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A LEAST SQUARES FINITE ELEMENT METHOD FOR THE EXTENDED MAXWELL SYSTEM

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Abstract—A finite element method based on the first order system LL^* (FOSLL^*) approach is derived for time harmonic Maxwell's equations in three dimensional domains. The finite element solution is a potential for the original field in a sense that the original field U is given by $U = L^*u$. The Maxwellian boundary data appears as natural boundary condition. Homogeneous Dirichlet boundary conditions for the potential must be imposed, and they are circumvented with weak enforcement of boundary conditions and it is proved that the sesquilinear form of the finite element system is elliptic in the space where the Dirichlet boundary conditions are satisfied weakly.

1. INTRODUCTION

The time-harmonic Maxwell's equations posed on normalized fields are given by

$$\begin{cases} \nabla \times \mathbf{E} - ik\mathbf{H} = 0 \\ -\nabla \times \mathbf{H} - ik\mathbf{E} = -\mathbf{J} \\ \nabla \cdot \mathbf{H} = 0 \\ \nabla \cdot \mathbf{E} = \rho \end{cases} \tag{1}$$

together with the boundary conditions

$$\mathbf{n} \times \mathbf{E} = -\mathbf{M}_s \quad \text{or} \quad \mathbf{n} \times \mathbf{H} = \mathbf{J}_s. \tag{2}$$

The normalized fields $\mathbf{E}$ and $\mathbf{H}$ relate to the physical fields $\mathcal{E}$ and $\mathcal{H}$ by $\mathbf{E} \sim \sqrt{\epsilon}\mathcal{E}$, $\mathbf{H} \sim \sqrt{\mu}\mathcal{H}$, where ϵ and μ are the permittivity and permeability of the medium, resp.

There are countless finite element (FE) formulations to deal with this problem (c.f. [1] and references therein), most prominent ones being derived from the curl-curl equation $\nabla \times \nabla \times \mathbf{E} - k^2\mathbf{E} = ik\mathbf{J}$.

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Discretizing the curl-curl equation above with the Galerkin's method requires specialized basis functions, usually the Nédélec's curl conforming functions, which are hard to come by in high orders, the arising system of linear equations is never positive-definite when k is above the lowest resonance number of the cavity and, moreover, the system becomes increasingly ill-conditioned when the frequency tends to zero. It should be also noted that the analysis of the method is rather involved [1].

Another way to discretize (1) is by viewing it as a first order PDE system and constructing a least squares functional whose minimizer solves (1). This approach leads to the class of first order system least squares (FOSLS) methods [2–5].

In the FOSLS methods, a given operator equation

$$Lu = f \tag{3}$$

is posed as a minimization problem

$$\min_u \|Lu - f\|^2 \tag{4}$$

in some Hilbert-space norm $\|\cdot\|$. Thus, the corresponding discrete system is hermitian and positive-semidefinite: If the space is finite dimensional, the minimization problem is equivalent with the normal equations $L^*Lu = L^*f$, where L^* is the adjoint of L, i.e., the hermitian transpose if L is a matrix and the norm is the Euclidean one.

However, the norm $\|\cdot\|$ above must be chosen such a way that it is computable and, that the resulting discrete method is convergent [2, 4]. The simplest choice is, of course, the L_2-norm, but negative Sobolev norms come in to question as well [6].

In [7], Cai et al. introduce an operator dependent norm which leads to a quadratic functional given by $\mathcal{J}(u) = \|L^*u^* - u\|_{L_2}^2$, here L^* is the L_2-adjoint of L. This method is coined as the first order system LL^*, or FOSLL^*, method since the corresponding finite dimensional system would be $LL^*u^* = f$.

In this paper, a FOSLL^* formulation for (1) is constructed and analyzed. We note that similar augmentation, as done in [8], of the adjoint operator to an elliptic one (in the sense of Petrovsky [9]), results in exactly the same extended Maxwell's system that Picard studied in [10, 11]. Thus, it is expected that the discrete system is stable at the zero-frequency limit.

Our method is almost identical to what Lee analyzed in the eddy current situation [8]. However, in the time-harmonic case, the stability of the finite element method rests on a spectral argument and, furthermore, we weaken certain homogeneous Dirichlet boundary conditions required by the method in the spirit of Lagrange multipliers as was proposed by Babuška in [12] by projecting away the part of the solution which does not satisfy the weakened Dirichlet conditions. Finally, we pose a non-homogeneous boundary condition for the original field.

The projection is implemented using the nullspace projection in the PETSc [13] Krylov subspace solver. In fact, because the method is of least-squares type, we are able to use conjugate gradient method as long we stay within the Krylov iteration in the subspace in which the weakened Dirichlet conditions are satisfied.

We need to discretize the system with nodal Lagrange H^1 basis functions which are very well known for arbitrary orders. This, however, leads to problems in domains with re-entrant corners [8, 14], where the H^1 elements fail to approximate the solution. However, in the paper [8], Lee obtains a convergent method with weighted Sobolev norms and H^1 elements.

The paper is organized as follows: First we will lay definitions and pose the extended Maxwell system. Following that we formulate the finite element system postponing most of the analysis to Appendix A. Next, we demonstrate the method on simple boundary value problems and, finally, we conclude the paper with discussion.

2. FORMULATION AND NOTATION

Let us take $\Omega \subset \mathbb{R}^3$ to be a convex polyhedral domain, denote its boundary by $\Gamma = \partial\Omega$. The space of square integrable complex valued functions on Ω are denoted by $L_2(\Omega)$, or by L_2 if there is no possibility of confusion on the domain. The norm of L_2 is given by

$$\|f\| := \sqrt{\int_\Omega |f(x)|^2 dx}.$$

The base Hilbert space we shall operate in is

$$\mathcal{H} := (L_2(\Omega))^8. \tag{5}$$

The norm of $\mathcal{H}$ is, likewise, denoted by $\|\cdot\|$: Let $F = (F_1 \dots F_8) \in \mathcal{H}$, then $\|F\| = \sqrt{\sum_{i=1}^8 \|F_i\|^2}$. We denote the complex conjugate of x by $\overline{x}$ and the inner product of two elements $E, F \in \mathcal{H}$ by $(E, F) = \int_\Omega \sum_i E_i(x)\overline{F_i(x)}dx$.

The extended Maxwell system corresponding to (1) is given [10, 15, 16] by

$$\left(\begin{bmatrix} 0 & 0 & \nabla\cdot & 0 \\ 0 & 0 & -\nabla\times & \nabla \\ \nabla & \nabla\times & 0 & 0 \\ 0 & \nabla\cdot & 0 & 0 \end{bmatrix} - ik \right) \begin{bmatrix} \Phi \\ \mathbf{E} \\ \mathbf{H} \\ \Psi \end{bmatrix} = \begin{bmatrix} 0 \\ -\mathbf{J} \\ 0 \\ \rho \end{bmatrix} \text{ in } \Omega. \tag{6}$$

Throughout the paper, we shall denote the matrix comprising of derivatives by $\mathcal{P}$, the unknown vector by U and the right hand size by F. Furthermore, we denote

$$L = \mathcal{P} - ik. \tag{7}$$

The trace operators corresponding to (2) with which the Dirichlet boundary data is prescribed are given by

$$\mathcal{B}_E U = \begin{bmatrix} -\mathbf{n}\cdot\mathbf{H} \\ -\Psi\mathbf{n} \\ -\mathbf{n}\times\mathbf{E} \\ 0 \end{bmatrix} \quad \text{or} \quad \mathcal{B}_M U = \begin{bmatrix} 0 \\ \mathbf{n}\times\mathbf{H} \\ -\Phi\mathbf{n} \\ -\mathbf{n}\cdot\mathbf{E} \end{bmatrix}. \tag{8}$$

We call these electric and, respectively, magnetic boundary operators. Corresponding to these operators we define two subspaces $\mathcal{D}_E$ and $\mathcal{D}_M$ coined the perfect electric conductor (PEC) subspace and the perfect magnetic conductor (PMC) subspace, respectively. They are defined as Cartesian products of Sobolev spaces by

$$\mathcal{D}_E(\Omega) \;=\; H^1(\Omega)/\mathbb{C} \times H_{D\mathring{C}}(\Omega) \times H_{\mathring{D}C}(\Omega) \times H^1_0(\Omega) \tag{9}$$

$$\mathcal{D}_M(\Omega) \;=\; H^1_0(\Omega) \times H_{\mathring{D}C}(\Omega) \times H_{D\mathring{C}}(\Omega) \times H^1(\Omega)/\mathbb{C}. \tag{10}$$

These spaces both come equipped with a norm $\|U\|_{\mathcal{D}}$ defined by

$$\|U\|^2_{\mathcal{D}} = \|U\|^2 + \|\mathcal{P}U\|^2. \tag{11}$$

We employ stand-in notation $\mathcal{B}_R$ and $\mathcal{D}_R$ with $R = E$ or $R = M$ since the theory is completely symmetric for both cases.

The Sobolev spaces $H_{D\mathring{C}}$ and $H_{\mathring{D}C}$ appearing above are defined by

$$\begin{cases} H_{\mathring{D}C} = H_0(\nabla\cdot) \cap H(\nabla\times) \quad \text{and} \\ H_{D\mathring{C}} = H(\nabla\cdot) \cap H_0(\nabla\times). \end{cases}$$

here $H_0(\nabla\cdot)$ consists of square integrable functions having an L_2 divergence and vanishing normal trace and $H_0(\nabla\times)$ consists of, likewise, $(L_2)^3$ vector fields with square integrable curl and vanishing tangential trace. These spaces without the subscript 0 refer to just spaces of square integrable vector fields with square integrable divergence or curl, respectively. The space H^1 is the usual subspace of L_2 whose functions have square integrable gradient and if $f \in H^1_0$, then the scalar trace of f on the boundary is zero. Finally, $H^1/\mathbb{C}$ is the space of H^1-functions having zero mean value.

The norm in $H_{\mathring{D}C}$ and $H_{D\mathring{C}}$ is given by $\|\boldsymbol{F}\|^2_V = \|\boldsymbol{F}\| + \|\nabla\times\boldsymbol{F}\|^2 + \|\nabla\cdot\boldsymbol{F}\|^2$, with V being either $H_{\mathring{D}C}$ or $H_{D\mathring{C}}$. With this convention it holds that the norm $\|\cdot\|_{\mathcal{D}}$ is equivalent with the graph-norm of $\mathcal{D}_R$, i.e.,

$$\begin{aligned}\|U\|_{\mathcal{D}}^2 = & \|U\|^2+\|\nabla\Phi\|^2+\|\nabla\Psi\|^2 \\ & +\|\nabla\times\mathbf{E}\|^2+\|\nabla\cdot\mathbf{E}\|^2+\|\nabla\times\mathbf{H}\|^2+\|\nabla\cdot\mathbf{H}\|^2. \quad (12)\end{aligned}$$

For sufficiently smooth functions we can relate the boundary trace operator

$$P(\mathbf{n}) = \begin{bmatrix} 0 & 0 & \mathbf{n}\cdot & 0 \\ 0 & 0 & -\mathbf{n}\times & n \\ \mathbf{n} & \mathbf{n}\times & 0 & 0 \\ 0 & \mathbf{n}\cdot & 0 & 0 \end{bmatrix} \quad (13)$$

with an integration by parts formula [15] for $\mathcal{P}$ by

$$\int_\Omega \mathcal{P}U\cdot\bar{V}+U\cdot\overline{\mathcal{P}V}dx = \int_\Gamma P(\mathbf{n})U\cdot\bar{V}d\sigma. \quad (14)$$

It holds that $-P(\mathbf{n}) = \mathcal{B}_E+\mathcal{B}_M$.

Thus for the Dirichlet data (8) we have the following integration by parts formulas

$$\begin{aligned} & -\langle\mathcal{B}_E U, [\varphi\ \mathbf{e}\ \mathbf{h}\ 0]\rangle \\ = & (\nabla\cdot\mathbf{H},\varphi)+(\mathbf{H},\nabla\varphi)+(\nabla\times\mathbf{E},\mathbf{h})-(\mathbf{E},\nabla\times\mathbf{h}) \\ & +(\nabla\Psi,\mathbf{e})+(\Psi,\nabla\cdot\mathbf{e}), \quad \forall(\phi,\mathbf{e},\mathbf{h})\in H^1\times H(\nabla\times)\times H(\nabla\cdot) \quad (15) \\ & -\langle\mathcal{B}_M U, [0\ \mathbf{e}\ \mathbf{h}\ \psi]\rangle \\ = & (-\nabla\times\mathbf{H},\mathbf{e})+(\mathbf{H},\nabla\times\mathbf{e})+(\nabla\Phi,\mathbf{e})+(\Phi,\nabla\cdot\mathbf{e}) \\ & +(\nabla\cdot\mathbf{E},\psi)+(\mathbf{E},\nabla\psi), \quad \forall(\mathbf{e},\mathbf{h},\psi)\in H(\nabla\cdot)\times H(\nabla\times)\times H^1. \quad (16)\end{aligned}$$

Since the domain Ω is convex it holds that [17] there exist constants c, $C>0$ such that

$$c\|u\|_{\mathcal{D}} \le \|u\|_{H^1} \le C\|u\|_{\mathcal{D}}. \quad (17)$$

We shall freely use this fact throughout the paper.

We shall also denote generic positive nonzero coefficients by c, C. Their values may change in the middle of a calculation, but they depend only on the domain and the wavenumber k.

It holds that $L:\mathcal{D}_R\to\mathcal{H}$ is a bounded operator. Furthermore, the unbounded operator $\mathcal{P}:\mathcal{D}_R\subset\mathcal{H}\to\mathcal{H}$ is *skew-self-adjoint* [16], meaning that $\mathcal{D}(\mathcal{P}^*)=\mathcal{D}(\mathcal{P})$ and $\mathcal{P}^*=-\mathcal{P}$. This translates directly to skew-self-adjointness of $L:\mathcal{D}_R\subset\mathcal{H}\to\mathcal{H}$, i.e.,

$$L^* = -P+ik. \quad (18)$$

2.1. Least Squares Finite Element Method

Instead of directly discretizing (6) we look for its *dual-potential* $u^* \in \mathcal{D}_R$ as a minimization problem

$$u^* = \arg\min_{u \in \mathcal{D}_R} \frac{1}{2}\|L^* u\|^2 - \operatorname{Re} l(u), \tag{19}$$

where

$$l(u) = \int_\Gamma \mathcal{B}_R U \cdot \bar{u} d\sigma - \int_\Omega F \cdot \bar{u} dx. \tag{20}$$

Note that

$$l(u) = (U, L^* u) \tag{21}$$

by integration by parts.

This approach is reversed to the usual one [7] where the u^* is the minimizer of $\|L^* u^* - U\|^2$, but if $U \in L_2$ we have

$$\begin{aligned} \arg\min_u \frac{1}{2}\|L^* u\|^2 - \operatorname{Re} l(u) &= \arg\min_u \frac{1}{2}\|L^* u\|^2 - \operatorname{Re} l(u) + \frac{1}{2}\|U\|^2 \\ &= \arg\min_u \frac{1}{2}\|L^* u - U\|^2. \end{aligned} \tag{22}$$

Thus $U = L^* u^*$.

The corresponding variational formulation for (19) is

$$\begin{aligned} \text{Find } u^* &\in \mathcal{D}_R, \text{ s.t.} \\ (L^* u^*, L^* v) &= l(v) \quad \forall v \in \mathcal{D}_R. \end{aligned} \tag{23}$$

We denote the sesquilinear form† appearing above by

$$a(u, v) = (L^* u, L^* v). \tag{24}$$

Clearly, if $L^* : \mathcal{D}_R \to \mathcal{H}$ has a bounded inverse it holds that a is $\mathcal{D}_R$-elliptic and, furthermore, if l is a bounded linear functional on $\mathcal{D}_R$, the minimizer of (19) exists, is unique and depends continuously on l. Furthermore if $\mathcal{D}_R^h$ is a finite dimensional subspace of $\mathcal{D}_R$ then we have, by Cea's lemma (see e.g., [18] or any other textbook on finite elements), the quasi-optimality

$$\|u_h^* - u^*\|_{\mathcal{D}} \le \frac{C}{\alpha} \inf_{v \in \mathcal{D}_R^h} \|v - u^*\|_{\mathcal{D}}, \tag{25}$$

where C and α are the continuity and ellipticity coefficients of a, respectively. Using this result and the boundedness of we get $L : \mathcal{D}_R \to \mathcal{H}$

$$\|L^* u_h^* - U\| = \|L^* u_h^* - L^* u^*\| \le c\|u_h^* - u^*\|_{\mathcal{D}} \le \frac{cC}{\alpha} \inf_{v \in \mathcal{D}_R^h} \|v - u^*\|_{\mathcal{D}}.$$

† a linear in first argument and conjugate linear in second argument

2.2. Discretization

We wish to discretize (23) with nodal H^1 conforming Lagrange elements, but constructing $\mathcal{D}_R$ conformal bases with such elements is complicated if the domain is not a rectangular one. More specifically, the spaces $H_{\overset{\circ}{DC}}$ and $H_{D\overset{\circ}{C}}$ pose problems as conformity in them translates in vanishing of normal or tangential component which in turn is difficult to accomplish with nodal vector elements. Furthermore, for first order elements, vanishing tangential or normal trace in, e.g., spherical domain implies that all components of the field vanish on the boundary.

As a remedy, in non-rectangular domains, we impose the boundary conditions for the dual potential in a discrete weak sense similarly as in [12]. However, instead of introducing Lagrange multipliers we restrict to the space where the boundary condition is satisfied weakly using the nullspace methods in PETSc Krylov space solver [13].

Suppose that Ω admits partition $\mathcal{T}$ by shape regular tetrahedrons. We denote the maximum circumference of the tetrahedrons by h. Let us denote the nodal Lagrange H^1 conforming finite element space consisting of piecewise polynomials of nth degree by S_h and the curl conforming nth order Nédélec space [19, 20] by N_h. Furthermore, let us define discrete boundary bilinear forms $b_R : (S_h)^8 \times S_h \times S_h \times N_h \to \mathbb{C}$ by

$$b_E(u, (\mu, \tau, \boldsymbol{\lambda})) : = \int_\Gamma \mathcal{B}_E \mathbf{u} \cdot [\mu \ \mathbf{n}\tau \ \boldsymbol{\lambda} \ 0]^T d\sigma, \tag{26}$$

$$b_M(u, (\mu, \tau, \boldsymbol{\lambda})) : = \int_\Gamma \mathcal{B}_E \mathbf{u} \cdot [0 \ \boldsymbol{\lambda} \ \mathbf{n}\tau \ \mu]^T d\sigma \tag{27}$$

The discrete spaces from which we look for the solution are now given by

$$V_h^R := \left\{ \mathbf{u} \in (S_h)^8 : b_R(\mathbf{u}, (\mu, \tau, \boldsymbol{\lambda})) = 0 \ \ \forall (\mu, \tau, \boldsymbol{\lambda}) \in S_h \times S_h \times N_h \right\}. \tag{28}$$

These are not necessarily subspaces of $\mathcal{D}_E$ and $\mathcal{D}_M$, respectively.

The main result of the paper, however, is that a is V_h^R-elliptic, i.e., there is a constant $\alpha > 0$ such that

$$\alpha \|u\|_{\mathcal{D}}^2 \leq a(u, u), \quad \forall u \in V_h^R. \tag{29}$$

The proof is provided in the Appendix A.

Thus, the discrete form of (23) for rectangular regions is given by

$$\begin{aligned} &\text{Find } u^* \in \mathcal{D}_R \cap (S_h)^8, \text{ s.t.} \\ &(L^* u^*, L^* v) = l(v) \quad \forall v \in \mathcal{D}_R \cap (S_h)^8, \end{aligned} \tag{30}$$

and for non-rectangular regions by

$$u^* = \arg \min_{u \in V_h^R} \frac{1}{2} \|L^* u\|^2 - \mathrm{Re}\ l(u). \tag{31}$$

3. NUMERICAL APPLICATIONS AND EXAMPLES

In this section, we shall present examples how the formulated FE system can be used. We will inspect how to recover an oblique plane-wave and, as a second example, we construct the perhaps simplest hybrid method to compute the reflection coefficient in a rectangular waveguide terminated by a PEC wall.

Example 1. In this example, we choose the electric boundary data to be that of a plane wave traveling in the direction $\phi = \frac{\pi}{4}$ (angle from x axis) and $\theta = \frac{\pi}{6}$ (angle from z axis) and we let the wave-number k vary.

In Fig. 1 we show the real parts of the electric and magnetic field computed with the discrete dual potential being in V_h^E. We used a single tetrahedron with 5th order polynomials as the FE basis.

The solution was forced to be in the V_h^E basis by computing the singular value decomposition of the matrix arising from the form b_E and giving the PETSc [13] conjugate gradient Krylov subspace solver those right singular vectors that correspond to singular values bigger than 10^{-10}. This tolerance can be chosen quite freely since the singular values are grouped in two clearly separated groups. The singular values are plotted in Fig. 2.

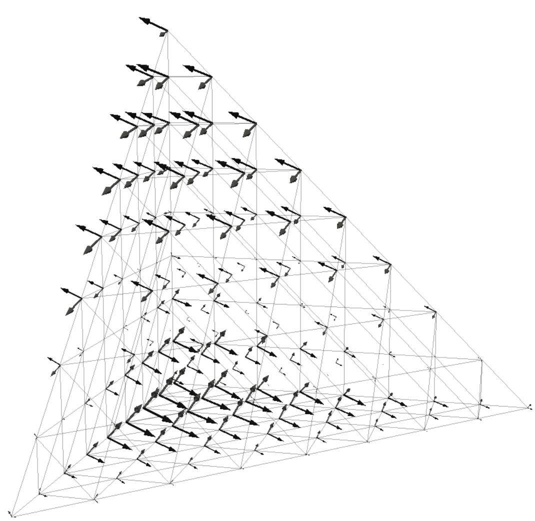

Figure 1. Real parts of electric field (dark gray arrow) and magnetic field (light gray arrow) in tetrahedron discretized with one 5th order element and interpolated to mesh in figure.

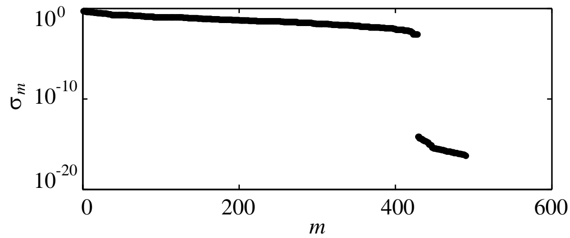

Figure 2. Singular values of the matrix corresponding to b_E on 5th order basis functions on tetrahedron.

It turns out that a is guaranteed to be $\mathcal{D}_R$-elliptic whenever $(ik)^{-1}$ is not in a spectrum of a compact operator $\mathcal{P}^{-1} : \mathcal{H} \to \mathcal{H}$. However, in a cubical domain we observe drop in the energy of the field solution at interior resonance frequencies (Fig. 3). Because the extended Maxwell system encodes both acoustic and electromagnetic equations [16] the system should have interior resonances at $k = \sqrt{m^2 + n^2 + l^2}\pi$, $m, n, l \in \mathbb{N}$.

Example 2. Next we compute the reflection coefficient from a rectangular waveguide which is terminated by a PEC wall and whose width is $a = 10\,\mathrm{mm}$ and height $b = 5\,\mathrm{mm}$. The length of the computational domain in z direction is $c = 20\,\mathrm{mm}$.

The primary field U_p traveling in the z direction is the TE_{10} mode given by ($k_c = \pi/a$, $\beta = \sqrt{k^2 - k_c^2}$)

$$\begin{cases} H_{z,p}(x,z) = \cos(k_c x)e^{i\beta z} \\ \mathbf{H}_{t,p}(x,z) = \dfrac{i\beta}{k_c}\mathbf{u}_x \sin(k_c x)e^{i\beta z} \\ \mathbf{E}_{t,p}(x,z) = \mathbf{u}_y \dfrac{ik}{k_c}\sin(k_c x)e^{i\beta z}. \end{cases} \tag{32}$$

The reflected field is given by

$$\begin{cases} H_{z,s}(x,z) = \rho\cos(k_c x)e^{-i\beta z} = \rho e^{-i2\beta z} H_{z,p}(x,z) \\ \mathbf{H}_{t,s}(x,z) = -\rho\dfrac{i\beta}{k_c}\mathbf{u}_x \sin(k_c x)e^{-i\beta z} = -\rho e^{-i2\beta z}\mathbf{H}_{t,p}(x,z) \\ \mathbf{E}_{t,s}(x,z) = \rho\dfrac{ik}{k_c}\mathbf{u}_y \sin(k_c x)e^{-i\beta z} = \rho e^{-i2\beta z}\mathbf{E}_{t,p}(x,z), \end{cases} \tag{33}$$

where ρ is an unknown reflection coefficient. We denote the computational domain by Ω, the input port at $z = 0$ by Γ and the field inside Ω by U_i.

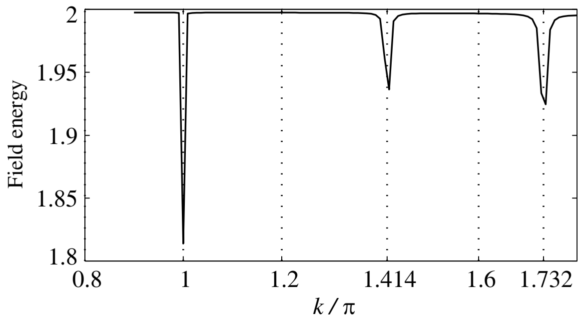

Figure 3. Energy of the FE solution against k/π. Near 1, $\sqrt{2}$ and $\sqrt{3}$ there is a strong deviation from the analytical value of 2. The direction of propagation of the plane wave is $\phi = \pi/4$ and $\theta = \pi/6$.

The interior field U_i satisfies the jump conditions

$$\mathcal{B}^s_R U_s + \mathcal{B}^i_R U_i = -\mathcal{B}^s_R U_p, \tag{34}$$

where $\mathcal{B}^i_R$ is the boundary trace operator from the computational domain to Γ and $\mathcal{B}^s_R$ from the scattering region to Γ and U_s is the reflected field and U_p is the impinging primary field. Note that at $\Gamma \mathcal{B}^s_R = -\mathcal{B}^i_R$. Now it holds that $\mathcal{B}^s_E U_1 = \rho e^{-i2\beta z}\mathcal{B}^s_E U_p$ and $\mathcal{B}^s_M U_s = -\rho e^{-i2\beta z}\mathcal{B}^1_M U_p$. Thus we arrive to the following set of equations.

$$\begin{aligned} \text{Find } u &\in \mathcal{D}_E, \text{ s.t.} \\ \langle L^* u, L^* v\rangle + \langle \rho e^{-i2\beta z}\mathcal{B}^s_E U_p, v\rangle &= -\langle \mathcal{B}^s_E U_p, v\rangle \quad \forall v \in \mathcal{D}_E \quad (35) \\ \mathcal{B}_M L^* u - \rho e^{-i2\beta z}\mathcal{B}^s_M U_p &= -\mathcal{B}^s_M U_p. \quad (36) \end{aligned}$$

We test the Equation (36) with one sine function and we arrive to following

$$\text{Find } u \in \mathcal{D}_E, \text{ s.t.}$$

$$\langle L^* u, L^* v\rangle + \langle \rho e^{-i2\beta z}\mathcal{B}^s_E U_p, v\rangle = -\langle \mathcal{B}^s_E U_p, v\rangle \ \forall v \in \mathcal{D}_E \tag{37}$$

$$\left(\mathcal{B}_M L^* u, \begin{bmatrix} 0 \\ \mathbf{u}_y \sin k_c x \\ 0 \\ 0 \end{bmatrix}\right)_{L_2(\Gamma)} - \rho \frac{ab}{2}\frac{i\beta}{k_c}e^{-i\beta z} = -\frac{ab}{2}\frac{i\beta}{k_c}e^{i\beta z} \tag{38}$$

The real part of the y component of the total electric field is shown in Fig. 4. The error of ρ compared to the analytical value is 9% (19106 DoFs) or 5% (16602 DoFs, refined mesh at port). The error of $|\rho|$ is of order 10^{-8}. In this calculation second order elements were used.

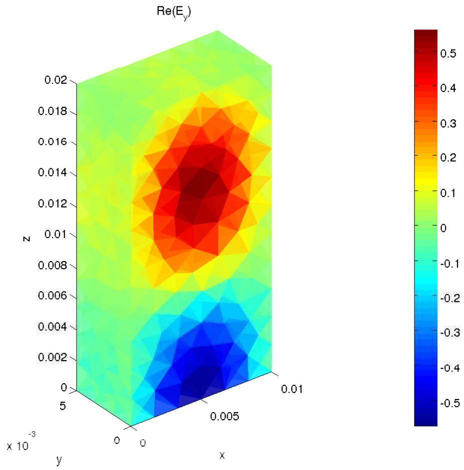

Figure 4. Real part of the total electric field inside the waveguide in Example 2.

4. DISCUSSION

We have constructed a symmetric finite element method based on the first order least squares LL^* approach for time harmonic extended Maxwell's equations. This allows us to use the conjugate gradient method to solve problems at frequencies where the usual curl-curl equation would be indefinite even with Hodge decompositions.

The Maxwellian boundary data is implemented in the system as a natural boundary condition on the right hand side of the equation. However, on the dual potential side, we must enforce boundary conditions in a discrete weak sense in order to handle more general geometries than rectangular ones. Thus, the constructed finite element method is non-conformal in a sense that the boundary conditions are not satisfied exactly.

It should be noted that the approximation in the Lemma 1 is probably not optimal as it only takes into account first order polynomials. Especially, in the Example 1, we obtained quite a good result with 5th order basis functions even though h is about 1. The convergence of the Krylov subspace solvers were not particularly studied since the examples were somewhat small. However, it took around 300 iterations for the conjugate gradient algorithm to reach relative error of 10^{-8} in the Example 1. In the Example 2 a direct solver was used.

The appearance of the div-curl in the formulation is dealt using nodal Lagrange H^1 elements which are readily available for very high orders, but on the other hand, they lead to convergence issues in non-convex non-smooth domains [8, 14]. However, the proposed method seems to be very suitable for domain decomposition methods, since we can always decompose any domain into convex subdomains and all communication between domains are handled through natural boundary conditions.

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APPENDIX A. ANALYSIS

The main result of this section, and the whole paper, is that a is V_R^h-elliptic and bounded. We prove it by first showing that a is $\mathcal{D}_R$ elliptic with existence of bounded inverse for $L^* : \mathcal{D}_R \to \mathcal{H}$. Following that, we establish an a-priori approximation estimate concerning the weak imposition of the Dirichlet boundary condition on the dual potential.

The existence of a bounded inverse for $L^* : \mathcal{D}_R \to \mathcal{H}$ implies $\mathcal{D}_R$-ellipticity of the sesquilinear form a:

Theorem 1. *The bilinear form $a : (u, v) \mapsto (L^*u, L^*v)$ is bounded and $\mathcal{D}_R$ elliptic save for countably many resonance numbers k.*

Proof. The boundedness is trivial since L^* and L are bounded as linear operators $\mathcal{D}_R \to \mathcal{H}$.

It holds that $\mathcal{P} : \mathcal{D}_R \to \mathcal{H}$ satisfies

$$c\|u\|_{\mathcal{D}} \le \|\mathcal{P}u\| \le C\|u\|_{\mathcal{D}},$$

for some $c, C > 0$ [10, 16]. Furthermore, the embedding $\mathcal{D}_R \hookrightarrow \mathcal{H}$ is compact by Maxwell compactness principle [21] and Rellich's selection principle [16, 18]. Thus, it holds that $\mathcal{P}^{-1} : \mathcal{H} \to \mathcal{H}$ is a compact operator and its spectrum $\Lambda(\mathcal{P}^{-1})$ is at most countable set having zero as the only point of accumulation. Thus, $\mathcal{P} - ik : \mathcal{D}_R \to \mathcal{H}$ has a bounded inverse whenever $(ik)^{-1} \notin \Lambda(\mathcal{P}^{-1})$.

The existence of the bounded inverse implies that there is $c_k > 0$ independent of $u \in \mathcal{D}_R$ s.t.

$$c_k\|u\|_{\mathcal{D}} \le \|L^*u\|,$$

thus c_k^2 suits as the ellipticity constant.

Near the resonance frequencies, the ellipticity constant is bounded above by $|1 - k/\lambda|^2/(\lambda^{-2} + 1)$, where $(i\lambda)^{-1} \in \Lambda(\mathcal{P}^{-1})$ minimizes $|1 - k/\lambda|/\sqrt{\lambda^{-2} + 1}$. This can be seen by taking v_λ to be the eigenvector corresponding to $(i\lambda)^{-1}$: $\|L^*\mathcal{P}^{-1}v_\lambda\| = \|L^*(i\lambda)^{-1}v_\lambda\| = \|(k/\lambda - 1)v_\lambda\| = |1 - k/\lambda|$, but $\|\mathcal{P}^{-1}v_\lambda\|_{\mathcal{D}} = \sqrt{\lambda^{-2}+1}$.

Let us now inspect V_h^R ellipticity of a by proving a controlling estimate for the size of the part of $u \in V_h^R$ which is not in $\mathcal{D}_R$:

Lemma 1. *Let $u \in V_h^E$ and let $\tilde{u} \in H^1$ be such that $L^*\tilde{u} = 0$ and $\mathcal{B}_E u = \mathcal{B}_E \tilde{u}$. Furthermore, let $v = [f\ \varepsilon\ \chi\ p] \in \mathcal{D}_E$ be the dual potential of $\tilde{u}$, i.e., $\tilde{u} = L^*v$.*

Then there is a constant $c > 0$ depending only on the geometry of Ω and the frequency such that

$$\|L^*v\|^2 \leq ch\|u\|_{\mathcal{D}}^2. \tag{A1}$$

Proof. Since v is the dual potential of $\tilde{u}$ and the vector field parts of u satisfy the discrete weak normal and tangential boundary conditions we can apply the approximation of S_h in H^1 and N_h in $H(\nabla\times)$ ([1] Remark 5.41). We make use of a Smith-Aronszajn inequality [22] $|u|_2^2 \leq c\left(\|\Delta u\|^2 + \|u\|^2\right)$.

Since v is the dual potential of $\tilde{u}$, by applying $-\mathcal{P}$ and $-ik$ on both sides of the equation $(-\mathcal{P} + ik)v = \tilde{u}$ and summing we get that $\Delta v = k^2 v$. Now

$$\begin{aligned}
|(L^*v, L^*v)| &= |\langle \mathcal{B}_E\tilde{u}, v\rangle| = |\langle \mathcal{B}_E u, v\rangle| \\
&= |\langle \mathbf{n}\cdot\mathbf{h}, f\rangle + \langle \mathbf{n}\times\varepsilon, \chi\rangle| \\
&= \inf_{(f_h,\lambda)\in S_h\times N_h} |\langle \mathbf{n}\cdot\mathbf{h}, f - f_h\rangle - \langle \mathbf{n}\times\mathbf{e}, \chi - \lambda\rangle| \\
&\leq \|u\|_{\mathcal{D}}\left(\inf_{f_h,\lambda}\|f - f_h\|_{H^1} - \|\chi - \lambda\|_{H(\nabla\times)}\right) \\
&\leq \|u\|_{\mathcal{D}}\left(C_1 h|f|_2 + C_2 h\|\nabla\times\chi\|_{H^1}\right) \\
&\leq \|u\|_{\mathcal{D}} Ch\left(|f|_2 + |\chi|_1 + |\chi|_2\right) \\
&\leq \|u\|_{\mathcal{D}} Ch\left(|v|_2 + \|v\|_{\mathcal{D}}\right) \\
&\leq \|u\|_{\mathcal{D}} Ch\left(\|v\|_{\mathcal{D}} + \sqrt{1+|k|^2}\|v\|\right) \\
&\leq \|u\|_{\mathcal{D}} Ch|v|_2 \leq \|u\|_{\mathcal{D}}\sqrt{1+|k|^2}\|v\| \\
&\leq \|u\|_{\mathcal{D}} Ch\|v\|_{\mathcal{D}} \leq ch\|u\|_{\mathcal{D}}^2.
\end{aligned}$$

The case of $u \in V_h^M$ is proved in the same way.

Using this lemma we can make a to be V_h^R elliptic by applying $\mathcal{D}_R$ ellipticity to $\|L^*(u - L^*v)\|^2$, where v is as in Lemma 1, and making use of the fact that $\mathcal{P}L^*v = ikL^*v$:

$$\begin{aligned}
\frac{1}{\alpha}\|L^*u\|^2 &= \frac{1}{\alpha}\|L^*(u - L^*v)\|^2 \\
&\geq \|u - L^*v\|_{\mathcal{D}}^2
\end{aligned}$$

$$
\begin{aligned}
&= \|u - L^*v\|^2 + \|\mathcal{P}u - \mathcal{P}L^*v\|^2 \\
&\geq \|u\|_{\mathcal{D}}^2 - \max\{1, |k|\} C' h^{\frac{1}{2}} 2 \|u\|_{\mathcal{D}}^2 \\
&\geq \left(1 - Ch^{\frac{1}{2}}\right) \|u\|_{\mathcal{D}}^2.
\end{aligned}
$$

Thus the following holds.

Theorem 2. *For small enough h, a is V_h^R-elliptic.*

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