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RIGOROUS CLOSED FORM EXPRESSIONS FOR THE INPUT ADMITTANCE OF A COAXIAL PROBE RADIATING INTO A LOSSY PARALLEL PLATE WAVEGUIDE. A DYADIC GREEN'S FUNCTION APPROACH

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Abstract—In this paper the derivation of a rigorous expression of the input admittance of a coaxially fed, infinite, lossy parallel plate waveguide (PPWG) is presented. The derivation makes use of the dyadic Green's function of the PPWG expressed as series a cylindrical wave-functions. Losses into the dielectric plates and on the conductors are considered rigorously. The approximation used in results presented in the past literature are critically discussed. Numerical experiments are performed to show the effects of the finite conductivity on the input impedance of the PPWG.

1. INTRODUCTION

The parallel plate waveguide, PPWG henceforth, is the simplest waveguiding structure available and is studied in every textbook of engineering electromagnetics. In spite of its simplicity, PPWG has been used to realize several microwave devices such as PPWG-based antennas [1–3]. Also, the PPWG theoretical model has been employed to ascertain equivalent circuits to be used in the design phase of complex devices [4–6]. Recently, the authors have modeled SIW (Substrate Integrated Waveguide) as a PPWG populated with metallic via holes [7, 8]. This method has been applied to the efficient analysis of

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antennas [9] and of resonators [10–12], for which the method presented in [13] was used. For many of this application the availability of an analytical expressions of the input admittance of a coaxially fed PPWG is of a primary importance. For this reason, analytical formulas of the input admittance have been presented in the past in a number of papers [14–18]. However, to the best of the authors' knowledge, the presence of losses has never been taken into account. In this paper, a rigorous derivation of the input admittance of a lossy, infinite PPWG fed by a coaxial probe is given. An approach which makes use of the dyadic Green's function follows. In fact, this method allows for a straightforward inclusion of both losses on the conductors and into the dielectric layer. However, notice that while the inclusion of losses into the dielectric substrate does not pose major problems, finite conductivity of top and bottom plates has to be carefully modeled. In fact, to rigorously take into account the presence of plates with finite conductivity, one should consider both equivalent electric and magnetic current sources to model the coaxial probe [19]. Notice that in [7, 8, 19] the Green's function is expressed first in the spectral domain and then transformed back into the spatial domain with the help of the theorem of residues. In the case at hand, losses on the top and bottom conducting plates directly affect the positions of the residues in the complex plane. In spite of the previous consideration, in [8] and [19] only magnetic current sources were considered, and a first order approximation was used for the residues. Results given in those papers were correct because good conductors were considered. In this paper, the input admittance will be studied widening the range of metals considered, including conductors with conductivity four order of magnitude lower than the one of copper, like low conductivity steel [20]. In the following, the dyadic Green's function of the parallel plate waveguide will be briefly discussed, and a rigorous expression of the input admittance of a coaxial probe radiating into the parallel plate will be calculated. Results from numerical experiment will be presented to evaluate the effects of conductivity and of the approximations introduced.

2. DYADIC GREEN'S FUNCTION OF THE LOSSY PARALLEL PLATES WAVEGUIDE

The dyadic Green's function of the parallel plate [7, 8, 19] is derived from a general expression in terms of cylindrical waves given for a

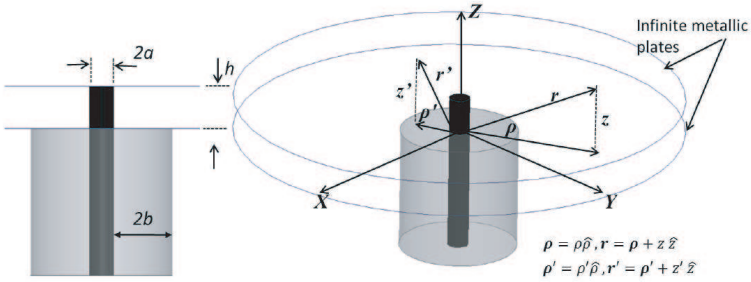
general multilayer structure

$$\bar{\bar{G}}_{PPW}(\mathbf{r},\mathbf{r}') = -\frac{j}{4\pi}\sum_n \int_0^\infty dk_\rho k_\rho$$
$$\left[(\nabla\times\hat{\mathbf{z}})\left(\nabla'\times\hat{\mathbf{z}}\right)\frac{\Phi_n\left(k_\rho,\rho,\phi\right)\Phi_{-n}\left(-k_\rho,\rho',\phi'\right)}{k_z k_\rho^2}F^{TM}\left(k_z,z,z'\right)\right.$$
$$\left.+(\nabla\times\nabla\times\hat{\mathbf{z}})\left(\nabla'\times\nabla'\times\hat{\mathbf{z}}\right)\frac{\Phi_n\left(k_\rho,\rho,\phi\right)\Phi_{-n}\left(-k_\rho,\rho',\phi'\right)}{k^2 k_z k_\rho^2}F^{TE}\left(k_z,z,z'\right)\right]$$
$$-\frac{1}{k^2}\hat{\mathbf{z}}\hat{\mathbf{z}}\delta\left(\mathbf{r}-\mathbf{r}'\right) \quad (1)$$

with $k=\omega\sqrt{\mu_0(\varepsilon_r'-j\varepsilon_r'')\varepsilon_0}$, $k_z=\sqrt{k^2-k_\rho^2}$ in which k_ρ is the spectral variable and correspond to the radial component of the propagation constant. Furthermore, $\omega=2\pi f$, where f is the frequency in Hz, μ_0 the vacuum permeability, ε_0 the vacuum permittivity, and $\varepsilon_r'-j\varepsilon_r''$ the complex permittivity of the dielectric between the conducting plates. Expression (1) refers to a cylindrical system of coordinates, like the one in Figure 1. Primed quantities are relevant to point of the space in which sources are present. Functions Φ in expression (1) have the following form:

$$\Phi_n\left(k_\rho,\rho,\phi\right)=J_n\left(k_\rho\rho\right)e^{-jn\phi} \quad (2)$$

where J_n is the Bessel function of order n.



(a) Side view (b) 3D view

Figure 1. Geometry of the coaxially fed parallel plate waveguide.

Functions F^{TM}, F^{TE} depend on the boundary conditions imposed on the top and bottom plates and on the position of the sources. In the present case only magnetic current sources placed in z' are considered. Functions F^{TM}, F^{TE} can be derived considering equivalent transmission lines, one for the TE mode and one for the

TM mode, closed on loads determined by the appropriate boundary conditions on the top and bottom plates. In particular F^{TM} is proportional to the current $I(z)$ in circuit in Figure 2(a) due to a unit voltage generator $\mathbf{v}$ placed in z'. Furthermore, F^{TE} is proportional to the voltage $V(z)$ in circuit in Figure 2(b) due to the current generator $\mathbf{i}$ in z'.

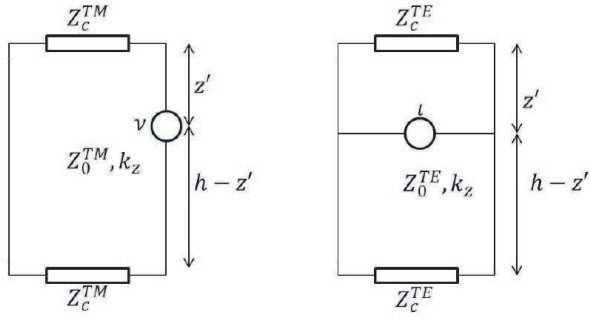


Figure 2. Equivalent transmission lines for determine functions F^{TM} and F^{TE}.

Functions F^{TM}, F^{TE}, for a magnetic source placed in $z' = 0$, have the following forms:

$$\begin{aligned}
F^{TM}\left(k_z, z, z'\right) &= 2Z_0^{TM} I\left(z\right) \\
&= \frac{\left(e^{jk_z z'} - \Gamma_c^{TM} e^{-jk_z z'}\right)\left(e^{-jk_z z} - \Gamma_c^{TM} e^{-2jk_z h} e^{jk_z z}\right)}{1 - \Gamma_c^{TM^2} e^{-2jk_z h}} \quad z > z' \\
F^{TM}\left(k_z, z, z'\right) &= 2Z_0^{TM} I\left(z\right) \\
&= \frac{\left(e^{jk_z z} - \Gamma_c^{TM} e^{-jk_z z}\right)\left(e^{-jk_z z'} - \Gamma_c^{TM} e^{-2jk_z h} e^{jk_z z'}\right)}{1 - \Gamma_c^{TM^2} e^{-2jk_z h}} \quad z < z'
\end{aligned} \tag{3}$$

$$\begin{aligned}
F^{TE}\left(k_z, z, z'\right) &= \frac{2V\left(z\right)}{Z_0^{TE}} \\
&= \frac{\left(e^{jk_z z'} + \Gamma_c^{TE} e^{-jk_z z'}\right)\left(e^{-jk_z z} + \Gamma_c^{TE} e^{-2jk_z h} e^{jk_z z}\right)}{1 - \Gamma_c^{TE^2} e^{-2jk_z h}} \quad z > z' \\
F^{TE}\left(k_z, z, z'\right) &= \frac{2V\left(z\right)}{Z_0^{TE}} \\
&= \frac{\left(e^{jk_z z} + \Gamma_c^{TE} e^{-jk_z z}\right)\left(e^{-jk_z z'} + \Gamma_c^{TE} e^{-2jk_z h} e^{jk_z z'}\right)}{1 - \Gamma_c^{TE^2} e^{-2jk_z h}} \quad z < z'
\end{aligned} \tag{4}$$

In the previous equations h is the thickness of the parallel plate and Γ_c^{TM}, Γ_c^{TE} are given by

$$\begin{aligned} \Gamma_c^{TM} &= \frac{Z_c^{TM} - Z_0^{TM}}{Z_c^{TM} + Z_0^{TM}} \\ \Gamma_c^{TE} &= \frac{Z_c^{TE} - Z_0^{TE}}{Z_c^{TE} + Z_0^{TE}} \end{aligned} \tag{5}$$

Impedances Z_c^{TM}, Z_c^{TE} depend on the boundary conditions imposed on the top and bottom plates. When plates are of a good conductor, they can be considered as semi-infinite layer of dielectric with dielectric constant $\varepsilon_c = \varepsilon_0(1 - j\sigma/(\omega\varepsilon_0))$, where σ is the conductivity of the conductor, and expressions (5) become

$$\begin{aligned} \Gamma_c^{TM} &= \frac{\dfrac{k_{zc}}{\omega\varepsilon_c} - \dfrac{k_z}{\omega\varepsilon}}{\dfrac{k_{zc}}{\omega\varepsilon_c} + \dfrac{k_z}{\omega\varepsilon}} = \frac{k_{zc}\varepsilon - k_z\varepsilon_c}{k_{zc}\varepsilon + k_z\varepsilon_c} \\ \Gamma_c^{TE} &= \frac{\dfrac{\omega\mu_0}{k_{zc}} - \dfrac{\omega\mu_0}{k_z}}{\dfrac{\omega\mu_0}{k_{zc}} + \dfrac{\omega\mu_0}{k_z}} = \frac{k_z - k_{zc}}{k_z + k_{zc}} \end{aligned} \tag{6}$$

with $\varepsilon = \varepsilon_r\varepsilon_0$ and $k_z = \sqrt{\omega^2\varepsilon\mu_0 - k_\rho^2}$, $k_{zc} = \sqrt{\omega^2\varepsilon_c\mu_0 - k_\rho^2}$, furthermore the following expressions have been used

$$\begin{aligned} Z_0^{TM} &= \frac{k_z}{\omega\varepsilon} \\ Z_0^{TE} &= \frac{\omega\mu_0}{k_z} \end{aligned} \tag{7}$$

It is easy to show that for a good conductor where $\sigma\omega/\varepsilon_0 \gg 1$, the Leontovich boundary condition holds, i.e.,

$$Z_c^{TM} = Z_c^{TE} \approx Z_s = (1 + j)/\sigma\delta \tag{8}$$

with $\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}}$. Expression (8) can be used in relations (6) to express Γ^{TM}, Γ^{TE}. Equation (1) is manipulated further extending the integration path to $[-\infty, +\infty]$ and introducing in the integrand Hankel's functions. With a proper choice of the integration path in the complex plane and by applying the residue theorem [19] one obtains

the following form for the Green's function:

$$\bar{\bar{G}}_{PPW}\left(\mathbf{r},\mathbf{r}'\right) = -\frac{1}{k^2}\hat{\mathbf{z}}\hat{\mathbf{z}}\delta\left(\mathbf{r}-\mathbf{r}'\right) - \frac{1}{4}\sum_m$$
$$\left[\left(\nabla\times\hat{\mathbf{z}}\right)\left(\nabla'\times\hat{\mathbf{z}}\right)H_0^{(2)}\left(k_{\rho m}\left|\boldsymbol{\rho}-\boldsymbol{\rho}'\right|\right)\right.$$
$$\frac{\left(e^{jk_{zm}z'}-\Gamma_c^{TM}e^{-jk_{zm}z'}\right)\left(e^{-jk_{zm}z}-\Gamma_c^{TM}e^{-2jk_{zm}h}e^{jk_{zm}z}\right)}{D^{TM'}\left(k_{\rho m}\right)} +$$
$$-\frac{1}{4}\sum_m\frac{1}{k^2}\left(\nabla\times\nabla\times\hat{\mathbf{z}}\right)\left(\nabla'\times\nabla'\times\hat{\mathbf{z}}\right)H_0^{(2)}\left(k_{\rho m}\left|\boldsymbol{\rho}-\boldsymbol{\rho}'\right|\right)$$
$$\left.\times\frac{\left(e^{jk_{zm}z'}+\Gamma_c^{TE}e^{-jk_{zm}z'}\right)\left(e^{-jk_zz}+\Gamma_c^{TE}e^{-2jk_{zm}h}e^{jk_{zm}z}\right)}{D^{TE'}\left(k_{\rho m}\right)}\right] \quad (9)$$

In Equation (9), $H_0^{(2)}$ is the Hankel function of second kind of zero order, m is the indexrelevant to the summation over the residues, $k_{\rho m}$ is the mth residue and $k_{zm}=\sqrt{k^2-k_{\rho m}^2}$. As shown in Equation (3), the expression valid for $z<z'$ is obtained interchanging z with z'. $D^{TM'}(k_\rho)$, $D^{TE'}(k_\rho)$ are the first derivatives of equations

$$\begin{aligned} D^{TM}\left(k_\rho\right) &= \left(1-\Gamma_c^{TM^2}e^{-2jk_zh}\right) \\ D^{TE}\left(k_\rho\right) &= \left(1-\Gamma_c^{TE^2}e^{-2jk_zh}\right) \end{aligned} \quad (10)$$

taken with respect to k_ρ and, when they are evaluated on residues $k_{\rho m}$, assume the following expressions:

$$D^{TM',TE'}(k_{\rho m}) = \frac{k_{zm}^2-k_{\rho m}}{k_{zm}}\left(\Gamma^{TM,TE^2}e^{-2jk_{zm}h}-1\right)+k_{\rho m}k_{zm}e^{-2jk_{zm}h}$$
$$\left(-2Z_0^{TM',TE'}\frac{\Gamma^{TM,TE}\left(1+\Gamma^{TM,TE}\right)}{Z_c^{TM,TE}+Z_0^{TM,TE}}+\frac{2jhk_{\rho m}}{k_{zm}}\Gamma^{TM,TE^2}\right) \quad (11)$$

in which

$$\begin{aligned} Z_0^{TM'} &= -\frac{k_{\rho m}}{k_{zm}}\frac{1}{\omega\varepsilon} \\ Z_0^{TE'} &= \frac{\omega\mu_0k_{\rho m}}{k_{zm}^3} \end{aligned} \quad (12)$$

Quantities $k_{\rho m}$, k_{zm} are found evaluating the roots of the following equations

$$\begin{aligned} D^{TM}\left(k_{\rho m}\right) &= 0 \\ D^{TE}\left(k_{\rho m}\right) &= 0 \end{aligned} \quad (13)$$

Roots of Equation (13) have to be computed numerically, however if top and bottom plates are made of a good conductor the following approximated analytical expression can be derived:

$$\begin{aligned} k_{zm} &= \frac{m\pi}{h} + \Delta k_z^{TM,TE} \quad \text{with} \\ \Delta k_z^{TM} &= \begin{cases} \sqrt{\frac{2jZ_s\omega\varepsilon}{h}} & m = 0 \\ \frac{2jZ_s\omega\varepsilon}{m\pi} & m = 1, 2, 3 \ldots \end{cases} \\ \Delta k_z^{TE} &= \frac{2jZ_s m\pi}{\omega\mu_0 h^2} \quad m = 1, 2, 3 \ldots \end{aligned} \tag{14}$$

The previous expressions have been found considering that $Z_s/Z_0^{TM,TE} \ll 1$ and under the hypothesis that $\Delta k_z^{TM,TE} \ll \pi/h$.

3. INPUT ADMITTANCE OF THE LOSSY PARALLEL PLATES WAVEGUIDE

For the case at hand, when perfectly conducting plates are considered, using the theorem of equivalence one can consider a ring of magnetic current placed in $z' = 0$ and directed along $\hat{\phi}$ as a source:

$$\mathbf{M}\left(\mathbf{r}'\right) = \mathbf{E_{coax}} \times \hat{\mathbf{z}} = -\frac{V}{\log\left(\frac{b}{a}\right)} \frac{1}{\rho'} \delta(z') \hat{\boldsymbol{\phi}} \tag{15}$$

Quantities a, b are the inner and outer radius of the coaxial probe as shown in Figure 1. In (15) a field $\mathbf{E_{coax}}$, $\mathbf{H_{coax}}$ relevant to the TEM mode is considered to be present in the coaxial cable. When a ground plane with finite conductivity is considered instead, the theorem of equivalence prescribes an additional electric current source. However, even under this condition, an equivalent magnetic current can be considered as [21]:

$$\mathbf{M}\left(\mathbf{r}'\right) = \left(\mathbf{E_{coax}} - Z_s \hat{\mathbf{z}} \times \mathbf{H_{coax}}\right) \times \hat{\mathbf{z}} \tag{16}$$

where Z_s is the impedance of the ground plane defined in Equation (8). When the expression of the electric and magnetic fields in a coaxial cable are substituted in (16) one has:

$$\mathbf{M}\left(\mathbf{r}'\right) = -\frac{V}{\ln\left(\frac{b}{a}\right)} \frac{1}{\rho'} \left(1 + \frac{Z_s}{\eta_0}\right) \delta(z') \hat{\boldsymbol{\phi}} \tag{17}$$

with $\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_{r_{coax}}\varepsilon_0}}$. The magnetic component of the electromagnetic field radiated into the PPWG by the source has the following form

$$\mathbf{H}\left(\mathbf{r}\right) = -j\omega\varepsilon \int_V \bar{\mathbf{G}}_{\mathbf{PPW}}(\mathbf{r},\mathbf{r}') \cdot \mathbf{M}\left(\mathbf{r}'\right) \mathbf{dr}' \tag{18}$$

which becomes

$$\mathbf{H}(\mathbf{r}) = j\omega\varepsilon \int_0^h dz' \int_a^b d\rho' \int_0^{2\pi} d\phi' \frac{1}{4}\sum_m \left[(\nabla \times \hat{\mathbf{z}}) (\nabla' \times \hat{\mathbf{z}}) H_0^{(2)} \left(k_{\rho m} \left|\boldsymbol{\rho} - \boldsymbol{\rho}'\right|\right) \frac{\left(e^{jk_{zm}z'} - \Gamma_c^{TM} e^{-jk_{zm}z'}\right)\left(e^{-jk_{zm}z} - \Gamma_c^{TM} e^{-2jk_{zm}h} e^{jk_{zm}z}\right)}{D^{TM'}(k_{\rho m})} \right] \cdot \hat{\boldsymbol{\phi}} \frac{V}{\log\left(\frac{b}{a}\right)} \left(1 + \frac{Z_s}{\eta_0}\right) \delta(z') \quad (19)$$

In expression (19) TE modes are omitted because of their convolution with the source gives zero contribution. Expression (19) gives the magnetic field due to the ring of the magnetic current in the PPWG which once solved gives the following expressions:

$$\mathbf{H}(\boldsymbol{\rho}) = \frac{V}{\log\left(\frac{b}{a}\right)} j\omega\varepsilon \frac{\pi}{2} \left(1 + \frac{Z_s}{\eta_0}\right) \sum_m (\nabla \times \hat{\mathbf{z}}) J_0(k_{\rho m}\rho) \left[H_0^{(2)}(k_{\rho m}a) - H_0^{(2)}(k_{\rho m}b)\right] \left[\frac{\left(1 - \Gamma_c^{TM}\right)\left(e^{-jk_{zm}z} - \Gamma_c^{TM} e^{-2jk_{zm}h} e^{jk_{zm}z}\right)}{D^{TM'}(k_{\rho m})}\right] \quad \rho < a \quad (20)$$

$$\mathbf{H}(\boldsymbol{\rho}) = \frac{V}{\log\left(\frac{b}{a}\right)} j\omega\varepsilon \frac{\pi}{2} \left(1 + \frac{Z_s}{\eta_0}\right) \sum_m \left\{(\nabla \times \hat{\mathbf{z}}) J_0(k_{\rho m}\rho) \left[H_0^{(2)}(k_{\rho m}\rho) - H_0^{(2)}(k_{\rho m}b)\right] + (\nabla \times \hat{\mathbf{z}}) H_0^{(2)}(k_{\rho m}\rho) \left[J_0(k_{\rho m}a) - J_0(k_{\rho m}\rho)\right]\right\} \left[\frac{\left(1 - \Gamma_c^{TM}\right)\left(e^{-jk_{zm}z} - \Gamma_c^{TM} e^{-2jk_{zm}h} e^{jk_{zm}z}\right)}{D^{TM'}(k_{\rho m})}\right] \quad b > \rho > a \quad (21)$$

$$\mathbf{H}(\boldsymbol{\rho}) = \frac{V}{\log\left(\frac{b}{a}\right)} j\omega\varepsilon \frac{\pi}{2} \left(1 + \frac{Z_s}{\eta_0}\right) \sum_m (\nabla \times \hat{\mathbf{z}}) H_0^{(2)}(k_{\rho m}\rho) \left[J_0(k_{\rho m}a) - J_0(k_{\rho m}b)\right] \left[\frac{\left(1 - \Gamma_c^{TM}\right)\left(e^{-jk_{zm}z} - \Gamma_c^{TM} e^{-2jk_{zm}h} e^{jk_{zm}z}\right)}{D^{TM'}(k_{\rho m})}\right] \quad \rho > b \quad (22)$$

Notice that in expression (21) the curl operator acts on the function out of the brackets. The presence of the inner conductor of the coaxial

probe extending into the PPWG is taken into account considering the scattering by the pin when the field given in Equation (20) (or (21)), considered for $\rho = a$, is impinging. To this end on the surface of the inner conductor is imposed the following condition:

$$\hat{\boldsymbol{\rho}} \times \nabla \times (\mathbf{H_i} + \mathbf{H_s}) = -j\omega\varepsilon_r\varepsilon_0 Z_s \left(\mathbf{H_i}+\mathbf{H_s}\right)_\mathbf{t} \tag{23}$$

where the pedice t indicates tangential component respect to $\hat{\rho}$, and pedices $\boldsymbol{i}$ and $\boldsymbol{s}$ indicate incident and scattered fields. The field scattered by the pinis expressed as a sum of outgoing cylindrical functions

$$\begin{aligned}\mathbf{H_s}\left(\mathbf{r}\right) \;=\; & \sum_{n,m} A_n^m \left(\nabla \times \hat{\mathbf{z}}\right) H_n^{(2)}\left(k_{\rho m}\rho\right) e^{-jn(\phi-\phi')} \\ & \left[\frac{\left(1-\Gamma_c^{TM}\right)\left(e^{-jk_{zm}z} - \Gamma_c^{TM} e^{-2jk_{zm}h} e^{jk_{zm}z}\right)}{D^{TM'}\left(k_{\rho m}\right)}\right]\end{aligned} \tag{24}$$

where the index n indicates the sum over cylindrical harmonics. In (24) the same dependence along z of the Green's function has been taken. Coefficients A_n^m are calculated testing both sides of Equation (22) with functions F^{TM}. One finds

$$\begin{aligned}A_n^m \;=\; & -\frac{V}{\log\left(\frac{b}{a}\right)} j\omega\varepsilon \frac{\pi}{2} \left(\frac{J_n\left(k_{\rho m}a\right) + Z_m J_n'(k_{\rho m}a)}{H_n^{(2)}\left(k_{\rho m}a\right) + Z_m H_n^{(2)'}\left(k_{\rho m}a\right)}\right) \\ & \left[H_0^{(2)}\left(k_{\rho m}a\right) - H_0^{(2)}\left(k_{\rho m}b\right)\right]\left(1 + \frac{Z_s}{\eta_0}\right)\end{aligned} \tag{25}$$

where

$$Z_m = \frac{j\omega\varepsilon_r\varepsilon_0}{k_{\rho m}} Z_s \tag{26}$$

The expression of the total field radiated by the probe into the parallel plate is the sum of (24) and (21) (for $b > \rho > a$) or (22) (for $\rho > b$).

Self admittance is computed considering the reaction between the total field and the magnetic current source

$$Y_{in} = -\frac{\displaystyle\int_{S_p} \mathbf{dr}\mathbf{H}\left(\mathbf{r}\right) \cdot \mathbf{M}\left(\mathbf{r}\right)}{|V|^2} \tag{27}$$

in which S_p is the annular region occupied by the magnetic current.

Expression (27) gives

$$
\begin{aligned}
Y_{in} \;=\; & \frac{j\omega\varepsilon_r\varepsilon_0\pi^2}{\log\left(\frac{b}{a}\right)^2}\left(1+\frac{Z_s}{\eta_0}\right)^2\sum_m\left\{-\frac{2j}{\pi}\log\left(\frac{b}{a}\right)\right. \\
& +H_0^{(2)}\left(k_{\rho m}b\right)\left[J_0\left(k_{\rho m}b\right)-J_0\left(k_{\rho m}a\right)\right] \\
& -J_0(k_{\rho m}a)\left[H_0^{(2)}\left(k_{\rho m}b\right)-H_0^{(2)}\left(k_{\rho m}a\right)\right] \\
& \left.-\frac{J_0\left(k_{\rho m}a\right)+Z_mJ_0'\left(k_{\rho m}a\right)}{H_0^{(2)}\left(k_{\rho m}a\right)+Z_mH_0^{(2)'}\left(k_{\rho m}a\right)}\left[H_0^{(2)}\left(k_{\rho m}b\right)-H_0^{(2)}\left(k_{\rho m}a\right)\right]^2\right\} \\
& \frac{\left(1-\Gamma_m^{TM}\right)\left(1-\Gamma_m^{TM}e^{-2jk_{zm}h}\right)}{D^{TM'}(k_{\rho m})}
\end{aligned}
\tag{28}
$$

Input impedance is straightforwardly computed as $Z_{in} = 1/Y_{in}$.

4. NUMERICAL RESULTS

In this section, numerical results for the input impedance of the parallel plate waveguide with different conductivities are presented and discussed. A first check for the expression (28) is obtained considering the lossless case, derived for $\sigma \to \infty$ and reported in Appendix A. As can be seen, the input impedance for the lossless case coincides with the one given in [14]. Formula (28) has been implemented in MATLAB code. Figure 3, Figure 4 and Figure 5 present the plots of the input impedance as a function of the thickness of the PPWG. Results refer to the case considered in [4], in which, however, only the lossless case was taken into account. A frequency of 2 GHz with $\varepsilon_r = 2.2$ was considered as in [4]. A 50 ohm coaxial probe with inner radius $a = 0.635$ mm, outer radius $b = 2.2$ mm and filled with a dielectric with $\varepsilon_r = 2.2$ was used to feed the PPWG. Figure 3 shows the comparison of the input impedance as a function of the thickness h simulated with HFSS and obtained using expression (28) and using roots of (13) calculated with a Muller search in the complex plane. Three values of conductivity have been taken ($5e7$ S/m, $5e4$ S/m, $5e3$ S/m) in the range measured in [20]. As can be seen, results are in very good agreement. Notice that in HFSS simulations, a layered impedance condition has been used to terminate the PPWG. As it was already pointed out in [4], this condition does not provide a perfect matching, so slightly different values are obtained for different PPWG radius. Plots in Figure 3 have been obtained with a radius of 300 mm, which roughly corresponds to three wavelengths at the considered frequency. Results in Figure 3 show that for lower values of σ the impedance experiences a more significant change. This can be explained considering that the penetration depth is inversely

proportional to $\sqrt{\sigma}$. So for lower conductivities the field penetrates considerably more into the conductor. In the analysis presented in the paper this effect is taken into account by the surface impedance Z_s. In fact, Z_s is inversely proportional to $\sqrt{\sigma}$ as well, and it also changes more significantly for lower values of σ. In the previous section, formula (28) gives a rigorous expression of the input admittance of the PPWG. However, two approximations can be considered. The first one is to use the approximated expressions of the roots presented in formula (14) in place of the exact roots obtained with a numerical method; the second one is to neglect the electric current contribution. To show the effects of these approximations and to determine their range of validity, Figure 4 presents the comparison with the input impedance, given by formula (28), using residues obtained numerically and the impedance using the approximated expression of the residues in (14). As it could be expected, differences are more evident for low conductivity. The same effect is observed in Figure 5 where we compare plots obtained with formula (28), in which both magnetic and electric sources are considered, with cases where the electric current contribution is neglected, i.e., in which $Z_s = 0$. Results in Figure 5 show that for σ smaller than $5e4\,\mathrm{S/m}$ electric currents are no more negligible.

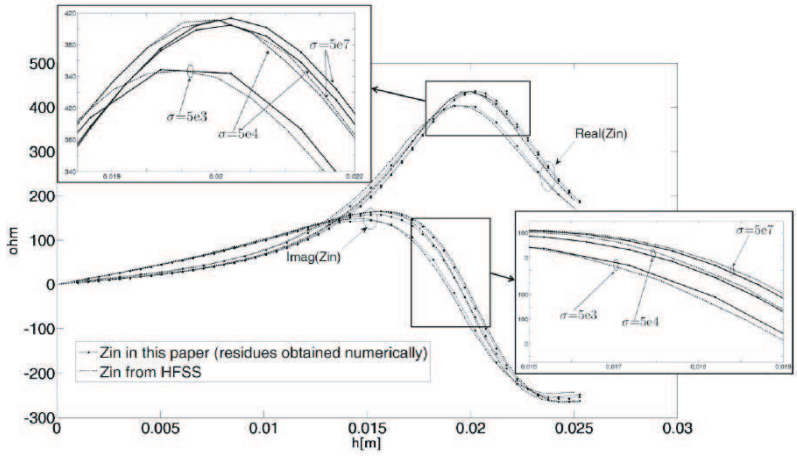


Figure 3. Plots of input impedance as a function of the PPWG thickness obtained with HFSS and expression (28). Residues are obtained solving Equation (13) numerically. Results were obtained at a frequency of 2 GHz and with $\varepsilon_r = 2.2$. Conductivity σ is in S/m.

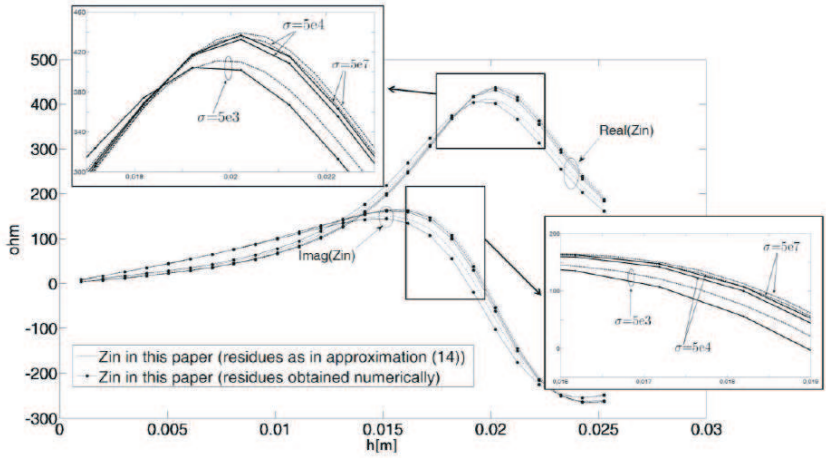


Figure 4. Plots of input impedance as a function of the thickness of the PPWG. Curves refer to approximated and numerical values of residues. Results were obtained at a frequency of 2 GHz and with $\varepsilon_r = 2.2$. Conductivity σ is in S/m.

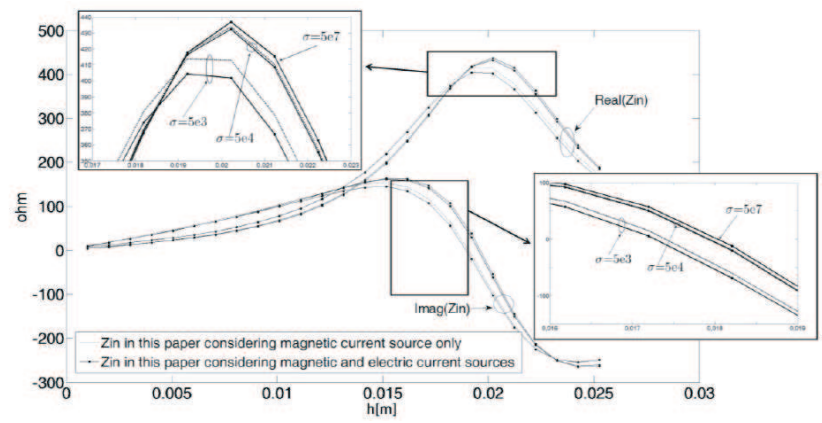


Figure 5. Plots of the input impedance as a function of the PPWG thickness. Comparison between the case with equivalent magnetic and electric current source and with magnetic current source only. Results were obtained at a frequency of 2 GHz and with $\varepsilon_r = 2.2$. Conductivity σ is in S/m.

5. CONCLUSIONS

A closed form expression for the input impedance of a coaxial probe radiating into a parallel plate waveguide has been present. At a variance of formulas known in literature, the expression takes into account both conducting and dielectric losses. The effects of neglecting the electric contribution to the equivalent source distribution have been pointed out. Also, the range of validity of a first approximation for the residues has been underlined. Results show that when conductors with a σ lower than 10^4 S/m are considered equivalent electric current have to be considered and a numerical evaluation of the residues is needed.

APPENDIX A. LOSSLESS CASE

Expression (28) is easily tested against and equivalent expression given in [14] for the lossless case considering the limit of (28) as $\sigma \to \infty$.

In particular considering that

$$\begin{aligned} &\Gamma^{TM} \to -1 \\ &k_{zm} \to \frac{m\pi}{h} \\ &k_{\rho m} \to \sqrt{k^2 - \left(\frac{m\pi}{h}\right)^2} \\ &D^{TM'}\left(k_{\rho m}\right) \to 2jk_{\rho m}^2 h \end{aligned}$$

One has

$$\begin{aligned} Y_{in} \to &-j\frac{\omega\varepsilon_r\varepsilon_0 2\pi^2}{\log\left(\frac{b}{a}\right)^2}\sum_m \frac{2j}{\pi}\log\left(\frac{b}{a}\right) \\ &+\left[J_0\left(k_{\rho m}b\right)Y_0\left(k_{\rho m}a\right) - J_0\left(k_{\rho m}a\right)Y_0\left(k_{\rho m}b\right)\right]\frac{H_0^{(2)}\left(k_{\rho m}b\right)}{H_0^{(2)}\left(k_{\rho m}a\right)} \end{aligned}$$

as was reported in [14].

REFERENCES

1. Ando, M., K. Sakurai, N. Goto, K. Arimura, and Y. Ito, "A radial line slot antenna for 12 GHz satellite TV reception," *IEEE Transactions on Antennas and Propagation*, Vol. 33, No. 12, 1347–1353, 1985.
2. Hashimoto, K., J. Hirokawa, and M. Ando, "A post-wall waveguide center-feed parallel plate slot array antenna in the

millimeter-wave band," *IEEE Transactions on Antennas and Propagation,* Vol. 58, No. 11, 3532–3538, 2010.

3. Amendola, G., E. Arnieri, L. Boccia, and V. Ziegler, "Annular ring slot radiating element for integrated millimeter wave arrays," *Proceedings of 6th European Conference on Antennas and Propagation, EuCAP 2012*, 3082–3085, 2012.
4. Hao, X., D. R. Jackson, and J. T. Williams, "Comparison of models for the probe inductance for a parallel-plate waveguide and a microstrip patch," *IEEE Transactions on Antennas and Propagation,* Vol. 53, No. 10, 3229–3235, 2005.
5. Hu, Y., Y. J. Zhang, and J. Fan, "Equivalent circuit model of coaxial probes for patch antennas," *Progress In Electromagnetics Research B*, Vol. 38, 281–296, 2012.
6. Xu, M. and T. H. Hubing, "The development of a closed-form expression for the input impedance of power-return plane structures," *IEEE Transactions on Electromagnetic Compatibility*, Vol. 45, No. 3, 478–485, 2003.
7. Arnieri, E. and G. Amendola, "Analysis of substrate integrated waveguide structures based on the parallel-plate waveguide Green's function," *IEEE Transactions on Microwave Theory and Techniques*, Vol. 56, No. 7, 1615–1623, 2008.
8. Amendola, G., E. Arnieri, and L. Boccia, "Analysis of lossy SIW structures based on the parallel plates waveguide Green's function," *Progress In Electromagnetics Research C*, Vol. 33, 157–169, 2012.
9. Arnieri, E. and G. Amendola, "Method of moments analysis of slotted substrate integrated waveguide arrays," *IEEE Transactions on Antennas and Propagation*, Vol. 59, No. 4, 1148–1154, 2011.
10. Amendola, G., G. Angiulli, E. Arnieri, and L. Boccia, "Resonant frequencies of circular substrate integrated resonators," *IEEE Microwave and Wireless Components Letters*, Vol. 18, No. 4, 239–241, 2008.
11. Amendola, G., G. Angiulli, E. Arnieri, and L. Boccia, "Computation of the resonant frequency and quality factor of lossy substrate integrated waveguide resonators by method of moments," *Progress In Electromagnetics Research Letters*, Vol. 40, 107–117, 2013
12. Angiulli, G., "Design of square substrate waveguide cavity resonators: Compensation of modelling errors by support vector regression machines," *American Journal of Applied Sciences*, Vol. 9, No. 11, 1872–1875, 2012.

13. Angiulli, G., "On the computation of nonlinear eigenvalues in electromagnetics problems," *Journal of Electromagnetic Waves and Applications*, Vol. 21, No. 4, 527–532, 2007.
14. Zheng, J.-X. and D. C. Chang, "End-correction network of a coaxial probe for microstrip patch antennas," *IEEE Transactions on Antennas and Propagation*, Vol. 39, No. 1, 115–118, 1991.
15. Williamson, A. G., "Equivalent circuit for radial-line/coaxial-line junction," *IET Electronics Letters*, Vol. 17, No. 8, 300–301, 1981.
16. Vandenbosch, G. A. E. and A. R. Van de Capelle, "Reduction of coaxial feed between two parallel conductors into finite number of voltage sources," *IET Electronics Letters*, Vol. 27, No. 25, 2387–2389, 1991.
17. Tomasic, B. and A. Hessel, "Electric and magnetic current sources in the parallel plate waveguide," *IEEE Transactions on Antennas and Propagation,* Vol. 35, No. 11, 1307–1310, 1987.
18. Wu, D. and C. Ruan, "Analysis on a radial-line/coaxial-line junction," *IET Electronics Letters*, Vol. 34, No. 10, 9944–996, 1998.
19. Huang, C., L. Tsang, and C. H. Chan, "Multiple scattering among vias in lossy planar waveguides using SMCG method," *IEEE Transaction on Advanced Packaging*, Vol. 25, No. 2, 181–188, 2002.
20. Otoshi, T. Y. and M. M. Franco, "The electrical conductivities of steel and other candidate materials for shrouds in a beam-waveguide antenna system," *IEEE Transaction on Instrumentation and Measurement*, Vol. 45, No. 1, 77–83, 1996.
21. Yoshitomi, K., "Equivalent currents for an aperture in an impedance surface," *IEEE Transaction on Antennas and Propagation*, Vol. 42, No. 11, 1554–1556, 1994.