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APPLICATION OF INTERVAL ANALYSIS ON ERROR ANALYSIS OF REFLECTION-ONLY MATERIAL CHARACTERIZATION METHODS

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Abstract—When performing electromagnetic material characterization, an error analysis should be performed to determine the sensitivity of the extracted permittivity and permeability. Traditional error analysis methods such as the error propagation method and Monte Carlo simulations can pose difficulties when analyzing free space material characterization methods. This paper thus shows how interval analysis can be implemented to perform error analysis on free space material characterization methods and provide an alternate means to perform error analysis. Background is presented on interval representations and interval functions, and a procedure for performing error analysis with interval analysis is presented. An error analysis is performed on the free space implementation of the layer-shift method with interval analysis and the subsequent standard deviations computed with interval analysis are compared to standard deviations computed through Monte Carlo simulation.

1. INTRODUCTION

Initially introduced in the thesis of R. E. Moore in [1], interval analysis is the mathematics of sets of numbers. One of the initial uses of interval analysis was to reduce error in computer calculations by bounding solutions. Today interval analysis is being used in a variety of

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applications. Interval analysis has been used to solve various computer graphic issues [2], design of analog integrated circuits [3], the study of utility economic analysis [4], and the design of robots [5].

An increasing area of interest for interval analysis is use of it for error analysis of measurement systems. Within electromagnetics, accurate error analysis is needed for a variety of applications as demonstrated in [6–8]. Traditional error analysis methods include, but are not limited to, the error propagation method and Monte Carlo simulations; both of these methods can suffer from several computational issues. The error propagation method can suffer from rapidly varying amplification factors which lead to an overestimation of measurement error [9]. Also, Monte Carlo simulations predict an overestimation of error when the extraction procedure is prone to the production of a significant number of extrema; an example of this occurrence is with the characterization of low-loss dielectrics with the dual polarization method [10]. In the instances the error propagation method and Monte Carlo simulations are not optimal, interval analysis is a suitable error analysis tool because the error analysis is performed through direct computation of the extraction equations with interval arithmetic. In this manner, the computation of the amplification factors for the error propagation method or the many repetitive computations with Monte Carlo simulations are avoided. Other advantages of interval analysis as an error analysis include: the error analysis requires only the extraction equations and the tolerances of the independent variables, time savings for computationally heavy extraction schemes, and explicit knowledge of the worst case extraction values.

The major contribution of the work is to demonstrate the use of interval analysis as a tool for error analysis for material characterization methods. When attempting to measure the permittivity and permeability of a material in a laboratory, error inserted into the extraction is unavoidable. Therefore, it is crucial to know to what degree are the extracted permittivity and permeability valid. With the use of interval analysis, the parameters used in the measurement setup can be formed into intervals based on the instrument tolerances and the extraction equations are adapted to form interval extensions. Then the final solutions of the permittivity and permeability will be intervals which give the exact bounds for the extracted permittivity and permeability. As an example of the use of interval analysis for error analysis of material characterization methods, error analysis is performed on the free space implementation of the layer-shift method. The resulting intervals are then compared to the standard deviations computed through Monte Carlo simulations.

2. BACKGROUND OF INTERVAL ANALYSIS

2.1. Interval Notation

A set or interval can be the set of all real numbers, a range of specified integers, the empty set, etc.. With a defined interval, interval analysis forms the rules of interval arithmetic and interval extensions. The basic concept of interval analysis is to perform computations with intervals on real numbers instead on real numbers themselves [11]. For example, the interval X can be represented as

$$[\underline{x}, \overline{x}] = \{x \in \mathbb{R}\colon a \leq x \leq b\}. \tag{1}$$

Intervals are always closed sets with the endpoints included within the interval. For more information on interval arithmetic, the reader is encouraged to consult [12].

2.1.1. Real Intervals

A real interval is an interval whose members belong to the field of all rational and irrational numbers. Real intervals are denoted with capital letters. There are two ways to represent a real interval which include infimum/supremum notation and midpoint/radius notation. Equation (1) is an example of infimum/supremum notation. The minimum bound is referred to as the infimum and the maximum bound is referred to as the supremum. The infimum and supremum are denoted with an underline and an overline respectively.

In addition to infimum/supremum notation, intervals can also be represented with their midpoint and radius. The midpoint of an interval is the point in the center of the interval or $m(X) = \frac{(\underline{X}+\overline{X})}{2}$. The radius is simply half the width of the interval, which is $r(X) = \frac{\overline{X}-\underline{X}}{2}$. The midpoint-radius form is thus

$$X = \langle m, r\rangle, \quad x \in \mathbb{R}. \tag{2}$$

2.1.2. Complex Intervals

Naturally, in the course of studying many engineering applications complex numbers will be encountered. This section reviews the three different ways to represent complex numbers in interval analysis which include rectangular, circular, and sector representation.

A rectangular interval is represented with intervals for the real and imaginary parts separately as in (3). With rectangular representation, a rectangle is constructed in the complex plane.

$$Z = X + jY = \{x + jy\colon x \in X, y \in Y\} \tag{3}$$

The second way to represent complex intervals is with circular representation [13]. In order to convert from a rectangular interval to a circular interval, one must calculate the midpoint, a, and radius, r, of the circle. This process is similar to converting from infimum/supremum representation to midpoint/radius representation. One then represents the complex interval as in Equation (4) which is portrayed as a circle in the complex plane.

$$Z = \langle a, r \rangle = \{z \in C \colon |z - a| \leq r\} \tag{4}$$

The last way to represent a complex interval is sector representation. Sector representation is similar to polar or magnitude/phase notation in conventional arithmetic. In complex sector representation, the interval is represented with a radius, r, and an angle, W [14]. Common notation for sectors is in Equation (5). Sectors form arcs in the complex plane. Utilization of complex sector intervals often involves converting to circular or rectangular complex intervals to perform arithmetic operations and then conversion back to sector intervals.

$$Z = R \cdot e^{j \cdot W} = \left\{r \cdot e^{jw} \colon r \in R, w \in W\right\} \tag{5}$$

2.2. Interval Functions

There are two main classifications of functions in interval analysis. The first classification of interval functions are *united extensions*. United extensions are functions that are created by taking a real-valued function f and computing the range of values $f(x)$ takes as x is varied through an interval X [12]. United extensions are denoted as $\bar{f}(X)$. United extensions result in the set image of $f(X)$ which is defined by

$$\bar{f}(X) = \{f(x) \colon x \in X\}. \tag{6}$$

The other type of interval functions are *interval-valued extensions* or shortly termed interval extensions. Interval extensions are created by directly extending an ordinary real-valued function to interval arguments [12]. Common notation for an interval extension of a function f is F. Although there is not a unique interval extension for a specific real-valued function, any interval extension is valid as long as that when a degenerate interval (an interval with the same infimum and supremum) is plugged into the interval extension the correct value for the real-valued function is retrieved, i.e.,

$$F\left([x, x]\right) = f(x). \tag{7}$$

Interval extensions will be exclusively used in this paper.

3. ERROR ANALYSIS OF THE LAYER-SHIFT METHOD USING INTERVAL ANALYSIS

3.1. The Layer-shift Method

The layer-shift method is a material characterization method which was originally developed in [15]. The layer-shift method is performed by measuring the reflection coefficient twice; once with the material under test (MUT) backed directly by a perfect electric conductor (PEC), and again with a space placed in between the MUT and the PEC. The layer-shift method can be performed with free space methods and with methods which use guided-wave applications, such as coaxial methods. The free space implementation of the layer-shift method is illustrated in Figure 1. In [16], the method is noted for being particularly competent in characterizing dielectric media.

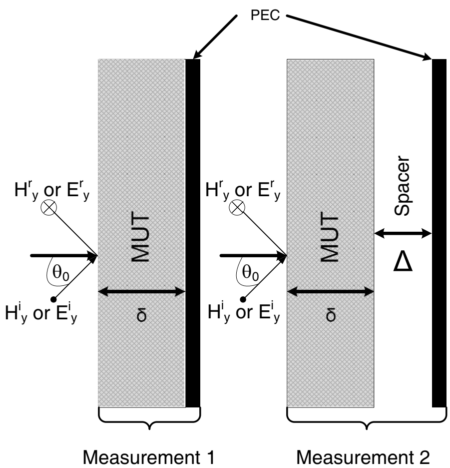

Figure 1. Layer-shift diagrams.

To demonstrate interval analysis as a tool to perform error analysis, the free space implementation of the layer-shift method is considered. From [17], the extraction equations for ϵ_r and μ_r are in (8) and (9) respectively. The free space version of the layer shift method can be executed with either TM (transverse magnetic) or TE (transverse electric) incidence; thus, (8) are the extraction equations for TM plane wave incidence and (9) are the extraction equations for

TE plane wave incidence.

$$\epsilon_r = \frac{k_{z2}\eta_0}{k_0 Z_2}, \quad \mu_r = \frac{k_{z2}^2 + k_0^2 \sin^2\theta}{k_0^2 \epsilon_r} \tag{8}$$

$$\mu_r = \frac{k_{z2} Z_2}{k_0 \eta_0}, \quad \epsilon_r = \frac{k_{z2}^2 + k_0^2 \sin^2\theta}{k_0^2 \mu_r} \tag{9}$$

In Equations (8)–(9), η_0 is the intrinsic impedance of free space and k_0 is the free space wavenumber. Additionally, Z_2 is the transverse impedance of the MUT defined in (10) and k_{z2} is the z-component of the wave vector defined in (11).

$$Z_2 = \frac{Z_3^b Z_1^a(z_1) Z_1^b(z_1)}{Z_3^b + Z_1^a(z_1) - Z_1^b(z_1)} \tag{10}$$

$$k_{z2} = \arctan\left(-j\frac{Z_1^a(z_1)}{Z_2}\right)/\delta \tag{11}$$

In (10) and (11), superscript a and b refer to measurements conducted with the MUT flush against the metal plate and with the spacer between the MUT and metal plate respectively. Furthermore, Z_1 and Z_3 are the transverse impedances in regions 1 (the free space region where the reflected field is measured) and region 2 (the spacer region) in 12 and 13, R is the reflection coefficient measured as depicted in Figure 1, δ is the thickness of the MUT and Δ is the thickness of the spacer. Note that $Z_3^a = 0$ equals zero because there is no spacer region and the MUT is flush against the PEC.

$$Z_1^a = Z_0\frac{1+R_a}{1-R_a}, \quad Z_1^b = Z_0\frac{1+R_b}{1-R_b} \tag{12}$$

$$Z_3^a = 0, \quad Z_3^b = jZ_3\tan(k_{z3}\Delta). \tag{13}$$

3.2. Error Analysis of the Layer-shift Method

The process of using interval analysis with material characterization methods is a multi-step process which is described below. The process is described within the context of the layer-shift method, but can be extended for error analysis of other material characterization methods.

Step 1 The first step in employing interval analysis is to determine the independent variables which propagate error into the extraction equations and form intervals based upon these independent variables. For the layer-shift method, these variables are the angle of incidence of the incident wave, θ, the thickness of the MUT, δ, the thickness of the spacer, Δ, and the magnitude and phase of the reflection coefficient, R. Due to the tolerance of the instruments used to measure these parameters, there is an associated amount of uncertainty for each variable. Using the tolerance of each measurement instrument, each of the stated independent variables can be transformed into an interval. For example, if $\theta = 45^\circ$ and the instrument used to measure the angle has a tolerance of $\pm 0.5^\circ$, the interval for θ will be $[44.5^\circ, 45.5^\circ]$. This process will be conducted for all of the independent variables.

Step 2 The second step is to form the interval extensions described in Section 2.2 of the extraction equations in (8) and (9). The interval extensions are created by taking Equations (8) and (9) and replacing the independent variables with the intervals described from step 1. In the case of the extraction equations from the layer-shift method, this will also make (12), (13), and Z_0 intervals.

Step 3 The last step of the error analysis is to evaluate the interval extensions using interval arithmetic. There are several software packages available that implement interval arithmetic. The software package used for this work was Intlab which is a Matlab extension [18]. Once the intervals are computed, the mid-point of the intervals will correspond to the nominal ϵ_r and μ_r values and the interval radii will correspond to the standard deviation associated with the extracted ϵ_r and μ_r.

Step 4 After the initial computation of the intervals, one should ensure that the intervals computed have the smallest radii possible. Often in the course of interval arithmetic, operations like multiplication and division produce infimum and supremum which are overestimations of the actual bounds of the interval extension. To ensure that the interval has the smallest radius a process called refinement is used. For example, say there is an interval extension F which is dependent on the interval X. Refinement is the process of computing the union of the interval values of F over the elements of a uniform subdivision of X into n smaller intervals represented by $X_1, X_2, \ldots, X_n$ [12]. The number of subdivision (i.e., the value of n) is determined by the user. Thus, to ensure that the smallest interval radius has been obtained, the value of n is increased until the infimum and supremum bounds of F no longer changes.

4. RESULTS

To test the usefulness of interval analysis as a method for error analysis, an error analysis was conducted with both interval analysis and Monte Carlo simulations on the layer-shift method used to characterize a commercial MagRAM (magnetic radar absorbing material) at 10 GHz with nominal $\epsilon_r = 10.65 - j1.50$ and $\mu_r = 1.65 - j0.9$. The values for the independent variables θ, δ, and Δ, R_a, and R_b with the corresponding instrument tolerances and related intervals are presented in Table 1.

Table 1. Independent variable values and independent variable intervals determined from instrument tolerances.

Independent Variable	Value	Instrument Tolerance	Independent Variable Interval
θ	$40°$	$\pm.5°$	$[39.5°, 40.5°]$
δ	57.16 mil	± 1 mil	$[56.16, 58.16]$ mil
Δ	230.74 mil	± 1 mil	$[229.74, 231.74]$ mil
$\lvert R_a \rvert$	0.1461	± 0.004	$[0.1421, 0.1501]$
$\angle R_a$	$66.14°$	$\pm 0.08°$	$[66.08°, 66.22°]$
$\lvert R_b \rvert$	0.6152	± 0.004	$[0.6112, 0.6192]$
$\angle R_b$	$-171.44°$	$\pm 0.08°$	$[-171.52°, -171.36°]$

Table 2 shows the mean ϵ_r and μ_r computed through 100,000 Monte Carlo simulations and the mid-points of the ϵ_r and μ_r intervals computed through interval analysis. The mid-points of the ϵ_r and μ_r intervals correspond closely to the nominal ϵ_r and μ_r values of the MagRAM sample. Therefore, the radii of the ϵ_r and μ_r intervals will be referenced from the nominal ϵ_r and μ_r values.

Table 2. Mean ϵ_r and μ_r predicted by Monte Carlo simulations and ϵ_r and μ_r interval mid-points computed through interval analysis.

Method	μ'_r	μ''_r	ϵ'_r	ϵ''_r
Monte Carlo	1.7135	-0.9394	10.6874	-1.5008
Interval Analysis	1.7157	-0.9420	10.6944	-1.5105

Table 3 shows the standard deviations for the total error propagated into the extracted μ_r and ϵ_r by running 100,000 Monte Carlo simulations and radii of the μ_r and ϵ_r intervals computed via interval analysis. Again, the interval radii are considered to be the standard deviations for the extracted μ_r and ϵ_r. At first glance, it appears that the interval analysis predictions are overestimated. However, one must remember that one standard deviation, σ, predicted

Table 3. Standard deviation of ϵ_r and μ_r predicted by Monte Carlo simulations and interval analysis at 10 GHz.

Method	$\sigma_{\epsilon'_r}$	$\sigma_{\epsilon''_r}$	$\sigma_{\mu'_r}$	$\sigma_{\mu''_r}$
Monte Carlo	0.2274	0.1091	0.0329	0.0226
Interval Analysis	2.6321	2.6321	1.0224	1.0224
Interval Analysis (Refined)	0.2194	0.0953	0.0333	0.0255

by Monte Carlo simulations is valid for only 34.1% of the extractions. For the standard deviations predicted with Monte Carlo simulations, a tolerance which represents 99.7% of possible extractions is 3σ wide; this corresponds to $\sigma_{\mu'_r} = 0.0987$, $\sigma_{\mu''_r} = 0.0678$, $\sigma_{\epsilon'_r} = 0.6822$, and $\sigma_{\epsilon''_r} = 0.3273$. Although the radii of the intervals are still larger than the 3σ tolerances predicted by the Monte Carlo simulations, it is emphasized that interval analysis gives absolute bounds for μ_r and ϵ_r. From this perspective, the interval analysis solutions provide useful information because the worst possible solution is explicitly stated. Nevertheless, the worst possible solutions are not the most probable bounds for μ_r and ϵ_r.

Using refinement as explained in Section 3.2, ϵ_r and μ_r intervals with smaller radii can be computed. Refinement was conducted for the ϵ_r and μ_r intervals for all five sources error (incidence angle, MUT thickness, spacer thickness, and magnitude and phase of the measured reflection coefficients) simultaneously. The intervals for ϵ'_r, ϵ''_r, μ'_r, and μ''_r were refined until the radii of the ϵ'_r, ϵ''_r, μ'_r, and μ''_r intervals no longer changed; for this refinement the smallest interval radii were achieved when $n = 30$. Accordingly, Table 3 shows the refined standard deviations (i.e., — interval radii) which show greater agreement to the standard deviations computed with Monte Carlo simulations. These results show how interval analysis can be implemented as a method for error analysis for electromagnetic material characterization.

5. CONCLUSIONS

An error analysis is performed on the free space implementation of the layer-shift method with both interval analysis and Monte Carlo simulations for a commercial MagRAM sample. The error analysis procedure using interval analysis includes forming interval extensions from the layer-shift extraction equations and evaluating the interval extensions in Intlab. The midpoint of the ϵ_r and μ_r intervals correspond to the nominal ϵ_r and μ_r values and radii of the intervals correspond to the ϵ_r and μ_r standard deviations. Initial standard deviations computed with interval analysis were larger than standard deviations computed with Monte Carlo simulations. However, use of refinement produced standard deviations which are in close agreement to standard deviations computed with Monte Carlo simulations. The results presented show that interval analysis can be used as an acceptable tool to perform error analysis of free space material characterization methods.

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