)$ can also be evaluated by surface integration of σE through the conductor's cross sectional area S_a

$$i(t) = \int_{S_a} \sigma E dS = 2\pi\sigma \int_0^a E(r,t) r dr$$

Substituting $E(r,t)$ by its expression in (16) yields

$$\begin{aligned} i(t) &= 2\pi\sigma \sum_{n=0}^{m} (1+n) g_n \tilde{e}_0^{(n)} \int_0^a \left(\frac{r}{a}\right)^{2n} r dr \\ &= 2\pi\sigma \frac{a^2}{2} \sum_{n=0}^{m} g_n \tilde{e}_0^{(n)} = \frac{1}{R_{dc}} \sum_{n=0}^{m} g_n \tilde{e}_0^{(n)} \end{aligned} \tag{25}$$

The results in (25) and (24) are consistent.

3.3.2. Stationary Regime

Consider the case of a stationary regime where $v(t) = V$, $i(t) = I$, and $\tilde{e}_0^{(n)} = 0$ for $n \geq 1$. Using (16) yields

$$\text{Stationary regime} \begin{cases} E(r) = E(a) = V = \sum\limits_{n=0}^{0} (n+1) g_n \tilde{e}_0^{(n)} = e_0 \\ H(r) = \frac{\sigma r}{2} \sum\limits_{n=0}^{0} g_n \tilde{e}_0^{(n)} = \frac{\sigma r}{2} e_0 \end{cases} \tag{26}$$

From $I = 2\pi a H(a)$, using (26), gives

$$I = \sigma \pi a^2 e_0 \;\rightarrow\; V = R_{dc} I, \;\; R_{dc} = 1/\left(\sigma \pi a^2\right) \tag{27}$$

The pul dc internal inductance L_i can be determined from the pul magnetic energy stored inside the conductor. Using $H = H(r)$ from (26), gives

$$L_i = \frac{2W_m}{I^2} = \frac{4\pi\mu}{I^2} \int_0^a H^2 r dr = \frac{\mu}{8\pi} \tag{28}$$

The results in (27) and (28) agree to well-known results valid for stationary regimes — see (8). It should be noted that the characteristic time constant τ in (3) and (15) corresponds to $\tau = 2L_i/R_{dc}$.

3.3.3. Time Harmonic Regime

Consider the case of a time-harmonic regime, where every sinusoidal function $f(t)$ can be expressed in the form $f(t) = \mathrm{Re}(\bar{F}\, e^{j\omega t})$, where $\bar{F}$ is a complex amplitude. Recalling that $\tilde{e}_o^{(n)} = \tau^n d^n e_0/dt^n$, that is, $\tilde{e}_o^{(n)} = \mathrm{Re}((j\omega\tau)^n \bar{e}_0\, e^{j\omega t})$, one gets from (16)

$$\text{Time-harmonic regime} \begin{cases} \bar{E}(r) = \displaystyle\sum_{n=0}^{m} (n+1) g_n (j\omega\tau)^n \bar{e}_0 \left(\frac{r}{a}\right)^{2n} \\ \bar{H}(r) = \dfrac{\sigma r}{2} \displaystyle\sum_{n=0}^{m} g_n (j\omega\tau)^n \bar{e}_0 \left(\frac{r}{a}\right)^{2n} \end{cases} \tag{29}$$

Therefore, the frequency-domain skin effect impedance can be determined from

$$\bar{Z}(\omega) = \frac{\bar{V}}{\bar{I}} = \frac{\bar{E}(a)}{2\pi a \bar{H}(a)} = R_{dc} \left(\frac{\sum\limits_{n=0}^{m} (n+1) g_n \,(j\omega\tau)^n}{\sum\limits_{n=0}^{m} g_n \,(j\omega\tau)^n} \right) \tag{30}$$

The result in (30) perfectly matches the frequency-domain result in (6) derived from the Bessel functions formalism.

3.3.4. Evaluation of the pul Conductor Voltage in the Time-domain

The time-varying pul voltage drop $v(t)$ is determined from (16), (19) and (23)

$$v(t) = E(a,t) = \sum_{n=0}^{m} (n+1) g_n\, \tilde{e}_0^{(n)} = \sum_{n=0}^{m} \frac{\tilde{e}_0^{(n)}}{n!n!} = R_{dc} \sum_{n=0}^{m} u_n\, \tilde{i}^{(n)} \tag{31}$$

where the dimensionless u_n coefficients are evaluated through

$$u_n = \sum_{k=0}^{n} \frac{a_k}{((n-k)!)^2} = a_n + \frac{a_{n-1}}{1!1!} + \frac{a_{n-2}}{2!2!} + \ldots + \frac{a_1}{(n-1)!(n-1)!} + \frac{1}{n!n!} \quad (32)$$

The first u_n coefficients are: $u_0 = 1$, $u_1 = 1/2$, $u_2 = -1/12$, $u_3 = 1/48$. Values of u_n, up to $n = 12$, are presented in Table 1. Table entries are consistent with values obtained in Eqs. (43)–(44) of [25], where a Laplace transform technique was used, and where the following identity should be observed $u_n = c_n/\omega_c^n$, with $4\omega_c = \tau$.

Table 1. Coefficients u_n.

n	u_n	n	u_n	n	u_n
1	$+5.0000 \times 10^{-1}$	5	$+1.5046 \times 10^{-3}$	9	$+8.2697 \times 10^{-6}$
2	-8.3333×10^{-2}	6	-4.0923×10^{-4}	10	-2.2530×10^{-6}
3	$+2.0833 \times 10^{-2}$	7	$+1.1143 \times 10^{-4}$	11	$+6.1381 \times 10^{-7}$
4	-5.5556×10^{-3}	8	-3.0355×10^{-5}	12	-1.6723×10^{-7}

Noting that $R_{dc}u_1\tau = L_i$, Eq. (31) can be rewritten as

$$v(t) = R_{dc}i(t) + L_i\frac{di(t)}{dt} + \left(R_{dc}\sum_{n=2}^{m} u_n\tau^n\frac{d^n i(t)}{dt^n}\right) \quad (33)$$

The result in (33) can be shown to be consistent with the frequency-domain analysis in Section 2.

Look back to the result in (6)

$$\tilde{V}(\omega) = R_{dc}\left(\frac{\sum\limits_{n=0}^{m}(n+1)g_n(j\omega\tau)^n}{\sum\limits_{n=0}^{m}g_n(j\omega\tau)^n}\right)\tilde{I}(\omega) \quad (34)$$

now, assume that the quotient of the two summations in parenthesis can be represented by another summation, with u_n coefficients:

$$\frac{\sum\limits_{n=0}^{m}(n+1)g_n(j\omega\tau)^n}{\sum\limits_{n=0}^{m}g_n(j\omega\tau)^n} = \sum_{n=0}^{m} u_n(j\omega\tau)^n \quad (35)$$

For (35) to hold true, the u_n coefficients must be given by

$$u_0 = 1, u_1 = g_1, u_2 = 2g_2 - u_1 g_1, \ldots, u_n = n g_n - \sum_{k=1}^{n-1} u_k g_{n-k} \tag{36}$$

Coherently with (32), the numerical evaluation of the u_n coefficients in (36) produces the same values that are shown in Table 1. Consequently, from (34)–(35), the following result is obtained

$$\tilde{V}(\omega) = R_{dc} \sum_{n=0}^{m} u_n \tau^n \left((j\omega)^n \tilde{I}(\omega) \right) \tag{37}$$

Noting that $\mathfrak{F}\left\{\frac{d^n}{dt^n} i(t)\right\} = \mathfrak{F}\left\{i^{(n)}\right\} = (j\omega)^n \tilde{I}(\omega)$, Eq. (37) can be rewritten as

$$\mathfrak{F}\{v(t)\} = R_{dc} \sum_{n=0}^{m} u_n \tau^n \mathfrak{F}\left\{i^{(n)}\right\} \tag{38}$$

The inverse Fourier transform of (38), if it exists, will yield

$$v(t) = R_{dc} \sum_{n=0}^{m} u_n \tau^n \, \frac{d^n i(t)}{dt^n}$$

The above result exactly confirms the time-domain expression of $v(t)$ in (33).

4. COMMENTS AND DISCUSSION

The developed full time-domain analytical formulation of skin effect theory, directly based on time-domain Maxwell's equations, contains no mistakes. Moreover, its coherence and consistency was checked through a number of examples.

The derivation of the results for the electric and magnetic fields in (16), with $\tilde{e}_0^{(n)}$ given by (23), as well as the result for the pul voltage drop $v(t)$ in (33), poses no special difficulties. Nonetheless, and quite strangely, the end-result in (33) has never been utilized in transient analysis, nor given special emphasis in the literature. However, when dealing with time-domain skin effect phenomena, the following approximation is often used

$$v(t) \approx Ri(t) + L\frac{di(t)}{dt} \tag{39}$$

The utilization of (39) is acceptable for waveforms of very low frequency, where $R = R_{dc}$ and $L = L_i$, or for waveforms of very narrow spectrum, where $R = \mathrm{Re}[\bar{Z}(\omega_0)]$ and $L = \mathrm{Im}[\bar{Z}(\omega_0)/\omega_0]$ — where ω_0 denotes the spectrum central frequency. For all the remaining cases the approximation in (39) is inadequate and fails to provide accurate results.

Therefore, the question, is, again, why the result in (33) has never been used.

There are two main reasons for this. One is linked with the fact that the definition of the generating function $\tilde{e}_0$ requires $i(t)$ to be a continuously differentiable function. The other is the convergence of the summation appearing in (33).

With the exception of the particular cases of direct current and alternating current, defined for $-\infty < t < \infty$, the vast majority of physical waveforms correspond to time-limited continuous functions, that is, $i(t)$ is a non-zero time-varying function without time discontinuities in a certain time interval, $i(t) \neq 0$ for $t \in [t_1, t_2]$, where t_2 can eventually go to infinity. The time-limited nature of physical waveforms necessarily gives rise to singularities in its high-order derivatives, at least at $t = t_1$. For example, consider that $i(t)$ is a double-exponential waveform — often used to model lightning impulses:

$$i(t) = \begin{cases} 0, & \text{for } t \leq 0 \\ I\left(e^{-k_1 t} - e^{-k_2 t}\right), & \text{for } t \geq 0 \end{cases}$$

The evaluation of the 1st and 2nd time-derivatives of $i(t)$ yields

$$\frac{di(t)}{dt} = \begin{cases} 0, & \text{for } t < 0 \\ \text{undefined}, & \text{for } t = 0 \\ I\left(k_2 e^{-k_2 t} - k_1 e^{-k_1 t}\right), & \text{for } t > 0 \end{cases},$$

$$\frac{d^2 i(t)}{dt^2} = \begin{cases} 0\,, & \text{for } t < 0 \\ I\left(k_2 - k_1\right)\delta(t), & \text{for } t = 0 \\ I\left(k_1^2 e^{-k_1 t} - k_2^2 e^{-k_2 t}\right), & \text{for } t > 0 \end{cases}$$

where $\delta(t)$ is the Dirac function, with $\lim_{\Delta t \to 0} \Delta t \times \delta(t) \to 1$.

The presence, at $t = 0$, of a singularity in the 2nd derivative of the current (as well as in all subsequent higher order derivatives) precludes the utilization of (33) otherwise unrealistic infinite voltage values would be obtained from (33).

A second reason for the apparent unsuccess of (33) is the divergence of the summation $\sum_n u_n \tau^n i^{(n)}$.

As was seen in Section 3.2, and also in Section 3.3.4, the above summation is obtained by inverting another summation. Even if the latter is convergent no guarantees exist that its inverse also converges.

Consider the following example

$$\begin{cases} S_1 = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots + (-1)^n \dfrac{x^{2n}}{(2n)!} + \ldots \\ S_2 = S_1^{-1} = 1 + \dfrac{x^2}{2} + \dfrac{5x^4}{24} + \ldots + E_n \dfrac{x^{2n}}{(2n)!} + \ldots \end{cases}$$

where the numbers E_n in S_2 are the so-called Euler numbers [28].

The summation S_1 is absolutely convergent for $-\infty < x < \infty$. However, its inverse S_2 diverges when $|x| > \pi/2$. In fact S_1 represents $\cos x$, whereas S_2 represents $\sec x$.

To make things clearer assume that the conductor current is a sinusoidal waveform, $i(t) = I\cos(\omega t)$. If R and $X = \omega L$ denote the real and imaginary parts of the pul complex impedance $\bar{Z}(\omega)$ then the corresponding sinusoidal voltage drop will be expressible as

$$v(t) = (R\cos(\omega t) - X\sin(\omega t))\, I = Ri(t) + L\frac{di(t)}{dt}$$

Noting that the nth derivative of $i(t)$ can be written as

For n even, $n = 2k$: $\frac{d^n i(t)}{dt^n} = (-1)^k \omega^{2k} I\cos(\omega t) = (-1)^k \omega^{2k} i(t)$

For n odd, $n = 2k+1$: $\frac{d^n i}{dt^n} = (-1)^{k+1} \omega^{2k+1} I\sin(\omega t) = (-1)^k \omega^{2k} \frac{di}{dt}$

the result in (33), particularized to sinusoidal regimes, can be put in the convenient form

$$v(t) = \underbrace{\left(\sum_{k=0}^{m} R_k\right)}_{R(\omega)} i(t) + \underbrace{\left(\sum_{k=0}^{m} L_k\right)}_{L(\omega)} \frac{di(t)}{dt};\ \begin{cases} R_k/R_{dc} = (-1)^k u_{2k}\,(\omega\tau)^{2k} \\ L_k/L_i = (-1)^k 2u_{2k+1}\,(\omega\tau)^{2k} \end{cases} \tag{40}$$

Successful convergence of the R and L summations is achieved if

$$\lim_{k\to\infty}\left|\frac{R_{k+1}}{R_k}\right| < 1, \quad \lim_{k\to\infty}\left|\frac{L_{k+1}}{L_k}\right| < 1 \quad \leftrightarrow \quad \omega\tau < \lim_{n\to\infty}\sqrt{\frac{u_n}{u_{n+2}}} \tag{41}$$

Therefore, from (36), or from Table 1, it can be shown that: $\lim_{n\to\infty}\sqrt{\frac{u_n}{u_{n+2}}} \approx 3.67$.

Taking into account that the skin penetration depth is related to the characteristic time constant τ through $2\omega\tau = (a/\delta_S)^2$, the convergence condition in (41) can be put as $a/\delta_S < 2.71 \approx e$. In conclusion, the R and L summations involved in the computation of $v(t)$ in (40) fail to converge before strong skin effect regime takes hold. For example, consider an aluminum wire of radius $a = 0.25\,\text{mm}$ ($\tau = 0.687\,\mu\text{s}$). Let us say that strong skin effect regime starts when $\delta_S < a/10$, that is, for frequencies exceeding 11.5 MHz. The convergence condition $a/\delta_S < e$ would only be obeyed for frequencies below 0.85 MHz — this constraint is a severe limitation to the applicability of (40).

5. CONCLUSION

A purely time-domain analytical formulation of skin effect theory for circular cylindrical conductors has been established based on time-domain Maxwell equations involving no frequency-domain concepts. The new formulation has been tested for consistency and coherence, and also, it has been validated against standard frequency-domain results arising from Bessel functions. As a result of the developed theory, a time-domain expression of the longitudinal voltage drop on a cylindrical conductor has been obtained in terms of the time-domain conductor current. Regrettably, the voltage vs current relationship showed to be of little use, the reasons for such being thoroughly analyzed and discussed: Firstly, continuously time-differentiable currents are required, and secondly, the analytical procedure employs two power series whose quotient is not absolutely convergent (for time harmonic regimes the series diverges when the skin depth becomes smaller than 37% of the conductor radius). In view of these remarks, our final conclusion is that the conditions necessary for the application of the developed full time-domain theory are very restrictive. Albeit the soundness of the full time-domain theory we concluded that, at present, the standard frequency-domain approach remains the best tool for dealing with skin effect problems.

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