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On Solution for the Longitudinal Electromagnetic Waves Derived from the Jefimenko Wave Equation and Its Application to Wireless Power Transfer

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Abstract—A goal of this work is to find a possible explanation of the experimental results of [1] where the wireless power transfer between two coils is investigated. We show that this wireless power transfer is provided by the longitudinal component of the EM field. Using Jefimenko's approach of solving the Maxwell equations we show that under specific conditions the longitudinal component drops with the distance as $1/R$. This result dependence can explain the experimentally detected dependence of the transferred power in the experiment.

1. INTRODUCTION

An experiment repeating one of the famous Tesla's experiments is reported in [1]. According to this work, high level of transferred power P_{tr} at a distance to 15 meters detected in the secondary coil is caused by specific mechanism of amplification of the current in this coil, so called 'inverse Liénard-Wiechert effect' considered in [1] too. But this mechanism is not sufficient to explain an experimentally detected dependence of P_{tr} on the distance R between the coils (Table 2 of [1]). This dependence allows one to state that propagation of the power between the coils cannot be described by dependence of the longitudinal component in the near zone of the antenna $E_{\|} \simeq 1/R^2$, even with consideration of the finite size of the radiating coil. So it is reasonable to suggest that, under certain conditions, the primary coil radiates longitudinal component of the EM field whose amplitude drops with distance as $1/R$, i.e., the longitudinal EM waves.

In this article we show that the longitudinal EM waves can exist although conditions for their radiation are very specific. However, even under these specific conditions the longitudinal EM waves can be used in communications.

A basic point of our study is the Jefimenko equation for the E field (Eq. (6.49) of [2]). Analysis of this equation allows one to conclude that the longitudinal EM waves are as physical as well investigated transversal EM waves.

It should be noted that the established opinion, according to which only transversal EM field can drop with the distance as $1/R$, is based on lack of experimental investigation of the longitudinal component and on the derivation of the transversality of the EM field of a *point charge*†. Here, we should clarify one point important for further investigation, i.e., regarding orders of the mathematical operations upon expressions containing singularities in the classical electrodynamics.

In frame of the classical electrodynamics, elementary charge is treated as a particle of infinite small radius. Because some unsolved problems of the classical electrodynamics, such as self-energy etc., arise from infinite small radius of the charge the following procedure of calculations of the electrodynamic quantities has been established: initially a radius of the charge r_{ch} is assumed to be finite, then the needed electrodynamic quantity is calculated and at the final stage a limit of this quantity at $r_{ch} \to 0$

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† The point laws of Liénard and Wiechert [3].

is computed. If a final value of the quantity is finite, this result is treated as true because this result is independent on specific value of r_{ch}. Obviously, Liénard's approach [4] of computing the EM fields of the elementary charge does not meet requirements of this procedure because the limit $r_{ch} \to 0$ is made at intermediate stage when the potentials but not the fields are calculated yet.

Moreover, Liénard's approach is not accepted to calculate the fields of extended source, for example, the current pulses in the antenna. It is caused by impossibility to calculate the retarded integrals of the extended source in the closed form. The latter is necessary since the next step in calculation of the EM fields is computing the partial derivatives of these integrals.

2. EXPRESSION FOR THE LONGITUDINAL COMPONENT AND ITS APPLICATION TO THE EXPERIMENT

It is known that retarded solutions of the Maxwell equations can be obtained directly for the fields, i.e., without introduction of the EM potential. This procedure has been developed by Jefimenko [5].

Before analysis of the Jefimenko expressions for the E field, we define the variables used in derivation of these expressions. As in [5], we will use the square brackets to note that the quantities depend on the retarded variables.

Jefimenko's equation for the E field (in SI units) is

$$\nabla^2 \bar{E} - \frac{1}{c^2}\frac{\partial^2 \bar{E}}{\partial t^2} = -\frac{1}{\varepsilon_0}\left(-\nabla\rho - \frac{1}{c^2}\frac{\partial \bar{J}}{\partial t}\right), \tag{1}$$

Using the superposition principle, we present this equation as

$$\bar{E} = \bar{E}_\rho + \bar{E}_J$$

$$\nabla^2 \bar{E}_\rho - \frac{1}{c^2}\frac{\partial^2 \bar{E}_\rho}{\partial t^2} = \frac{1}{\varepsilon_0}\nabla\rho, \tag{2}$$

$$\nabla^2 \bar{E}_J - \frac{1}{c^2}\frac{\partial^2 \bar{E}_J}{\partial t^2} = \frac{1}{\varepsilon_0 c^2}\frac{\partial \bar{J}}{\partial t}, \tag{3}$$

The key feature of the retarded solutions of the above equations is that a direction of the partial vector $\bar{E}$ is determined in each case by the direction of the partial source vector. So we cannot separate the propagating E waves, described by these equations, into the transversal and longitudinal waves. However, there is a question: how does the original Eq. (1) yield the set of transversal and longitudinal waves?

The answer is: the transversal and longitudinal EM waves are described by the superposition of the solutions of Eqs. (2) and (3)[‡]. So decomposition of the vector $\bar{E}$ in Eq. (1) onto the components, perpendicular and parallel to the wave vector $\bar{k}$ of propagation of the EM waves gives the complete set of the transversal and longitudinal waves. Thus, we are able to conclude that the law of attenuation of each kind of EM wave is determined only by the properties of the sources in Eq. (1) and longitudinal EM waves are as physical as transversal waves.

So the other question can be stated: is it possible to choose conditions on the sources of the EM field such that the conditions of existence of the longitudinal EM waves falling off with the distance as $E \simeq 1/R$ can be realized? To study this problem, let us consider properties of the sources of Eqs. (2) and (3) in more detail.

Actually, the solution of Eq. (1) is the superposition of solutions of two equations with independent sources $\nabla\rho$ and $(1/c^2)\partial\bar{J}/\partial t$[§]. Both solutions have the spatial dependence $E \simeq 1/R$ but for the point-like source, in Liénard approach, in the direction of motion of the charge (the longitudinal direction) the solutions are in counter-phase, i.e., they mutually compensate each the other, and the spatial dependence of the E field degenerates into

$$E_\parallel = E_\rho + E_J \simeq \frac{1}{R^2}. \tag{4}$$

[‡] Section 2 of [5].

[§] Despite ρ and $\bar{J}$ being connected by the total charge conservation law, their partial derivatives are not, so they can be treated as independent.

In the transversal direction to the motion of the charge there is no such compensation so

$$E_\perp = E_J \simeq \frac{1}{R}. \tag{5}$$

Therefore, to find conditions for longitudinal EM waves with the dependence $E \simeq 1/R$ to exist, one should find when the source of Eq. (1) has the needed property. According to Jefimenko, a solution of Eq. (1) is of the form[‖]

$$\bar{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\left(\bar{R}-\bar{r}\right)}{\left|\bar{R}-\bar{r}\right|^2 c}\left[\frac{\partial\rho}{\partial t}\right] d^3r - \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\left|\bar{R}-\bar{r}\right| c^2}\left[\frac{\partial\bar{J}}{\partial t}\right] d^3r, \tag{6}$$

where only terms $\simeq 1/R$ of Jefimenko's complete expression are quoted.

The following step in Jefimenko's procedure is transformation of the second term in the first integral of the *rhs* of the above equation as

$$\int \frac{[\partial\rho/\partial t]}{c|\bar{R}-\bar{r}|^2}\left(\bar{R}-\bar{r}\right) d^3r = -\int \frac{\left[\nabla'\cdot\bar{J}\right]}{c\left|\bar{R}-\bar{r}\right|^2}\left(\bar{R}-\bar{r}\right) d^3r = \int\left(\frac{\left(\nabla'\cdot\left[\bar{J}\right]\right)}{c\left|\bar{R}-\bar{r}\right|^2} - \frac{\left(\left(\bar{R}-\bar{r}\right)\cdot\left[\partial\bar{J}/\partial t\right]\right)}{c^2\left|\bar{R}-\bar{r}\right|^3}\right)\left(\bar{R}-\bar{r}\right) d^3r. \tag{7}$$

In the first transformation of Eq. (7) the equality $[\partial\rho/\partial t] = -[\nabla\cdot\bar{J}]$ is used, and next, the vector identity

$$\left(\nabla'\cdot\left[\bar{J}\right]\right) = \left[\left(\nabla'\cdot\bar{J}\right)\right] + \left[\frac{\partial\bar{J}}{\partial t}\right]\cdot\nabla'[t]$$

is done to express $[(\nabla'\cdot\bar{J}])]$ as

$$\left[\left(\nabla'\cdot\bar{J}\right)\right] = \left(\nabla'\cdot\left[\bar{J}\right]\right) - \left[\frac{\partial\bar{J}}{\partial t}\right]\cdot\nabla'[t]$$

But the law of the charge conservation puts some limitation on the temporary dependence of the functions ρ and $\bar{J}$. For the elementary charge this dependence should be

$$\rho(\bar{r},t) = \rho(\bar{r}-\bar{r}_0(t))\,; \quad \bar{J} = \frac{d\bar{r}_0(t)}{dt}\cdot\rho(\bar{r}-\bar{r}_0(t)), \tag{8}$$

where $\bar{r}_0(t)$ is the radius vector of the center of the charge. If the charge density depends on the time *directly*, $\rho = \rho(t;\bar{r}-\bar{r}_0(t))$, we would have deviation from the charge conservation law because in the co-moving frame where the charge is being at rest $\mathtt{d}\rho/\mathtt{d}t \neq 0$. For extended source, namely, the current pulse, the temporary dependence can be more complex but even in this case we are able to consider the current pulse as an assemble of moving electrons on a background the positive ions of the lattice so the temporary dependence represented by Eq. (8) is correct. So the specific form of the temporary dependence transforms the above vector identity into

$$\left(\nabla'\cdot\left[\bar{J}\right]\right) = \left[\left(\nabla'\cdot\bar{J}\right)\right] - \left[\left(\nabla'\cdot\bar{J}\right]\right)\cdot\left(\frac{d\bar{r}_0(t_{ret})}{dt_{ret}}\cdot\nabla' t_{ret}\right) = \left[\left(\nabla'\cdot\bar{J}\right)\right]\cdot\left(1-\frac{d\bar{r}_0(t_{ret})}{dt_{ret}}\cdot\nabla' t_{ret}\right)$$

So

$$\left[\left(\nabla'\cdot\bar{J}\right)\right] = \frac{\left(\nabla'\cdot\left[\bar{J}\right]\right)}{\left(1-\frac{d\bar{r}_0(t_{ret})}{dt_{ret}}\cdot\nabla' t_{ret}\right)}. \tag{9}$$

It essentially complicates the procedure of calculations because one cannot go to the quantity $(\nabla'\cdot[\bar{J}])$ without appearance of a denominator in the integrand. Therefore we cannot repeat next steps given in Section 2-2 of [5] for our case.

To continue a procedure of calculations, we accept the model (8) for the charge density of the extended source. Such a model of the source is applicable to describe the short current pulse on the wire assuming that a decay time of the pulse is much longer than the time of propagation of the pulse

[‖] Eqs. (2)–(2.4) of [5]; we use Jefimenko's notation too, i.e., the quantities in the square brackets [...] are calculated at the retarded time.

in the antenna. In the other words, this model is correct if the pulse is not suffered by deformation of its shape.

It allows to compute the quantities entering into the square brackets

$$\left[\frac{\partial\rho}{\partial t}\right] = -\left(\frac{d\bar{r}_0(t_{ret})}{dt}\cdot[\nabla\rho]\right);$$
$$\left[\frac{\partial\bar{J}}{\partial t}\right] = \frac{d^2\bar{r}_0(t_{ret})}{dt^2}\cdot[\rho] - \left(\frac{d\bar{r}_0(t_{ret})}{dt}\cdot[\nabla\rho]\right)\frac{d\bar{r}_0(t_{ret})}{dt}$$

and, therefore, the expression for the E field takes a form

$$\bar{E} = \int\left\{\frac{[\rho]}{\left|\bar{R}-\bar{r}\right|^3} - \frac{1}{c\left|\bar{R}-\bar{r}\right|^2}\left(\frac{d\bar{r}_0(t_{ret})}{dt}\cdot[\nabla\rho]\right)\right\}\left(\bar{R}-\bar{r}\right)d^3r$$
$$+\int\frac{d\bar{r}_0(t_{ret})/dt}{c^2\left|\bar{R}-\bar{r}\right|}\left(\frac{d\bar{r}_0(t_{ret})}{dt}\cdot[\nabla\rho]\right)d^3r - \frac{1}{c^2}\int\frac{[\rho]\left(d^2\bar{r}_0(t_{ret})/dt^2\right)}{\left|\bar{R}-\bar{r}\right|}d^3r. \tag{10}$$

We do not intend to make a complete analysis of Eq. (10) but only in application to the experiment. In the experiment, the primary coil was loaded by a sine like signal. This signal can be treated as a set of current pulses traveling with constant velocity close to the speed of light. If we are able to treat the velocity of the source as a constant in Eq. (10) during some period of time, i.e., $v = const$ and

$$\rho = \rho(x-vt,y,z); \quad \frac{\partial\rho}{\partial t} = -v\frac{\partial\rho}{\partial x}; \quad J_{\|} = v\rho(x-vt;y;z); \quad \frac{\partial J_{\|}}{\partial t} = -v^2\frac{\partial\rho}{\partial x}$$

then, retaining the terms $\simeq 1/R$ in (10), we have

$$E_{\|} = -\frac{1}{4\pi\varepsilon_0}\left(1-\frac{v}{c}\right)\int\frac{v}{c|\bar{R}-\bar{r}|}\left[\frac{\partial\rho}{\partial x'}\right]d^3r. \tag{11}$$

The source $[\partial\rho/\partial x']$ can be processed as a function of the retarded variables, which gives the spatial dependence

$$E_{\|} = -\left(1-\frac{v}{c}\right)\frac{v\alpha}{cR}. \tag{12}$$

where α is some constant.

A dependence of the form (12) could explain the behavior of the power decay with the distance in [1]. But first we should consider if the conditions of radiation of such a longitudinal component, i.e., traveling of the charged object in a straight line with constant velocity during some period of time, are realized in the experiment described in [1].

The primary coil was loaded by sinusoidal current of frequency equal to the resonant frequency 1.7 MHz of the radiating system. Because of the resonance the current exists in the coil as a set of standing or traveling waves; the needed mode can be controlled by the phase of the input signal. The wavelength λ of the frequency 1.7 MHz is about 176 m, the sizes of the coils are $L = 1.3$ m and $d = 0.18$ m, the number of turns N is greater than 2000 so the conditions for the traveling waves mode $L_{wire} = \pi dN \gg \lambda$ are satisfied. In opposite to dipole antenna, there is no stopping of the current pulses at the end of the monopoles. So the velocity of the traveling pulses can be considered with good approximation as constant.

The path of the current pulses is a circle in the cross section plane (the turns of the coil). Consequently, motion of a charged element of the pulse in this circle is described by the coordinate dependence

$$x(t) = \frac{d}{2}\sin\left(\frac{2vt}{d}\right); \quad y(t) = \frac{d}{2}\cos\left(\frac{2vt}{d}\right),$$

so for times required the condition $2vt \approx id$, $i = 0, \pm1, \pm2, \ldots$ or $t \approx id/2v$, the x coordinate of the pulse is

$$x(t) = \frac{d}{2}\sin\left(\frac{2vt}{d}\right) \approx vt,$$

For all windings of the primary coil the rectilinear motion of the current pulse can be assumed to exist during the time periods t_{rec}

$$\frac{n\pi d}{2v} < t_{rec} < \frac{n\pi d}{2v} + \Delta t, \quad n = 0, \pm 1, \pm 2, \ldots, N-1, \tag{13}$$

where $\Delta t \ll d/(2v)$ is the time limit of rectilinear motion of the pulse in the single turn. So for lower and upper segments of the turns,

$$x(t) = \frac{d}{2}\sin\left(n\pi + 2vt/d\right) \approx \frac{d}{2}\sin\left(\frac{2vt}{d}\right)\cos(n\pi) \approx \pm vt$$

and the pulses can be treated as moving in a straight line (Figure 1) for some distance.

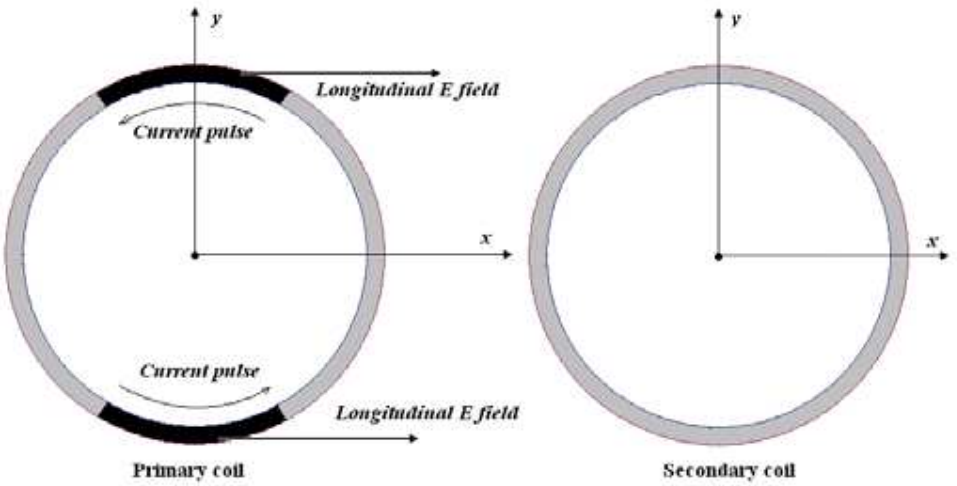

Figure 1. Schematic drawing of the coil alignment in experiments of [1]. The sections of the primary coil, where the current pulses radiate the far longitudinal field, are marked.

We suggest that while the current pulses move in the sections of the turns, where the x coordinate of the element of the pulse satisfies the condition (13), the longitudinal E field of required property is generated by the primary coil.

It is necessary to note why the longitudinal field $\simeq 1/R$ has not been detected in other experiments. It is caused by a factor $(1 - v/c)$ in Eq. (12) which suppresses the radiating component so the latter is much weaker than corresponding transversal component. It is in an agreement with the result of detection of the longitudinal component in [6], where this component is too weak that could not be detected at the distance greater 3 m from the radiator.

However, in the experiments of [1] the current in the secondary coil is amplified by the factor $1/(1 - v'/c)$, because of the inverse Liénard-Wiechert effect. Here, v' is the traveling velocity of the current pulses in the secondary coil. Both coils have identical resonance frequency so the velocities of the pulses in them are equal or close. Therefore, both factors mutually compensate each other.

We do not analyze the application of the complete expression (10) to describe the results of the experiment. As it is noted above, the aim of this work is to show a possibility for the far longitudinal E field to exist if this field is created by a distributed (non point-like) source.

Also we shortly consider a difference between Jefimenko's and Liénard's approaches for the classical (point-like) charge.

We have in Jefimenko's approach,

$$\mathbf{E}_{\varphi}(\mathbf{R}, t) = \lim_{\rho \to \delta} \int \frac{\nabla_r \rho(\mathbf{r}; t_{ret})}{|\mathbf{R} - \mathbf{r}|} \mathrm{d}^3 r = \frac{e\bar{R}_{ret}}{R_{ret}^3 \left[1 - (\bar{v}_{ret} \cdot \bar{R}_{ret})/(cR_{ret})\right]} + \frac{\partial}{\partial t}\left[\frac{e}{R_{ret} - (\bar{v}_{ret} \cdot \bar{R}_{ret})/c}\right], \tag{14}$$

and in Liénard's approach,

$$\mathbf{E}_{\varphi}(\mathbf{R}, t) = \nabla_R \int \lim_{\rho \to \delta} \frac{\rho(\mathbf{r}; t_{ret})}{|\mathbf{R} - \mathbf{r}|} \mathrm{d}^3 r = \nabla_R \frac{e}{R_{ret} - (\bar{v}_{ret} \cdot \bar{R}_{ret})/c}, \tag{15}$$

where lower index *ret* means that a quantity is calculated at the retarded time. So the longitudinal component $\simeq 1/R$ should exist in (14) and does not exist in (15). Since in most of electrodynamical calculations the Liénard approach is used, it can explain why expressions for the longitudinal EM wave have not been obtained before.

It should be noted that the expression for the E field of the extended source, containing no longitudinal component $\simeq 1/R$, was firstly given in [7] (Eqs. (12)–(14)). But in Panosfky-Phillips textbook, a special type of the source is considered, namely, $\bar{j} = \bar{j}_0 \cdot e^{i\omega t}$ and $\rho = \rho_0 \cdot e^{i\omega t}$. It means that the spatial and temporary variables are separated in the source. But even a dependence $\rho(\bar{r}, t) = q\delta(\bar{r} - \bar{v}t)$ does not allow to separate the variables. So if we repeat a calculation procedure of [7] with the source of the latter form we obtain divergent integrals because the Fourier method developed in [7] cannot be directly applied to the singular function describing the classical charge. So Panofsky-Phillips procedure is not applicable to compute correct expression for the longitudinal E field.

3. CONCLUSION

In this work we analyze Jefimenko's approach of solving the Maxwell equations. Decomposition of the wave equation for the E field onto two wave equations with independent sources allows us to conclude that the actual electric field is a superposition of two vectorial spherical waves, created by these independent sources. Analysis of Jefimenko's solution of the Maxwell equations in case the source is a current pulse traveling in a straight wire with constant velocity shows that the longitudinal EM wave should drop with the distance as $1/R$, i.e., they behave as transversal EM waves.

We do not make complete analyses of all possible conditions for longitudinal EM waves to exist, but limit ourselves to only one case. In other words, we analyze specific conditions when the source of the EM wave moves with constant velocity and the path of the source can be treated as a straight line at least for some distance, i.e., in application to the experiment [1].

Since in practical applications extended sources are current pulses in the antenna systems, the velocities of such pulses are close to the speed of light. Correspondingly, intensity of the longitudinal component is sufficiently small which is caused by the factor $(1 - v/c)$. This can serve as explanation of why the longitudinal EM waves are hard to detect. But if the receiving antenna is of the same design as described in [1], the inverse Liénard-Wiechert effect provides amplification of the received longitudinal component. So the longitudinal E field can be used for communication. To demonstrate it, the authors of [1] intend to make a series of experiments to investigate how a level of EM power transferred from the primary coil to the secondary one drops with the distances up to 100 meters.

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