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Simple Methods to Raise the Robustness and Efficiency of the Incomplete Cholesky Preconditioners for FEM Simulation of Electromagnetic Problems

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Abstract—In this paper, the finite element method (FEM) is applied to the analysis of three-dimensional (3D) electromagnetic structures. The incomplete Cholesky (IC) preconditioner based on shifted operators is used to solve the finite element linear systems. Several strategies are adopted to raise the efficiency and robustness of the preconditioner. Numerical experiments for several microwave devices demonstrate the superior numerical convergence and robustness of the proposed preocnditioner.

1. INTRODUCTION

In full-wave electromagnetic (EM) simulations, the finite element method (FEM) [1, 2], finite difference time domain method (FDTD) [3], and method of moment [4, 5] are three most popular methods in the past few decades, and each of them has its advantages and disadvantages in solving different EM problems. Among the three methods, FEM has most wide applications as it can cope with complex materials and structures in a natural way. As a result, it has gained much attention. Till now, with the effort of many researchers, FEM has gained great advance in theory, and has been successfully used to solve many practical problems, such as eddy-current problems [6], waveguide discontinuities [7–11], electromagnetic compatibility (EMC) problems [12], and scattering/ radiation problems [13, 14].

The major disadvantage of FEM is that when three-dimensional (3D) problems are solved, the application of FEM will generate a rather ill-conditioned, highly sparse linear system, and the dimension of which is proportional to the cube of the electric-size of the problem. Solution of the linear system is the most time-consuming step in the whole FEM simulation process, which greatly affects the efficiency and ability of FEM in solving large or complex problems. Till now, there are many published research papers about the solution of the FEM linear system. The most popular method is the preconditioned Krylov subspace iterative method, such as ICCG [15], SSORCG [16], FSAICG [17], etc. As the FEM matrix is rather ill-conditioned, the construction of highly efficient preconditioners using conventional methods is very difficult. The most popular incomplete Cholosky preconditioner always breaks down during factorization, or may be very inefficient even if the factorization is completed successfully. Even if a stabilization strategy is adopted, the number of iterations is still large [15]. In [18], the eigenvalue distribution of the FEM matrix is thoroughly investigated, which shows that the FEM matrix is highly indefinite and rather ill-conditioned. This explained why it is so hard to construct efficient preconditioners. In that paper, a joint vector and scalar potential formulation is proposed. This method can alleviate the low-frequency instability of FEM to some extent. The main disadvantage of this method is that the nonzero elements in the FEM matrix will be doubled. As a result, the solution time cannot be saved much though the iteration time is reduced. Moreover, the efficiency of this method is not so obvious when frequency is high.

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In [19], incomplete Cholesky factorization preconditioners based on shifted operators is proposed (which is called SL-IF for short) to solve microwave structures. Numerical results show that this proconditioner is more efficient and robust than conventional IC preconditoners. In this paper, this method is further discussed. Several strategies are proposed to improve the efficiency and robustness of the SL-IF preconditioner. In the next section, the details of the algorithm are briefly reviewed.

2. FORMULATION

The computational domain of a typical multiport microwave device is illustrated in Figure 1. To truncate the computational domain, the wave port is filled with perfectly matched layers (PMLs) [20], which can be conveniently interpreted as a diagonal anisotropic medium. The PMLs are backed with perfectly electric conductor (PEC) or absorbing boundary conditions (ABCs).

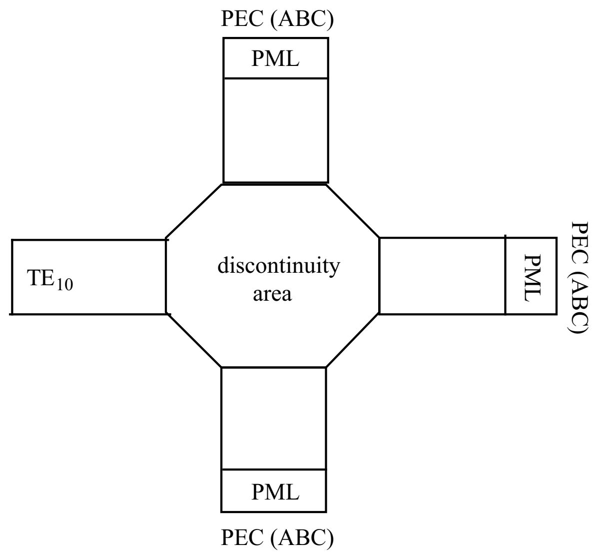

Figure 1. The finite element model of multiport microwave devices.

The corresponding functional is:

$$F(\mathbf{E}) = \frac{1}{2}\int_V \frac{1}{\mu_r}(\nabla \times \mathbf{E}) \cdot \bar{\bar{\Lambda}} \cdot (\nabla \times \mathbf{E}) - k_0^2 \varepsilon_r \mathbf{E} \cdot \bar{\bar{\Lambda}} \cdot \mathbf{E} dV \tag{1}$$

where k_0 is the wave number and V the FEM computational domain.

Applying the Rayleigh-Ritz variational method, a large sparse linear system is obtained:

$$\left[\mathbf{F} - K_0^2 \mathbf{G}\right] \cdot e = r \tag{2}$$

where **F** and **G** are sparse matrices with the same nonzero patterns, the explicit expressions of them are:

$$\mathbf{F}_{ij} = \int_v \frac{1}{\mu_r}(\nabla \times \mathbf{N}_i) \cdot \bar{\bar{\Lambda}} \cdot (\nabla \times \mathbf{N}_j) dV \tag{3}$$

$$\mathbf{G}_{ij} = \int_V \varepsilon_r \mathbf{N}_i \cdot \bar{\bar{\Lambda}} \cdot \mathbf{N}_j dV \tag{4}$$

As analyzed in [18], **F** is semi-definite and **G** positive definite. As a result, the FEM matrix is indefinite and rather ill-conditioned with more than K negative eigenvalues (here K is the number of nodes). For convenience, the stiff matrix can be expressed as:

$$\mathbf{A} = \mathbf{F} - K_0^2 \mathbf{G} \tag{5}$$

To construct the preconditioning matrix, the following shifted operator is constructed [19]:

$$\nabla \times \left(\mu_r^{-1}\bar{\bar{\Lambda}} \cdot \nabla \times \check{\mathbf{E}}(r)\right) + (\alpha + j\beta)k_0^2\varepsilon_r\bar{\bar{\Lambda}} \cdot \check{\mathbf{E}}(r) = 0 \text{ in } \Omega \tag{6}$$

with $\alpha^2 + \beta^2 = 1$. Applying the FEM scheme, the generated FEM matrix corresponding to (6) is

$$\mathbf{M} = \left[\mathbf{F} + (\alpha + j\beta)k_0^2\mathbf{G}\right] \tag{7}$$

$\mathbf{M}$ is a good preconditioner of the FEM matrix. It is demonstrated that choosing $\alpha = 0.0$, $\beta = 1.0$ is preferred to get better preconditioning efficiency. As the inverse of $\mathbf{M}$ is uneasy to be obtained, IC factorization with diagonal perturbation [15] to $\mathbf{M}$ is applied. Finally, a lower triangular matrix L will be obtained:

$$LL^T \approx \mathbf{M} \tag{8}$$

Numerical tests show that the efficiency and robustness of this SL-IF preconditioning scheme are much better than the conventional IC preconditioner for the waveguide discontinuity and patch antenna problems [19]. In the following, several strategies are adopted to further improve the efficiency and robustness of the SL-IF preconditioner.

Firstly, permutation strategies are adopted to improve the quality of the SL-IF preconditioner. The efficiency of symmetric reverse Cuthill-McKee (SYMRCM) permutation and approximate symmetric minimum degree (SYMAMD) permutation [21] is considered and compared. Supposing that the permutaion matrix is P, the matrix $\mathbf{M}$ after permutaion is transposed to

$$\mathbf{M}_1 = P^T\mathbf{M}P \tag{9}$$

As the preconditioner obtained from (9) cannot be directly applied to solve (2), the linear system (2) is also permuted with P.

$$P^T\mathbf{A}P[P^Te] = P^Tr \tag{10}$$

For convenience, we define

$$\mathbf{A}_1 = P^T\mathbf{A}P, \quad e_1 = P^Te, \quad r_1 = P^Tr \tag{11}$$

Then (10) can be written for short as:

$$\mathbf{A}_1e_1 = r_1 \tag{12}$$

To further improve the efficiency and robustness of the SL-IF preconditioner, Jacobi scaling is applied to M_1 before factorization:

$$\mathbf{M}_2 = G^{-1}\mathbf{M}_1G^{-1} \tag{13}$$

where $G = D^{1/2}$, D is the diagonal of $\mathbf{M}_1$. Through the above scaling, the symmetric property of $\mathbf{M}_1$ is reserved, while all diagonal entries become one, thus the condition number of the matrix $\mathbf{M}_2$ is improved. The computational overhead of this operation is only equal to one matrix-vector multiplication as only half of the matrix needs to be handled by considering the symmetric property of the matrix.

As known, it cannot be guaranteed that IC factorization is robust even for SPD matrix, and a diagonal perturbation matrix is often needed to stabilize the algorithm. Different from [19], here the following perturbation strategy is adopted:

$$M_3 = M_2 + \gamma\mathbf{I} \tag{14}$$

where γ is a real positive parameter and $\mathbf{I}$ the unit matrix. Comparatively, this perturbation strategy is more robust, and the robustness of which can be guaranteed in theory. A larger γ will make the factorization process more robust, though the efficiency of the generated preconditioner may not necessarily better. As can be seen in the numerical results in the next section, a very small γ is needed in most cases. If enough fill-in elements in $\mathbf{L}$ are reserved, the perturbation matrix is not even needed, which is a great advantage compared with the IC preconditioner.

After processing IC factorization to M_3, a lower triangular matrix L_3 is obtained:

$$L_3 L_3^T \approx M_3 \tag{15}$$

The final preconditioning matrix L, which is a good approximation of $\mathbf{M}_1$, is obtained through the following operation:

$$L = GL_3 \tag{16}$$

As G is a diagonal matrix, the above operation is quite easy to apply. The preconditioned linear system is

$$L^{-1}\mathbf{A}_1 L^{-T}\left(L^T e_1\right) = L^{-1} r_1 \tag{17}$$

To distinguish this preconditioner with the SL-IF preconditioner proposed in [19], it is noted as SIC for short in this paper. In the next section, several numerical models are presented to testify the efficiency of the SIC preconditioner.

3. NUMERICAL RESULTS AND DISCUSSIONS

In our numerical results, all initial vectors were set to zero. The parameters α, β in the preconditioner are chosen as $\alpha = 0.0$, $\beta = 1.0$. The first example analyzed is a short circuited E-plane slot-coupled T-junction [22]. The illustrated figure and dimensions (in mm) of the problem is shown in Figure 2. By considering the symmetric property of the problem, only half of the structure should be modeled. The problem is discretized into 41226 tetrahedrons containing 8696 nodes. As a result, a total of 45348 unknowns are generated with edge-based FEM. Our calculations of the amplitude of the S_{11} parameters are shown in Figure 2(b). Compared with the measured results in [22], it is found that the results match the experimental data well.

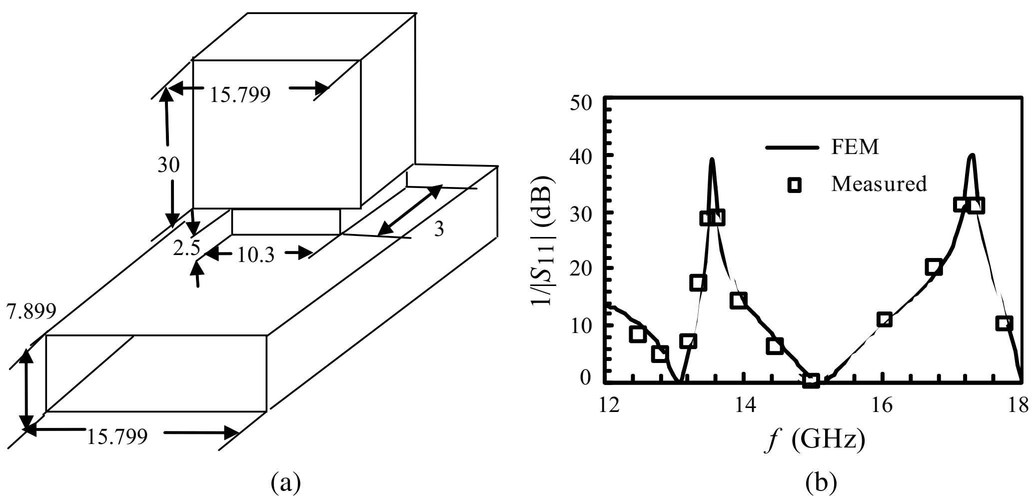

Figure 2. The short circuited E-plane slot-coupled T-junction. (a) Geometry and dimensions. (b) Magnitude of scattering parameters.

Figure 3 illustrates the sparsity pattern of the FEM matrix without permutation, with SYMRCM permutation and SYMAMD permutation. As can be seen from the figure, the nonzero element in the original matrix is distributed in a very wide band. As a result, applying Cholesky factorization to the original matrix will generate many fill-in elements.

The influence of permutation on the efficiency of preconditioners is firstly tested. The maximum number of nonzero elements reserved per row (denoted as p) in the preconditioning matrix L is taken to be $p = 20$. Firstly, Jacobi scaling is not considered in construction of the SIC preconditioner. Figure 4 illustrates the variation of iteration number and CPU time of SICCG with perturbation parameter for different permutation strategies. As the number of iterations of SICCG without permutation is too large, it is not shown in the figure. In our experiments, SYMRCM permutation shows a

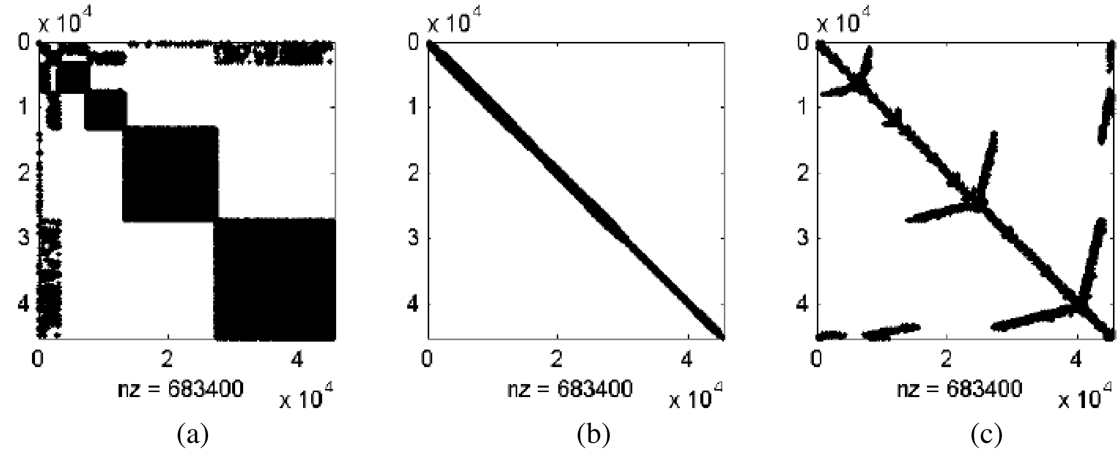

Figure 3. Sparsity patterns of the FEM matrices for the T-junction. (a) Without permutation. (b) SYMRCM permutation. (c) SYMAMD permutation.

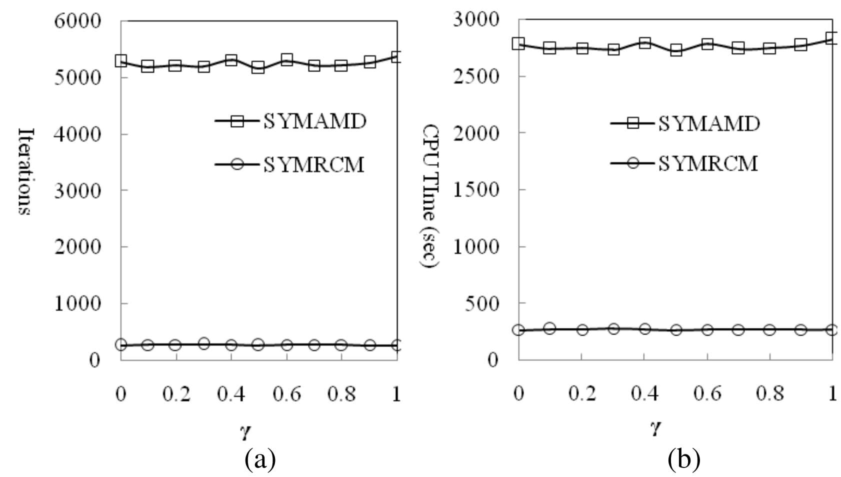

Figure 4. Iteration behavior of SICCG (without Jacobi scaling) for different permutation strategies. (a) Iterations. (b) CPU Time.

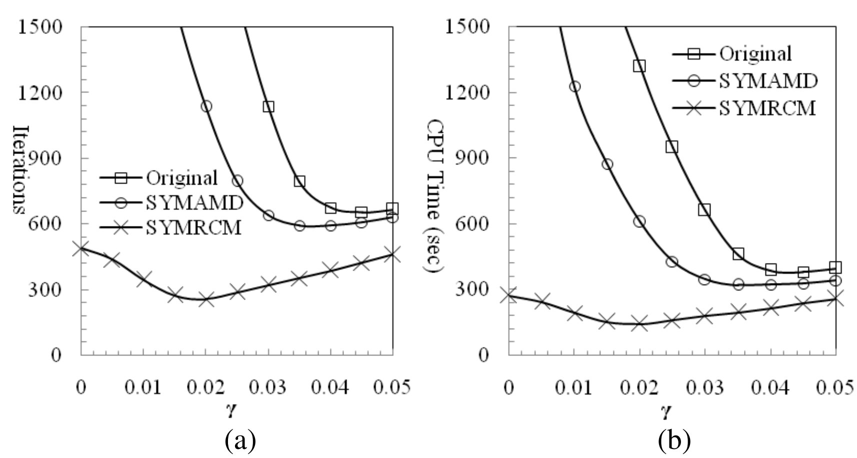

Figure 5. Iteration behavior of SICCG (with Jacobi scaling) for different permutation strategies. (a) Iterations. (b) CPU Time.

much better performance than SYMAMD permutation for SICCG. This illustrates that the SYMAMD algorithm designed for complete Cholesky factorization is not necessarily the best permutation strategy for constructing incomplete factorization preconditioners.

Then, the effectiveness of Jacobi scaling applied to the preconditioner is tested. Figure 5 shows the iteration behavior of SICCG with Jacobi scaling for different permutation strategies. After Jacobi scaling, the smallest number of CG iterations is 257, which appears at $\gamma = 0.02$. Without Jacobi scaling, the smallest number of iterations is 475, which appears at $\gamma = 0.15$. Comparatively, more than 200 iterations can be saved with Jacobi scaling. This shows that the robustness and efficiency of the preconditioner are further improved after applying Jacobi scaling before factorization. From the figures, it can also be seen that when SYMRCM is used, a very small γ is needed to guarantee the quality of the SIC preconditioner. Thus, the robustness of the preconditioner is greatly enhanced.

Next, the number of fill-in elements per row of SIC is varied from 15 to 40. Figure 6 shows the number of iterations and CPU time used for different p and γ. From Figure 6(a), it can be seen that the convergence behavior of SICCG improves with the increasing of p. Especially when $p < 20$, the iteration number of SICCG is sensitive to p. However, a denser preconditioner requires more CPU time per iteration due to increased costs. Through studying Figure 6(b), it is found that when $p \geq 20$, $\lambda \geq 0.02$, CPU time increases gradually with the increasing of p. The least CPU time used is 125 s when $p = 35$ and $\lambda = 0.01$. However, a larger p will need more memory and SIC factorization time. Usually, we can take $p = 20$ or $p = 25$ to get a robust and efficient preconditioner.

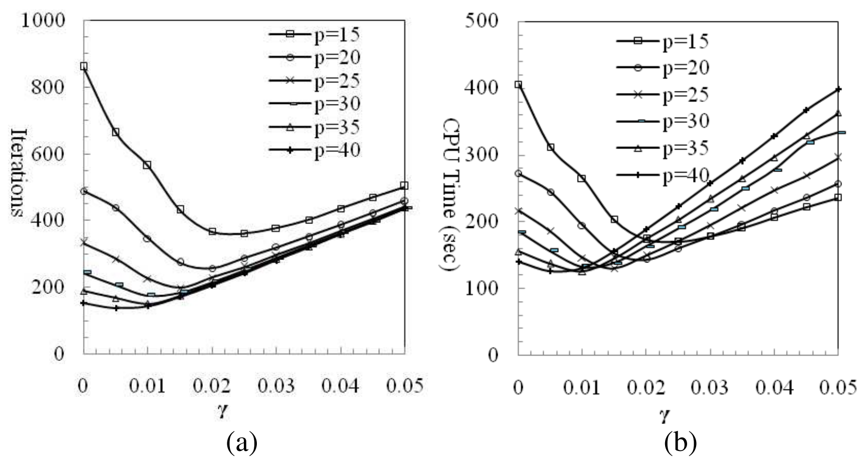

Figure 6. Iteration behavior of SICCG with SYMRCM permutation for different number of fill-in elements. (a) Iterations. (b) CPU time

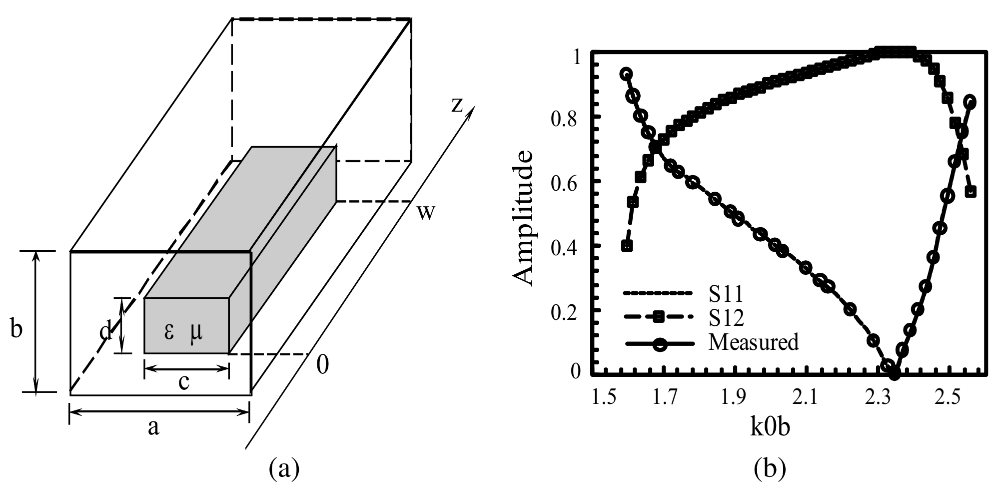

Figure 7. The rectangular dielectric-loaded waveguide. (a) Configuration. (b) Scattering parameters.

The second example tested is a waveguide partially filled with dielectric. The configuration of the waveguide is shown in Figure 7(a). The rectangular waveguide has a width of $a = 2\,\mathrm{cm}$ and height of $b = 1\,\mathrm{cm}$. The inserted dielectric material slab has dimensions $c = 0.888\,\mathrm{cm}$, $d = 0.399\,\mathrm{cm}$ and $w = 0.8\,\mathrm{cm}$, and the dielectric constant is $\varepsilon = 6\varepsilon_0$. With edge-based FEM, a sparse linear system with about 24561 unknown edges is obtained. The scattering parameters simulated with FEM and the measured results [23] are plotted in Figure 7(b). The two results agree quite well.

Next, the operating frequency is set to $f = 9.0\,\mathrm{GHz}$. The maximum number of fill-in elements per row (p) in the preconditioner is taken to be $p = 20$. Figure 8 shows the iterations behaviors of SICCG with Jacobi scaling and different permutations. For this problem, the least iteration number is 198 for $r = 0.015$. For this problem, the SICCG with SYMRCM permutation still shows an excellent convergence behavior.

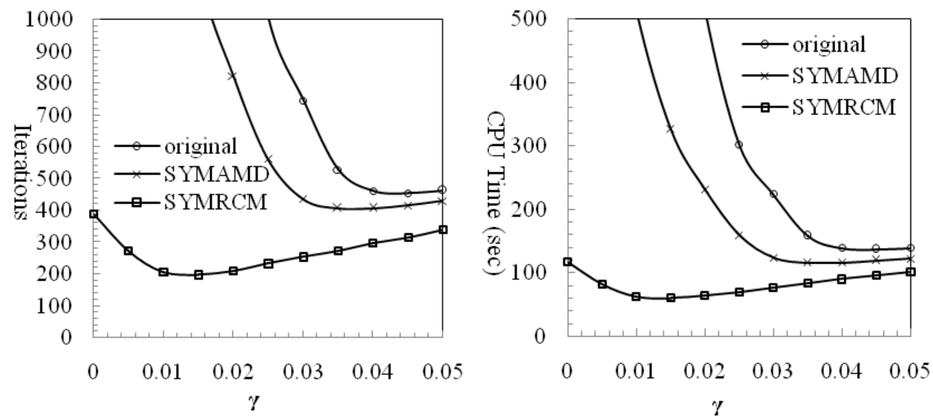

Figure 8. Iteration behavior of SICCG (with Jacobi scaling) for different permutation strategies. (a) Iterations. (b) CPU Time.

4. CONCLUSIONS

In this paper, an incomplete Cholesky preconditioner based on a shifted operator is applied to the full-wave finite-element analysis. Some strategies including SYMRCM permutation, Jacobi scaling before factorization and efficient diagonal perturbation are adopted to improve the robustness and efficiency of the preconditioner. Through the numerical examples, it is shown that these strategies are very effective in improving the robustness and efficiency of the preconditioner.

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