Numerical Investigation on the Spectral Properties of One-Dimensional Triadic-Cantor Quasi-Periodic Structure

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Abstract—We numerically investigate the optical spectra of a photonic band gap material realized by one-dimensional Triadic-Cantor quasi-periodic structure. The studied system is composed of two elementary layers H and L with refractive indices $n_L = 1.45$ (SiO_2) and $n_H = 2.3$ (TiO_2), respectively. Analytical calculations using a trace and antitrace maps approach have been used to find the reflection and transmission theoretical expressions in visible range under quarter wavelength condition. In our results we present the effect of iteration order of Triadic-Cantor sequence on the optical properties of these multilayer systems, namely the photonic band gap behavior and the optical windows presence, which makes this type of structures good candidates for interesting applications in the field of the nano-optical Engineering.

1. INTRODUCTION

In 1970s, the progress achieved for reducing the dimensions of semiconductor structures in the nanoscale [1] helped to create thin films with controlled thicknesses allowing strong electrons confinement in a direction of space.

Later, similar discoveries were required to control the optical properties of these materials and inhibit the light propagation. In other words, it comes to redirecting the propagation in directions for certain frequencies or localize the light in some region. In this perspective, exploiting existing formal analogy between the electron and photon, in 1987 Yablonovitch [2] and John [3] simultaneously proposed the first concept of an artificial structure so-called "photonic crystal", whose dielectric constant is modulated periodically. Many studies have shown that these materials allow the control and manipulation of light through Photonic Band Gaps (PBGs) inhibiting the electromagnetic waves propagation in certain directions of space. These properties make the photonic crystal interesting for many potential applications in integrated optics.

Furthermore, the technological need continues to increase in order to find materials with broader PBG and multi-band materials. The first manifestation of this objective was the creation of a novel class of artificial crystal called "quasi-periodic photonic crystals" by Daniel Schechtman [4] who was awarded the Nobel Prize in Chemistry in 2011 for this discovery. These structures are formed by stacking two or more materials deposited according to recursive inflation rule preset, so that they can be considered as intermediate systems between a periodic crystal and random amorphous solid.

The aim of this work focuses on the contribution to the study of optical properties of quasi-periodic one-dimensional multilayer structures according to a so-called Triadic-Cantor distribution, where we have used the numerical simulation method called "trace and antitrace maps" for the extraction of the spectral responses of these structures.

2. PHOTONIC SYSTEM PRESENTATION

The Cantor set was discovered in 1875 by the British mathematician Henry John Stephen Smith [5] and studied and introduced for the first time in 1883 by the German mathematician Georg Cantor [6–8]. The Cantor set acquires a very important role in many branches of mathematics, mainly in set theory and fractal theory [9, 10]. Although the Cantor set is defined in a general manner, the interesting building, best known in point of view photonic applications, is the Triadic-Cantor [11] which will be detailed in the following.

2.1. The Triadic Cantor Set Construction

The Cantor-Triadic distribution can be obtained by repetition of basic operation on a increasingly smaller scale from one iteration (k) to another $(k + 1)$. At every step (growth stage), a fractal configuration is obtained, wherein the previous scaled copies may be recognized. The stopping of the build process, after a finite number of steps generates a Triadic-Cantor pre-fractal [12]. To build this set (Fig. 1), we choose a segment (the initiator) of length L while for any stage of growth, the generator consists of one-third of the central portion of each segment in the structure at the previous stage [13-15]. Triadic Cantor sets at different stages of growth are shown in Fig. 1.

2.2. The Quasi-Periodic Multilayer Structures According to the Triadic-Cantor Distribution

The one-dimensional multilayer system distributed according to the Triadic-Cantor sequence is a structure with fractal morphology [16], based on Triadic-Cantor sequence Fig. 1. It is generated by a simple substitution rule: $H \mapsto HLH$, $L \mapsto LLL$ with H and L being two different elementary materials [17, 18]. If the sequence begins with the generator H, the first generations become what are shown in Table 1.

Table 1. The first generations of the 1D multilayer structure according to the Triadic-Cantor distribution.

Iteration order	The multilayer structure
0	H
1	HLH
2	$HLHLLLHLH$
3	$HLHLLLHLHLLLLLLLLLHLHLLLHLH$
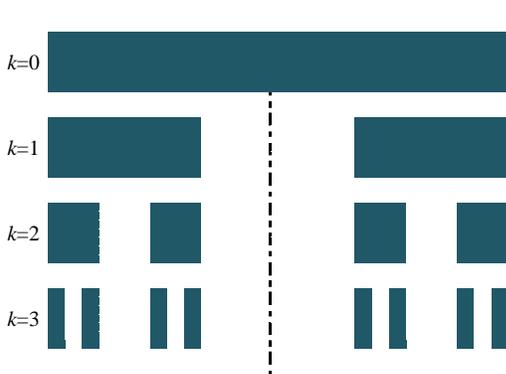
Figure 1. The Triadic-Cantor set presentation for iteration: $k = 0$, $k = 1$, $k = 2$ and $k = 3$.
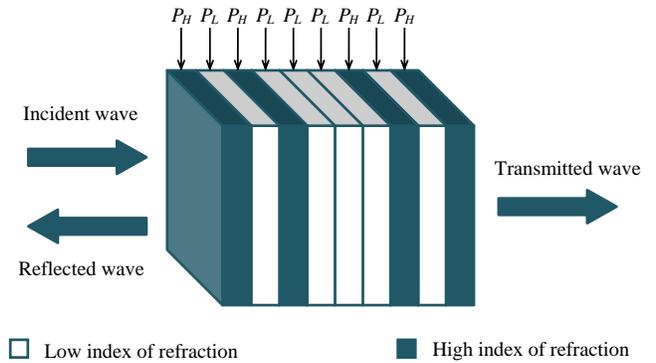
Figure 2. The one-dimensional multilayer dielectric structure according to the Triadic-Cantor distribution for the 2nd iteration ($HLHLLLHLH$).

In this work, we choose a multilayer photonic system distributed according to the Triadic-Cantor sequence, with a dielectric layers arrangement of high (H) and low (L), where n_L and n_H are the refractive indexes, respectively, and optical thicknesses $\lambda_0/4n_H$ and $\lambda_0/4n_L$, where λ_0 is the reference wavelength and $\lambda_0/4n_H = \lambda_0/4n_L$: quarter wavelength condition.

3. CALCULATION METHOD

In order to calculate the optical transmission by the quasi-periodic photonic systems, we use a trace and antitrace maps approach for the electromagnetic fields. For the studied structure (Fig. 2), each layer is associated to a propagation matrix P_i where $i = \{H, L\}$ given by:

$$P_i = \begin{pmatrix} \cos \delta_i & \sin \delta_i \\ \sin \delta_i & \cos \delta_i \end{pmatrix}$$

In the case with normal incidence and polarization parallel to the multilayer surfaces, the transmission through the interface $j \leftarrow i$ is given by the transfer matrix [21]:

$$P_{ji} = \begin{pmatrix} 1 & 0 \\ o & n_i/n_j \end{pmatrix}$$

where $d_{H(L)} = kn_{H(L)}d_{H(L)}$. $n_{H(L)}$ is the refractive index of media $H(L)$, $d_{H(L)}$ the layer thicknesses, and k the wave number in vacuum. Generally, we choose appropriate layer thicknesses d_H and d_L to make $n_H d_H = n_L d_L$. Then we have $\delta_H = \delta_L = \delta$.
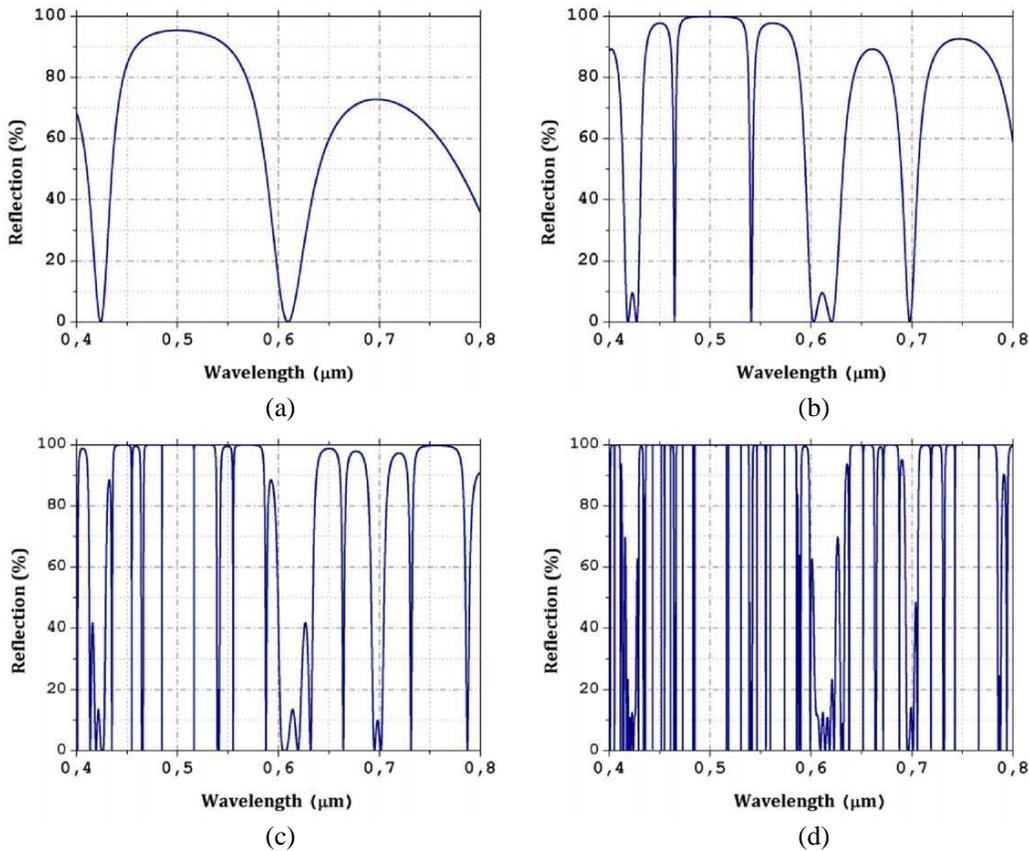
Figure 3. Spectra reflections of quasi-periodic multilayer structures according to the Triadic-Cantor distribution for different iterations: (a) 2nd iteration, (b) 3rd iteration, (c) 4th iteration, (d) 5th iteration.

Therefore, the whole multilayer is represented by a product matrix P relating the incoming and reflected waves to the transmitted wave. The total transmission matrix P has the form:

$$\begin{cases} P_0 = P_H, \bar{P}_0 = P_L \\ P_1 = P_0\bar{P}_0P_0, \bar{P}_0 = \bar{P}_0\bar{P}_0\bar{P}_0 \\ \quad\vdots \\ P_k = P_{k-1}\bar{P}_{k-1}P_{k-1} \end{cases}$$

The transmission coefficient for all the structure (Fig. 2) is given by:

$$T_k = \frac{4}{\mid P_k \mid^2 + 2} \tag{1}$$

where $|P_k|^2$ is the sum of squares of the four elements of the matrix P_k. Since the transfer matrix is unimodular, the transmission coefficient can be written as:

$$T_k = \frac{4}{x_k^2 + y_k^2} \tag{2}$$

where x_k and y_k denote respectively the trace and antitrace of the transfer matrix P_k. Basing on the trace antitrace method we make an iterative algorithm calculation which enables us to determine the transmission spectras of the studied structures [19, 20].
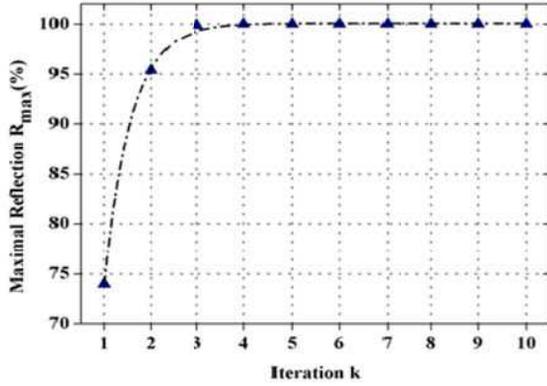
Figure 4. The maximal reflection $R_{\max}$ (%) Variation according to the iteration number (k).
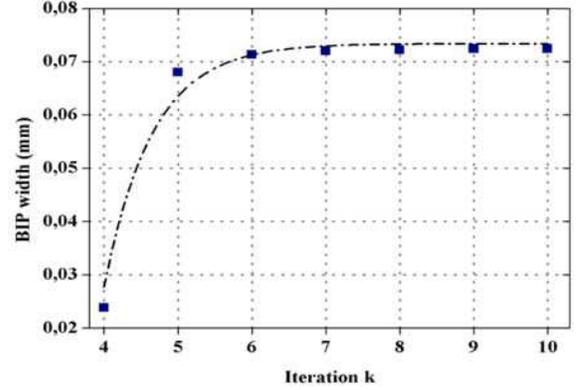
Figure 5. The maximal reflection $R_{\max}$ (%) Variation according to the iteration number (k).
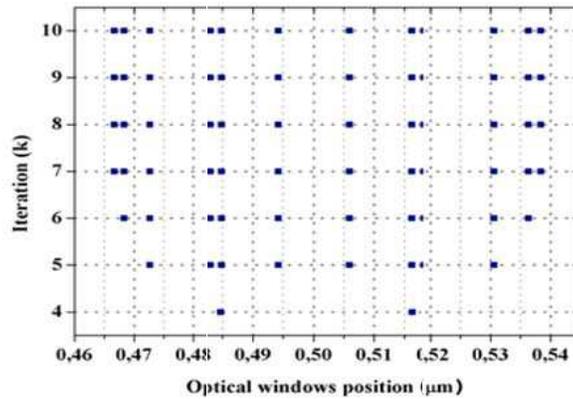
Figure 6. Number and position of the optical windows according to the iteration number (k).

4. NUMERICAL RESULTS AND DISCUSSION

The multilayer quasi-periodic structures constructed according to the Triadic-Cantor model are presented in Section 2.2, where we choose as basic layers of the stack: TiO_2 ($n_H = 2.3$) et SiO_2 ($n_L = 1.45$). Fig. 3 shows the optical reflections spectra in the visible range, for normal incidence with quasi-periodic multilayer structures corresponding to iterations from 1 to 4.

    The spectra in the previous figure show the performance of the multilayer system, namely a total reflection of 100% for the entire visible range increases with the number of iterations. The peaks become increasingly fine, and the number and width of the photonic band gap increase with the iteration of the sequence system generator. In fact, we find from the 4th iteration the increase of the maximal reflection on all the visible range, hence the appearance of PBG covering the range studied (Fig. 4). One of these PBGs is centered around $\lambda_0 = 0.5\,\mu m$, which widens exponentially with the iteration number (k) (Fig. 5) before reaching a limit.

Table 2. Widths and limits of PBG, optical windows number and their positions, according to the iteration number.

k	Number	Positions
5	8	0.4727–0.4830–0.4848–0.4942–0.5059–0.5162–0.5182–0.5306
6	10	0.4683–0.4727–0.4830–0.4848–0.4942–0.5059–0.5162–0.5182–0.5306–0.5363
8	12	0.4667–0.4683–0.4727–0.4830–0.4848–0.4942–0.5059–0.5162–0.5182–0.5306–0.5363–0.5384
9	12	0.4667–0.4683–0.4727–0.4830–0.4848–0.4942–0.5059–0.5162–0.5182–0.5306–0.5363–0.5384
10	12	0.4667–0.4683–0.4727–0.4830–0.4848–0.4942–0.5059–0.5162–0.5182–0.5306–0.5363–0.5384
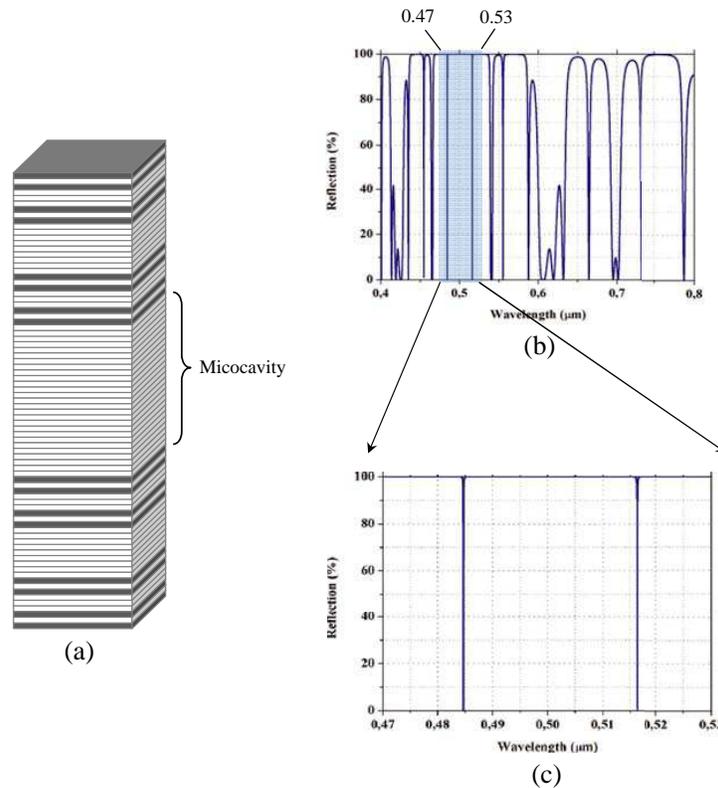
Figure 7. (a) The configuration of a multilayer structure according to the Triadic-Cantor distribution for the 4th iteration, (b) reflection spectrum of a multilayer structure according to Triadic-Cantor distribution between $0.4\,\mu m$ and $0.8\,\mu m$, (c) reflection spectrum of a multilayer structure according to Triadic-Cantor distribution between $0.47\,\mu m$ and $0.53\,\mu m$.

    One the other hand, it is possible to have one or more peaks or optical transmission windows from the 4th iteration. These peaks are located within the PBGs, hence a strong light confinement for certain wavelengths in these spectral regions (Table 2).

    The Fig. 6 illustrates the positions of such optical windows in central PBG, as well as their number according to the structure generation order (k). Also we notice that they are symmetrical with respect to the reference wavelength $\lambda_0 = 0.5\,\mu m$.

    This phenomenon is explained by the symmetrical and fractal geometry of Triadic-Cantor distribution, where the structure in question is considered as a set of optical cavities with a low refractive index n_L sandwiched between planes parallel dielectric mirrors symmetrical relative to the structure center (Fig. 7). For any generation order (k), the multilayer structure keeps its symmetry (Table 1), and the dielectric volume and number of cavities increase from one iteration to another for achieving the minimum value so that the cavities become capable of resonate certain wavelengths (eigenmodes): the width of the cavity must be equal to an integer number of incident wave half-length (Bragg condition) [22].

5. CONCLUSION

In this work, for normal incidence of light we numerically investigate the reflection properties through the photonic multilayer systems according to the Triadic-Cantor distribution as a function of the iteration order of the generator sequence (k). From the photometric responses simulation it is clear that the optical performance of the studied systems increase with the iteration (k) namely the increase of the widths of the PBGs and their numbers. This property is due to the increased number of layers from one iteration to another, causing multiple internal reflections in the structure, giving rise to further constructive interferences, which prevent certain wavelengths to propagate in the system by increasing the PBG width. Moreover, we observe the presence of the optical windows from the 4th iteration. The appearance of these transmission peaks is explained by the formation of resonant microcavities in multilayer structures which confine the light for the wavelengths situated in the range of the PBG.

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