

# Differential Forms and Electromagnetic Field Theory

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*(Invited Paper)*

**Abstract**—Mathematical frameworks for representing fields and waves and expressing Maxwell’s equations of electromagnetism include vector calculus, differential forms, dyadics, bivectors, tensors, quaternions, and Clifford algebras. Vector notation is by far the most widely used, particularly in applications. Of the more sophisticated notations, differential forms stand out as being close enough to vectors that most practitioners can readily understand the notation, yet at the same time offering unique visualization tools and graphical insight into the behavior of fields and waves. We survey recent papers and book on differential forms and review the basic concepts, notation, graphical representations, and key applications of the differential forms notation to Maxwell’s equations and electromagnetic field theory.

## 1. INTRODUCTION

Since James Clerk Maxwell’s discovery of the full set of mathematical laws that govern electromagnetic fields, other mathematicians, physicists, and engineers have proposed a surprisingly large number of mathematical frameworks for representing fields and waves and working with electromagnetic theory. Vector notation is by far the most common in textbooks and engineering work, and has the advantage of being widely taught and learned. More advanced analytical tools have been developed, including dyadics, bivectors, tensors, quaternions, and Clifford algebras. The common theme of all of these mathematical notations is to hide the complexity of the set of 20 coupled differential equations in Maxwell’s original papers [1, 2] using high level abstractions for field and source quantities. Increasing the sophistication of the notation simplifies the appearance of the governing equations, revealing hidden symmetries and deeper meaning in the equations of electromagnetism.

For various reasons, however, few of the available higher order mathematical notations are actually used by applied practitioners and engineers. Tensors, Clifford algebras, and other related analysis tools are not widely taught, particularly in engineering programs. The conundrum is that regardless of the elegance of the formulation, to solve concrete problems the complexity of the mathematical symbols must be unwound to obtain formulas involving components of the electromagnetic field in a specific coordinate system. There is something of a balancing act when choosing a mathematical formalism — if the notation is too abstract, it is hard to solve practical problems, but at the same time a high-level representation can greatly aid in learning and understanding the underlying principles of electromagnetic theory. For day-to-day work in electromagnetics engineering, Heaviside’s vector analysis [3] is a near-perfect compromise between abstraction and concreteness, and when lightly flavored with dyadic notation, adequate for nearly all practical problems.

Bearing in mind this need for balance between concreteness and abstraction, among the non-traditional notations for electromagnetic theory there is one that stands out: the calculus of differential

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forms. Differential forms have the advantage of behaving much like vectors, making the notation far easier to learn than, say, tensor analysis. At the same time, differential forms notation simplifies some calculations and allows for elegant pictures that clarify some of the non-intuitive aspects of electromagnetic fields. This makes differential forms an attractive supplement to vector notation, and worth spending some time with to enhance one's deeper understanding of electromagnetic theory.

Differential forms originated in the work of Hermann Günter Grassmann and Élie Cartan. In 1844, Hermann Günter Grassmann published his book *Die lineale Ausdehnungslehre, ein neuer Zweig der Mathematik* [4], in which he developed the idea of an algebra in which the symbols representing geometric entities such as points, lines, and planes are manipulated using certain rules. Grassmann introduced what is now called exterior algebra, based upon the *exterior product*

$$a \wedge b = -b \wedge a \quad (1)$$

Based on Grassmann's exterior algebra Cartan [5] developed the exterior calculus. Exterior calculus has proven to be the "natural language" of field theory since it yields simple and compact mathematical formulae, and simplifies the solution of field theoretical problems.

Differential forms establish a direct connection to geometrical images and provide additional physical insight into electromagnetism. Electromagnetic theory merges physical, mathematical, and geometrical ideas. In this complex environment creative thinking and the construction of new concepts is supported by imagery as provided by geometric models [6]. Since the introduction of the differential forms notation or exterior calculus, the connection between the mathematical notation and geometric interpretations of field and source quantities has been well established in the literature on electromagnetic field theory.

Early books that popularized the use of differential forms for electromagnetism include the old-school texts by Flanders [7] and the surprisingly fun general relativity textbook by Misner et al. [8], which combined rigorous mathematics with clever and interesting graphical illustrations. Another early text that adopted the notation was Thirring [9]. The well-known paper by Deschamps [10] did much to entice students of electromagnetism to this notation. Later came the counterculture text by Burke [11]. Early papers on various aspects of differential forms and electromagnetics include [12–19].

Since those pioneering works, a number of intermediate and advanced electromagnetics textbooks using differential forms have been published [20–22]. Mathematically oriented books include [23, 24]. One of the authors has published an introduction into basic electromagnetics principles and their applications to antenna and microwave circuit design based on exterior differential calculus [25, 26], and both authors have added the publication of a problem solving book presenting a wide range of real-world electromagnetics problems [27].

Hehl and Obukhov have published an introduction into electrodynamics based on exterior calculus [28], and Lindell has published a comprehensive treatment of differential forms in electromagnetics [29]. The breadth of the topic has been expanded by treatments of boundary conditions [30], electromagnetics in two and four dimensions [31], applications of Green's theorem [32], Green's functions [33, 34], and constitutive relations [35]. The use of differential forms in undergraduate teaching was promoted in [36]. Other relevant papers over the last two decades include [37, 38].

Numerical methods based on finite elements and other mathematical frameworks have been heavily impacted by differential forms [39–41]. In applications to numerical analysis, discrete differential forms is an area of particular noteworthiness [42–44]. Absorbing boundary conditions were treated by Teixeira and Chew [45]. Transformation optics also has a close relationship to differential forms. Theoretical treatments prefigured these results, including [46]. A formula that related the Hodge star operator in non-Euclidean geometry to the permittivity and permeability of an inhomogeneous, anisotropic medium first appeared in [47]. This formula is the basis for a relationship between the permittivity and permeability tensors needed to achieve ray propagation along prescribed paths using a curvilinear coordinate transformation [48]. Differential forms have impacted other related topics such as inverse scattering [49] and network theory [50].

In this paper, we review the notation, graphical representations, and key results that have arisen from the use of differential forms notation with Maxwell's equations and electromagnetic theory. We provide a basic introduction which largely follows the presentation in some of the pedagogically oriented papers and books cited above, and discuss some of the insights that differential forms offer relative to

key concepts from electromagnetic field theory. Finally, advanced topics including Green’s functions, potentials, and aperiodic spherical waves are treated.

## 2. DIFFERENTIAL FORMS

The scalar and vector fields used in electromagnetic theory may be represented by exterior differential forms. Differential forms are an extension of the vector concept. The use of differential forms does not necessarily replace vector analysis. The differential form representation supplies additional physical insight in addition to the conventional vector picture, and once one is familiar with the notation, it is easy to translate back and forth between vectors and differential forms.

One way to understand the difference between vector fields and differential forms is to contrast the two possible interpretations of the electric field as a force on a test charge (force picture, represented by vectors) or the change in energy experienced by the test charge as it moves through the field (energy picture, represented by differential forms). The two interpretations are not in contradiction, but complement each other in jointly providing a deep understanding of a fundamental physical property of the universe.

The energy  $W_{21}$  required to move a particle with electric charge  $q$  in an electric field with components  $E_x$ ,  $E_y$ , and  $E_z$ , from point  $\mathbf{r}_1$  to  $\mathbf{r}_2$  is obtained by the integral

$$W_{21}(t) = -q \int_{\mathbf{r}_1}^{\mathbf{r}_2} \bar{E} \cdot d\mathbf{l} \tag{2}$$

Figure 1(a) shows the path of integration for the definition of the voltage  $v_{21}$  from node 2 to node 1. The vector path integral sums up the projection of the field vector on the vectorial path element. The contribution of the integrand over each small length of the integration path is proportional to the product of the magnitudes of field vector with the infinitesimal path element and the cosine of the angle enclosed between them.

Instead of using a vector, the electric field can be represented by a linear combination of differentials, according to

$$\mathcal{E} = E_x(x, y, z, t)dx + E_y(x, y, z, t)dy + E_z(x, y, z, t)dz \tag{3}$$

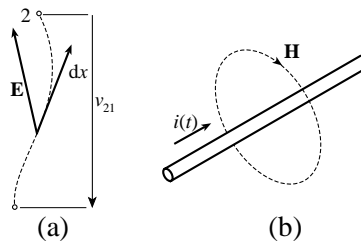
The object on the right hand side is known as a one-form, and has three independent components, similar to a vector. The key difference is that the unit vectors  $\hat{x}$ ,  $\hat{y}$ , and  $\hat{z}$  are replaced by differentials  $dx$ ,  $dy$ , and  $dz$ . The energy can be expressed in terms of the differential form as

$$W_{21}(t) = -q \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathcal{E} \tag{4}$$

which is simpler than the vector line integral, as there is no dot product or differential line element. In a sense, the dot product in (2) converts the vector into a differential that can be integrated. The integrand in (2) is already given in terms of differentials, so the dot product is not needed. In this sense, the differential form representation is more natural than the vector representation.

In the differential forms notation, the electric field is represented by a one-form. A one-form is a linear combination of differentials of each coordinate. An arbitrary one-form can be written

$$A = A_1dx + A_2dy + A_3dz. \tag{5}$$



**Figure 1.** Path of integration, (a) for the definition of the voltage and (b) for the definition of the current.

The three quantities  $A_1$ ,  $A_2$ , and  $A_3$  are the components of the one-form. Two one-forms  $A$  and  $B$  can be added, so that

$$A + B = (A_1 + B_1)dx + (A_2 + B_2)dy + (A_3 + B_3)dz. \quad (6)$$

One-forms can be integrated over paths. As shown in the introduction, we graphically represent a one-form as surfaces. The one-form  $dx$  has surfaces perpendicular to the  $x$ -axis spaced a unit distance apart. These surfaces are infinite in the  $y$  and  $z$  directions. The integral of  $dx$  over a path from the point  $(0, 0, 0)$  to  $(4, 0, 0)$  is

$$\int_0^4 dx = 4.$$

This matches the graphical representation in Figure 2(a), since the path shown in the figure crosses four surfaces. If the path were not of integer length, we would have to imagine fractional surfaces in between the unit spaced surfaces. A path from  $(0, 0, 0)$  to  $(0.25, 0, 0)$ , for example, crosses 0.25 surfaces.

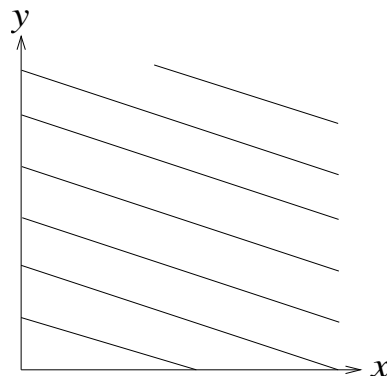
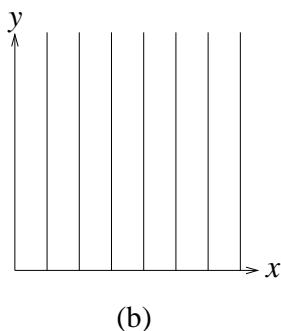
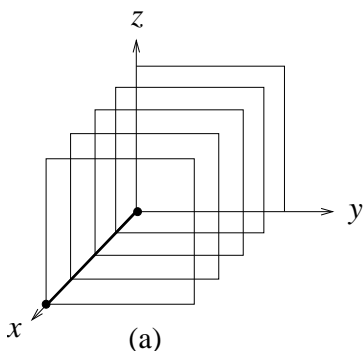
We can also think of  $dx$  as a one-form in the plane. In this case, the picture becomes a series of lines perpendicular to the  $x$ -axis spaced a unit distance apart, as shown in Figure 2(b). Graphically, integrals in the plane are similar to integrals in three dimensions: the value of a path integral is the number of lines pierced by the path.

In order to graphically integrate a one-form properly, we also have to think of the surfaces as having an orientation. The integral of the one-form  $-dx$  over a path from  $(0, 0, 0)$  to  $(4, 0, 0)$  is  $-4$ . Thus, when we count surfaces pierced by a path, we have to compare the sign of the one-form with the direction of the path in order to determine whether the surface contributes positively or negatively. The orientation of surfaces can be indicated using an arrowhead on each surface, but since the orientation is usually clear from context, to reduce clutter we do not indicate it in figures.

A more complicated one-form, such as  $3dx + 5dy$ , has surfaces that are oblique to the coordinate axes. This one-form is shown in Figure 3. The greater the magnitude of the components of a one-form, the closer the surfaces are spaced. For one-forms with components that are not constant, these surfaces can be curved. The surfaces can also originate along a line or curve and extend away to infinity, or the surfaces may be finite. In this case, the integral over a path is still the number of surfaces or fractional surfaces pierced by the path. There are one-forms that cannot be drawn as surfaces in three dimensions, but in most practical situations, the surface picture can be used.

A one-form represents a quantity which is integrated over a path, whereas vector represents a quantity with a magnitude and direction, such as displacement or velocity. Despite this difference, both types of quantities have three independent components, and can be used interchangeably in describing electromagnetic field quantities. Mathematically, vectors and differential forms are closely related. In Euclidean coordinates, we can make a correspondence between vectors and forms. The one-form  $A$  and the vector  $\mathbf{A}$  are equivalent if they have the same components:

$$A_1 dx + A_2 dy + A_3 dz \leftrightarrow A_1 \hat{\mathbf{x}} + A_2 \hat{\mathbf{y}} + A_3 \hat{\mathbf{z}}. \quad (7)$$



**Figure 2.** (a) The one-form  $dx$  integrated over a path from the point  $(0, 0, 0)$  to  $(4, 0, 0)$ . (b) The one-form  $dx$  in the plane.

**Figure 3.** The one-form  $3dx + 5dy$ .

We say that the one-form  $A$  and the vector  $\mathbf{A}$  are dual. Since it is easy to convert between the differential form and vector representations, one can choose the quantity which best suits a particular problem. We will see in the next section that in coordinate systems other than Euclidean, the duality relationship between forms and vectors changes.

The common physical interpretation of the electric field is related to the force on a point-like unit charge. This force picture yields in a natural way to the vector representation and to the visualization of the electric field via field lines. Another viewpoint is to consider the energy of a charge moved through the field. We can visualize the field via the change of the energy of a test charge moved through the field. This energy picture is more related to differential forms. Figure 4(a) shows the representation of the field via the surfaces of constant test charge energy or constant electric potential. For an electrostatic field the surfaces associated with the one-form  $\mathcal{E}$  are equipotentials. The voltage between two points 2 and 1 is given by

$$v_{21} = - \int_1^2 \mathcal{E}. \tag{8}$$

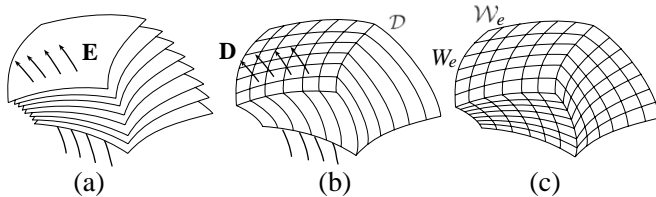
For a static electric field, the surfaces of the one-form never meet or end. In general the surfaces of a one-form also may end or meet each other, which corresponds to fields produced by sources or time-varying fields.

The magnetic field intensity can be represented also by a one-form according to

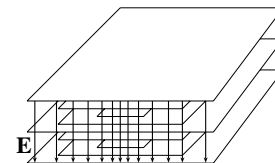
$$\mathcal{H} = H_x(x, y, z, t)dx + H_y(x, y, z, t)dy + H_z(x, y, z, t)dz \tag{9}$$

The dimension of the differential form  $\mathcal{E}$  is V and  $\mathcal{H}$  has the dimension A. The coefficients  $E_x$ ,  $E_y$ , and  $E_z$  have units of V/m, similar to the electric field intensity vector. The differentials  $dx$ ,  $dy$ , and  $dz$  have units of length, which combines with the coefficients so that the one-form  $\mathcal{E}$  has units of V. Mathematically, the differential forms  $\mathcal{E}$  and  $\mathcal{H}$  express the changes of the electric and magnetic potentials over an infinitesimal path element.

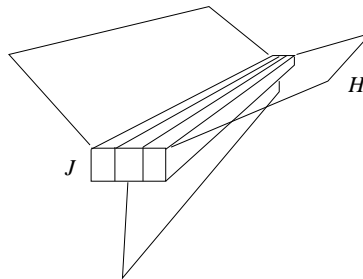
Figure 5 shows a situation we encounter in time variable fields. In the center of the structure the field intensity is higher than at its edges. New one-form surfaces appear as we move closer to the center



**Figure 4.** Geometric representation of (a) a one-form, (b) a two-form, (c) a three-form. The one-form is represented by surfaces, the two-form as tubes of flux or flow, and the three-form as cubes representing volumetric density.



**Figure 5.** One-form with ending surfaces representing a nonconservative field.



**Figure 6.** A graphical representation of the physical behavior of magnetic field and electric current: tubes of  $J$  produce surfaces of  $H$ .

of the structure. In this case the integral (8) will depend on the path from  $\mathbf{x}_1$  to  $\mathbf{x}_2$  and we cannot assign a scalar potential to the field. This type of field is sometimes referred to as nonconservative. With the vector picture, it can be challenging to determine if a particular field is nonconservative, but with the one-form surface representation, it is visually obvious.

The magnetic field produced by a line current behaves similarly, as new one-form surfaces of  $\mathcal{H}$  are created by the presence of an electric current. one-form surfaces radiate outward from the line current. Figure 1(b) shows the path of integration for the definition of the current  $i$ . The integration path passes through a number of one-form surfaces equal to the amount of current flowing through the path. The differential forms picture of the line current and associated magnetic field is shown in Figure 6. The surfaces of  $\mathcal{H}$  emanate from the tubes of electric current  $\mathcal{J}$ . With the vector picture, the vector field corresponding to the magnetic field intensity appear to rotate around the line current everywhere in space, but the curl of the vector field is actually zero everywhere except at the location of the current. The differential forms picture makes it clear that the surfaces of the magnetic field intensity one-form are created by the tubes of current.

## 2.1. Integrating One-Forms over Paths

The laws of electromagnetics are expressed in terms of integrals of fields represented by differential forms. To apply these laws, we must be able to compute the values of integrals of differential forms. Since one-forms are by definition mathematical quantities which are integrated over paths, the process of evaluating an integral of a one-form is quite natural. The key idea is that we can replace the coordinates  $x$ ,  $y$ , and  $z$  (or  $u$ ,  $v$ ,  $w$  in curvilinear coordinates) with a parametric equation for the integration path.

A parametric representation of an integration path has the form  $[f(t), g(t), h(t)]$ , so that the functions  $f$ ,  $g$ , and  $h$  give the coordinates of a point on the path for each value of the parameter  $t$ . We replace the coordinates in a one-form by these functions, and then the integral can be evaluated. For a differential, when the coordinate is replaced by a function defining the path, we then take the derivative by  $t$  to produce a new differential in the variable  $t$ . For example,  $dx$  becomes  $df = f'(t)dt$ , where the prime denotes the derivative of the function  $f(t)$  by  $t$ . The differential form now has a single differential,  $dt$ , and the integral can be performed using standard rules of calculus. Two-forms can be integrated similarly, except that the integration surface requires two parameters, and the final integrand over the parametric surface is a double integral.

## 2.2. Two-Forms, Three-Forms, and the Exterior Product

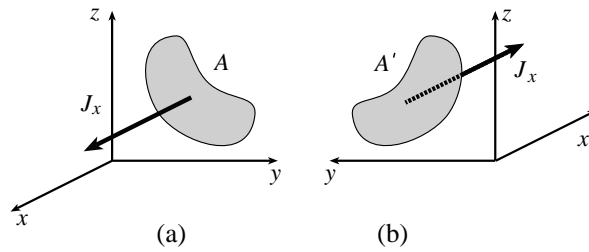
To introduce the concept of higher order differential forms, let us consider the current  $i$  flowing in  $x$ -direction through the surface  $A$  in Figure 7(a). To compute the current we have to integrate the  $x$ -component  $J_x$  of the current density over the surface  $A$  in the  $yz$ -plane

$$i = \int_A J_x dy dz. \quad (10)$$

If we integrate a current density over an area we have to consider the orientation of the area. If in Figure 7(a) the current density  $J_x$  is positive, the current  $i$  also will be positive. Inverting the direction of  $J_x$  will yield a negative current. This inversion may be performed by mirroring the coordinates with respect to the  $yz$ -plane. How do we know whether a surface integral is positive or negative? The answer is: we have to define a positive orientation. A positive oriented or right-handed Cartesian coordinate system is specified as follows: if we are looking in  $z$ -direction on the  $xy$ -plane the  $x$ -axis may be rotated clockwise by  $90^\circ$  into the  $y$ -axis. In Figure 7(a) the vector component  $J_x$  is pointing in positive orientation. In Figure 7(b) the coordinate system as well as the vector field were rotated by  $180^\circ$  around the  $z$ -axis. Physically nothing has changed. In the left figure, the vector pointing towards the observer is positive, whereas in the right figure the vector pointing away from the observer is positive.

Exterior differential forms allow one to represent the orientation of a coordinate system in a rigorous way. We introduce the exterior product or wedge product  $dy \wedge dz$  with the property

$$dy \wedge dz = -dz \wedge dy \quad (11)$$



**Figure 7.** The orientation of an area.

An exterior differential form is the exterior product of differential forms. Exterior differential forms consisting wedge products of two differentials or sums of such products are called two-forms. We may decide either  $dy \wedge dz = dydz$  or  $dy \wedge dz = -dydz$ . Deciding that

$$dy \wedge dz = dydz \tag{12}$$

assigns to  $dy \wedge dz$  the positive orientation and to  $dz \wedge dy$  the negative orientation. The integral (10) can now be written in the orientation-independent form

$$i = \int_A J_x dy \wedge dz \tag{13}$$

Physically, a two-form is a quantity which is integrated over a two-dimensional surface. A quantity representing flow of a fluid, for example, has units of flow rate per area, and would be integrated over a surface to find the total flow rate through the surface. Similarly, the integral of electric flux density over a surface is the total flux through the surface, which has units of charge. A general two-form  $C$  is written as

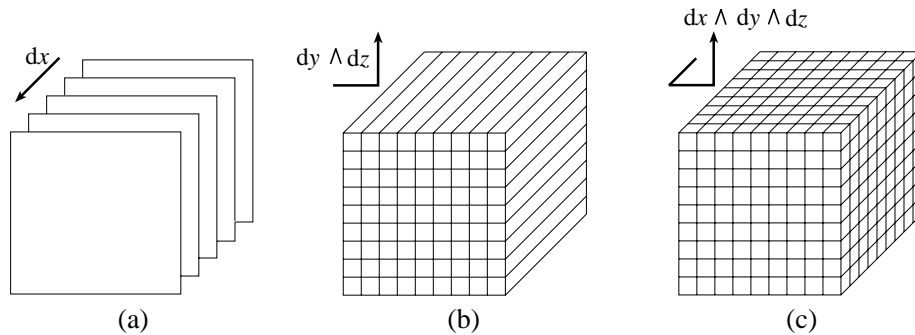
$$C = C_1 dy \wedge dz + C_2 dz \wedge dx + C_3 dx \wedge dy. \tag{14}$$

For convenience, we usually use the antisymmetry of the exterior product to put the differentials of two-forms into right cyclic order, as in (14).

We can use the exterior product to create differential forms of higher order as well. An exterior differential form of order  $p$  is called a  $p$ -form and has  $p$  differentials in each term. In  $n$ -dimensional space the order of a differential form may assume values  $0, \dots, n$ . In differential form notation a clear distinction between scalars, pseudoscalars, polar vectors and axial vectors is made. Scalars are represented by zero-forms, pseudoscalars by three-forms, polar vectors by one-forms and axial vectors by two-forms. For a  $p$ -form  $\mathcal{U}$  and a  $q$ -form  $\mathcal{V}$  the commutation relation for the exterior product is

$$\mathcal{U} \wedge \mathcal{V} = (-1)^{p+q+1} \mathcal{V} \wedge \mathcal{U} \tag{15}$$

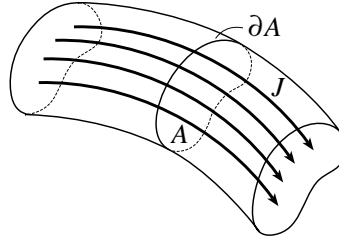
Figure 8 shows the fundamental 1-, 2- and three-forms in Cartesian coordinates.



**Figure 8.** The fundamental (a) one-form, (b) two-form, (c) three-form in Cartesian coordinates.

### 2.3. Current Density Two-Form

We can describe the flow of current in a conductor by a current density vector field  $\mathbf{J}(\mathbf{x}) = [J_x(\mathbf{x}), J_y(\mathbf{x}), J_z(\mathbf{x})]^T$ . The current flows along the current density field lines going through the boundary  $\partial A$  of the area  $A$  as shown in Figure 9. In the differential forms representation, the current is depicted as tubes of flow. Figure 4(b) shows the tube representation of a two-form. The two-form is visualized as a bundle of tubes carrying the current. Each tube carries a unit amount of current, and the current density is inversely proportional to the cross-sectional area of the tubes. Figure 8(b) shows the tube representations of the fundamental two-form  $dy \wedge dz$ . The tubes are made up of the one-form surfaces of  $dy$  and  $dz$ .



**Figure 9.** Current flow.

If the surface  $S$  is an arbitrarily oriented curved surface in three-dimensional space and the current density vector has the  $x$ -,  $y$ - and  $z$ -components  $J_x$ ,  $J_y$  and  $J_z$ , the total current flowing through  $S$  is

$$i = \int_S J_x dy \wedge dz + J_y dz \wedge dx + J_z dx \wedge dy. \quad (16)$$

The first term of the integrand concerns the integration of the  $x$ -component of the current density over the projection of the surface  $S$  on the  $yz$ -plane and so forth.

Let us introduce the current density two-form  $\mathcal{J}$

$$\mathcal{J} = J_x dy \wedge dz + J_y dz \wedge dx + J_z dx \wedge dy \quad (17)$$

The total current  $i$  flowing through  $S$  may be expressed in a compact notation as the integral of the differential form  $\mathcal{J}$

$$i = \int_S \mathcal{J} \quad (18)$$

The dimension of the current density differential form  $\mathcal{J}$  is A. The electric flux density two-form  $\mathcal{D}$  has the dimension As, and the magnetic flux density two-form  $\mathcal{B}$  has the dimension Vs. These differential forms represent the current or the flux through an infinitesimal area element.

### 2.4. Charge Density Three-Form

The electric charge  $q$  is given by the volume integral of the electric charge density  $\rho$ . For the electric charge density we introduce the electric charge density three-form

$$\mathcal{Q} = \rho dx \wedge dy \wedge dz \quad (19)$$

Figure 4(c) shows the graphic visualization of a three-form by subdividing the volume into cells. The cell volume is inversely proportional to the charge density. The charge density form  $\mathcal{Q}$  with the dimension As represents the charge in an infinitesimal volume element. We obtain the total charge  $q$  by performing the volume integral over the three-form  $\mathcal{Q}$ :

$$q = \int_V \mathcal{Q} \quad (20)$$



### 2.5. Electromagnetic Fields and Sources

Tables 1 and 2 summarize the  $p$ -forms that describe key electromagnetic field and source quantities. The electric and magnetic fields are represented by both a one-form and a two-form. The one-form and two-form representations capture different physical properties of the field. The one-form represents the energy picture and shows the change in potential energy as a test charge moves across the one-form surfaces. The two-form shows the flux of the field that extends from positive charges to negative charges.

**Table 1.** Differential forms that represent electromagnetic fields and sources with units and the corresponding vector quantities.

Quantity	Form	Type	Units	Vector/Scalar
Electric Field Intensity	$\mathcal{E}$	one-form	V	<b>E</b>
Magnetic Field Intensity	$\mathcal{H}$	one-form	A	<b>H</b>
Electric Flux Density	$\mathcal{D}$	two-form	C	<b>D</b>
Magnetic Flux Density	$\mathcal{B}$	two-form	Wb	<b>B</b>
Electric Current Density	$\mathcal{J}$	two-form	A	<b>J</b>
Electric Charge Density	$\mathcal{Q}$	three-form	C	$\rho$

**Table 2.** Differential forms of electromagnetism expanded in components.

$$\begin{aligned}
 \mathcal{E} &= E_x dx + E_y dy + E_z dz \\
 \mathcal{H} &= H_x dx + H_y dy + H_z dz \\
 \mathcal{D} &= D_x dy \wedge dz + D_y dz \wedge dx + D_z dx \wedge dy \\
 \mathcal{B} &= B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy \\
 \mathcal{J} &= J_x dy \wedge dz + J_y dz \wedge dx + J_z dx \wedge dy \\
 \mathcal{S} &= S_x dy \wedge dz + S_y dz \wedge dx + S_z dx \wedge dy \\
 \mathcal{Q} &= \rho dx \wedge dy \wedge dz
 \end{aligned}$$

### 2.6. Hodge Star Operator and the Constitutive Relations

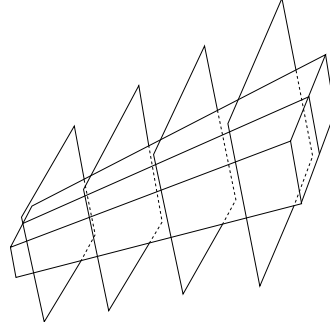
Since the electric field intensity  $\mathcal{E}$  and the electric flux density  $\mathcal{D}$  represent the same field, it is clear that there must be some relationship between  $\mathcal{E}$  and  $\mathcal{D}$ . This relationship is expressed using the Hodge star operator. Simply stated, the star operator works by taking a differential form and converting it to a new form with the “missing” differentials. For one-forms and two-forms,

$$\begin{aligned}
 \star dx &= dydz, & \star dydz &= dx \\
 \star dy &= dzdx, & \star dzdx &= dy \\
 \star dz &= dxdy, & \star dxdy &= dz
 \end{aligned}$$

The one-form  $\mathcal{A}$  and the two-form  $\star\mathcal{A}$  are both dual to the same vector. The star operator applied twice is the identity operator, so that  $\star\star\mathcal{A} = \mathcal{A}$  for any form  $\mathcal{A}$ .

The star operator has a simple graphical interpretation. Considering the fundamental one-form  $dx$ , the surfaces of  $dx$  are perpendicular to the  $x$  direction. Applying the star operator gives  $dydz$ , which has tubes in the  $x$  direction, so that the surfaces of  $dx$  are perpendicular to the tubes of  $\star dx$ . In general, the star operator acting on one-forms and two-forms converts surfaces into perpendicular tubes and tubes into perpendicular surfaces. This is illustrated in Figure 10.

The relation between the electric field intensity one-form  $\mathcal{E}$  and the electric flux density  $\mathcal{D}$  depends on the material properties of the medium and the metric properties of space. In an isotropic medium, the electric field intensity vector and the flux density vector are proportional, and it is not always clear why the two different quantities are needed for the electric field. With differential forms notation,



**Figure 10.** The star operator converts one-form surfaces into perpendicular two-form tubes.

field intensity and flux density are represented by different mathematical objects. The mathematical relationship between  $\mathcal{E}$  and  $\mathcal{D}$  is known as a constitutive relation. The constitutive relation is written using the Hodge star operator as

$$\mathcal{D} = \epsilon \star \mathcal{E} \quad (21)$$

where  $\epsilon$  is the permittivity of a medium. The permittivity  $\epsilon$  accounts for the material properties whereas the Hodge operator  $\star$  accounts for the metric properties of the space or the chosen coordinate system, respectively. This equation shows that surfaces of  $\mathcal{E}$  yield perpendicular tubes of  $\mathcal{D}$  (see Figure 10). The constant scales the sizes of the tubes without changing their direction. The magnetic field constitutive relation is

$$\mathcal{B} = \mu \star \mathcal{H}. \quad (22)$$

where  $\mu$  is the permeability of the medium. The differential forms notation helps to elucidate why two different representations are used for one physical field. With vector notation, flux and field intensity are both vectors, but with differential forms, they become one-forms and two-forms, with different geometrical representations and physical interpretations.

### 3. MAXWELL'S EQUATIONS IN INTEGRAL FORM

The integral form of Maxwell's equations is given by:

$$\oint_{\partial S} \mathcal{H} = \frac{d}{dt} \int_S \mathcal{D} + \int_S \mathcal{J}, \quad \text{Ampère's Law} \quad (23)$$

$$\oint_{\partial S} \mathcal{E} = -\frac{d}{dt} \int_S \mathcal{B}, \quad \text{Faraday's Law} \quad (24)$$

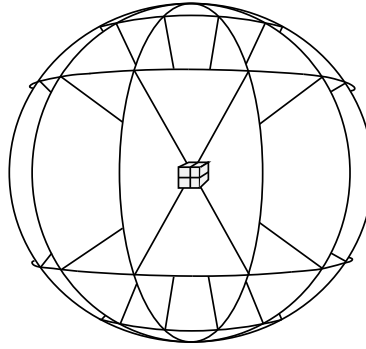
$$\oint_{\partial V} \mathcal{B} = 0, \quad \text{Magnetic Flux Continuity} \quad (25)$$

$$\oint_{\partial V} \mathcal{D} = \int_V \mathcal{Q}. \quad \text{Gauss' Law} \quad (26)$$

Ampère's law relates current to magnetic field. It states that the sum of conduction and displacement currents through an area  $S$  equals the magnetic tension around the boundary  $\partial S$ . Faraday's law relates the magnetic flux to the electric field. It says that the time derivative of the magnetic flux through an area  $S$  equals the electric tension around the boundary  $\partial S$ . Both equations are tied together via the constitutive equations relating flux densities to field intensities.

#### 3.1. Point Charge

As a simple example of the application of Maxwell's laws in integral form, we consider the electric flux produced by a point charge. The tubes of flux from a point charge  $Q$  extend out radially from the



**Figure 11.** Electric flux density due to a point charge. Tubes of  $D$  extend out radially from the charge.

charge (Figure 11). Thus, the flux density two-form  $D$  has to be a multiple of the differentials  $d\theta d\phi$  in the spherical coordinate system. The tubes of  $d\theta d\phi$  are denser near the poles  $\theta = 0$  and  $\theta = \pi$  than at the equator. We need to include the correction factors  $r^2 \sin \theta$  to have tubes with the same density everywhere in space. Thus,  $\mathcal{D}$  has the form

$$\mathcal{D} = D_0 r^2 \sin \theta d\theta d\phi \tag{27}$$

where  $D_0$  is a constant.

To apply Gauss law, we will choose the surface  $S$  to be a sphere around the charge. The right-hand side is

$$\int_V \mathcal{Q} = Q \tag{28}$$

where  $V$  is the volume inside the sphere. The left-hand side is

$$\int_S \mathcal{D} = \int_0^{2\pi} \int_0^\pi D_0 r^2 \sin \theta d\theta d\phi = 4\pi r^2 D_0.$$

By Gauss’s law, we know that  $4\pi r^2 D_0 = Q$ . Solving for  $D_0$  and substituting into (27), we obtain

$$\mathcal{D} = \frac{Q}{4\pi} \sin \theta d\theta d\phi \tag{29}$$

for the electric flux density due to the point charge. Since  $4\pi$  is the total amount of solid angle for a sphere and  $\sin \theta d\theta d\phi$  is the differential element of solid angle, this expression implies that the amount of flux per solid angle is the same at any distance from the charge.

#### 4. THE EXTERIOR DERIVATIVE

In this section, we define the exterior derivative operator, which will allow us to express Maxwell’s laws as differential equations. This operator has the symbol  $d$ , and acts on a  $p$ -form to produce a new form with degree  $p + 1$ . This  $(p + 1)$ -form characterizes the spatial variation of the  $p$ -form.

The exterior derivative operator can be written as

$$d = \left( \frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz \right) \wedge \tag{30}$$

The  $d$  operator is similar to a one-form, except that the coefficients are partial derivative operators instead of functions. When this operator is applied to a differential form, the derivatives act on the coefficients of the form, and the differentials combine with those of the form according to the properties of the exterior product.

The sum and product rules for exterior differentiation are:

$$d(\mathcal{U} + \mathcal{V}) = d\mathcal{U} + d\mathcal{V}, \tag{31a}$$

$$d(\mathcal{U} \wedge \mathcal{V}) = d\mathcal{U} \wedge \mathcal{V} + (-1)^{(\deg \mathcal{U})} \mathcal{U} \wedge d\mathcal{V}. \tag{31b}$$

The exterior derivatives of  $p$ -forms are

zero-form:

$$df(\mathbf{x}) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz,$$

one-form:

$$d\mathcal{U}(\mathbf{x}) = \left( \frac{\partial U_z}{\partial y} - \frac{\partial U_y}{\partial z} \right) dy \wedge dz + \left( \frac{\partial U_x}{\partial z} - \frac{\partial U_z}{\partial x} \right) dz \wedge dx + \left( \frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y} \right) dx \wedge dy,$$

two-form:

$$d\mathcal{V}(\mathbf{x}) = \left( \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right) dx \wedge dy \wedge dz,$$

three-form:

$$d\mathcal{Q}(\mathbf{x}) = 0.$$

Computing exterior derivatives is straightforward. One takes the partial derivative of a differential form by  $x$  and adds the differential  $dx$  from the left, repeats for  $y$  and  $z$ , and adds the three results. This process is very similar to implicit differentiation, except that one must combine new differentials with existing differentials using the exterior product. When taking the exterior derivative of a one-form or two-form, some terms may drop out due to repeated differentials. Table 3 shows the correspondence between several vector derivative and exterior derivative operations.

**Table 3.** Differential operators.

<i>Vector Differential Operator</i>	<i>Exterior Differential Operator</i>
grad $f$	$df$
curl $\mathbf{A}$	$d\mathcal{A}$
div $\mathbf{B}$	$d\mathcal{B}$
curl grad $f = \mathbf{0}$	$d df = 0$
div curl $\mathbf{A} = 0$	$d d\mathcal{A} = 0$
div grad $f$	$d \star df$ or $\star d \star df$
curl curl $\mathbf{A}$	$d \star d\mathcal{A}$ or $\star d \star d\mathcal{A}$

## 5. POINCARÉ'S LEMMA

A form  $\mathcal{V}$  for which  $d\mathcal{V} = 0$  is said to be closed, and a form  $\mathcal{V}$  for which  $\mathcal{V} = d\mathcal{U}$  is said to be exact. For differential forms the statement  $\mathcal{V} = d\mathcal{U}$  implies  $d\mathcal{V} = 0$ . The relation

$$dd\mathcal{U} = 0 \tag{32}$$

may be verified easily. In conventional vector notation this corresponds to  $\text{curl grad} = \mathbf{0}$  and  $\text{div curl} = 0$ . All exact forms are closed. However it may also be shown that all closed forms are exact. Poincaré's lemma states that

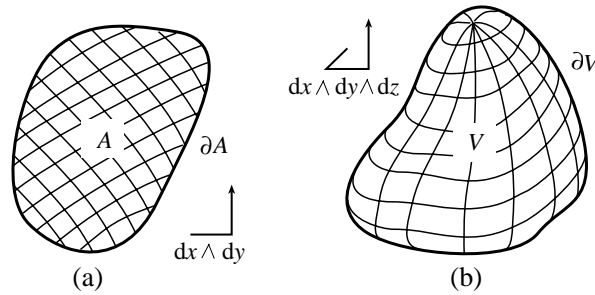
$$d\mathcal{V} = 0 \quad \Leftrightarrow \quad \mathcal{V} = d\mathcal{U}. \tag{33}$$

If a differential form has zero exterior derivative, then it is itself the exterior derivative of another form.

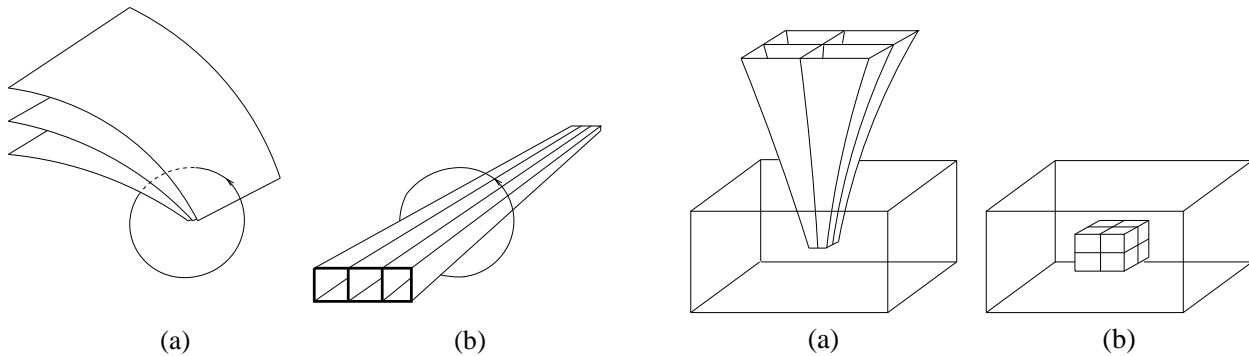
## 6. STOKES' THEOREM

In (23) and (24) line integrals over the boundary of the surface  $A$  are related to surface integrals over  $A$ . Figure 12(a) shows the relation between the orientation of the area  $A$  and the boundary  $\partial A$ . The line integral over the closed contour  $\partial A$  is called circulation. In (25) and (26) the surface integrals are performed over the boundary  $\partial V$  of the volume  $V$ . Figure 12(b) shows the orientation of the boundary surface  $\partial V$ . The Stokes' theorem relates the integration of a  $p$ -form  $\mathcal{U}$  over the closed  $p$ -dimensional boundary  $\partial V$  of a  $(p+1)$ -dimensional volume  $V$  to the volume integral of  $d\mathcal{U}$  over  $V$  via

$$\oint_{\partial V} \mathcal{U} = \int_V d\mathcal{U}. \tag{34}$$



**Figure 12.** (a) Area  $A$  with boundary  $\partial A$  and (b) volume  $V$  with boundary  $\partial V$ .



**Figure 13.** Stokes theorem for  $\mathcal{U}$  a one-form. (a) The loop  $\partial V$  pierces three of the surfaces of  $\mathcal{U}$ . (b) Three tubes of  $d\mathcal{U}$  pass through a surface  $V$  bounded by the loop  $\partial V$ .

**Figure 14.** Stokes theorem for  $\mathcal{U}$  a two-form. (a) Four tubes of  $\mathcal{U}$  pass through the closed, rectangular surface  $\partial V$ . (b) Four boxes of the three-form  $d\mathcal{U}$  lie inside the surface.

This summarizes the Stokes' theorem and the Gauss' theorem of conventional vector notation.

The property of Stokes' theorem to relate the integration of a  $p$ -form over a closed boundary of a volume to the integration of the exterior derivative of the  $p$ -form over the volume enclosed by that boundary yields an interesting geometric interpretation of Poincaré's lemma: since a boundary has no boundary the iterated application of the exterior derivative yields zero value.

If  $\mathcal{U}$  is a zero-form, then Stokes' theorem states that  $\int_a^b d\mathcal{U} = f(b) - f(a)$ . This is the fundamental theorem of calculus. If  $\mathcal{U}$  is a one-form, then  $\partial V$  has to be a closed path.  $V$  is a surface that has the path as its boundary. Graphically, Stokes' theorem says that the number of surfaces of  $\mathcal{U}$  pierced by the path is equal to the number of tubes of the two-form  $d\mathcal{U}$  that pass through the path (Figure 13).

If  $\mathcal{U}$  is a two-form, then  $\partial V$  is a closed surface and  $V$  is the volume inside it. Stokes' theorem requires that the number of tubes of  $\mathcal{U}$  that cross the surface is equal to the number of boxes of  $d\mathcal{U}$  inside the surface, as shown in Figure 14.

## 7. CURVILINEAR COORDINATES

In exterior calculus the field equations may be formulated without reference to a specific coordinate system. Depending on the problem the choice of a specific coordinate system may simplify the problem solution considerably. We introduce an orthogonal curvilinear coordinate system

$$u = u(x, y, z), \quad v = v(x, y, z), \quad w = w(x, y, z). \tag{35}$$

The coordinate curves are obtained by setting two of the three coordinates  $u, v, w$  constant. Coordinate surfaces are defined by setting one of the three coordinates constant. In an orthogonal coordinate system in any point (except singular points) of the space the three coordinate curves are orthogonal. The same

holds for the three coordinate surfaces going through any point. The differentials  $dx$ ,  $dy$ ,  $dz$  by the differentials  $du$ ,  $dv$ ,  $dw$  are related to

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv + \frac{\partial x}{\partial w} dw, \quad (36)$$

$$dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv + \frac{\partial y}{\partial w} dw, \quad (37)$$

$$dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv + \frac{\partial z}{\partial w} dw. \quad (38)$$

The rules for transformation of the Cartesian basis two-forms  $dx \wedge dy$ ,  $dy \wedge dz$ ,  $dz \wedge dx$  and the Cartesian basis three-form  $dx \wedge dy \wedge dz$  follow directly from the above equations by applying the rules of the exterior product. Using the metric coefficients  $g_1$ ,  $g_2$  and  $g_3$

$$g_1^2 = \frac{\partial \mathbf{x}}{\partial u} \cdot \frac{\partial \mathbf{x}}{\partial u}, \quad g_2^2 = \frac{\partial \mathbf{x}}{\partial v} \cdot \frac{\partial \mathbf{x}}{\partial v}, \quad g_3^2 = \frac{\partial \mathbf{x}}{\partial w} \cdot \frac{\partial \mathbf{x}}{\partial w} \quad (39)$$

we introduce the unit one-forms

$$\mathbf{s}_1 = g_1 du, \quad \mathbf{s}_2 = g_2 dv, \quad \mathbf{s}_3 = g_3 dw. \quad (40)$$

The integral of  $\mathbf{s}_1 = g_1 du$  along any path with  $v$  and  $w$  constant yields the length of the path.

For the curvilinear unit differentials the Hodge star operator is

$$\begin{aligned} \star f &= f \mathbf{s}_1 \wedge \mathbf{s}_2 \wedge \mathbf{s}_3, \\ \star (A_u \mathbf{s}_1 + A_v \mathbf{s}_2 + A_w \mathbf{s}_3) &= A_u \mathbf{s}_2 \wedge \mathbf{s}_3 + A_v \mathbf{s}_3 \wedge \mathbf{s}_1 + A_w \mathbf{s}_1 \wedge \mathbf{s}_2, \\ \star (A_u \mathbf{s}_2 \wedge \mathbf{s}_3 + A_v \mathbf{s}_3 \wedge \mathbf{s}_1 + A_w \mathbf{s}_1 \wedge \mathbf{s}_2) &= A_u \mathbf{s}_1 + A_v \mathbf{s}_2 + A_w \mathbf{s}_3, \\ \star (f \mathbf{s}_1 \wedge \mathbf{s}_2 \wedge \mathbf{s}_3) &= f. \end{aligned} \quad (41)$$

The application of the Hodge operator fully accounts for the metric properties of the used coordinate system. In contrast to this the exterior derivative is independent of the metric. This is a substantial advantage of exterior calculus.

### 7.1. Cylindrical Coordinates

In the cylindrical coordinate system, a point in space is specified by the radial distance of its  $x$ ,  $y$  coordinates  $\rho = \sqrt{x^2 + y^2}$ , angle from the  $+x$  axis in the  $x$ - $y$  plane  $\phi$ , and height in the  $z$  direction. The differentials of the cylindrical coordinate system are  $d\rho$ ,  $d\phi$  and  $dz$ . To convert forms into unit vectors, the angular differential  $d\phi$  must be made into a unit differential  $\rho d\phi$ . One-forms correspond to vectors by the rules

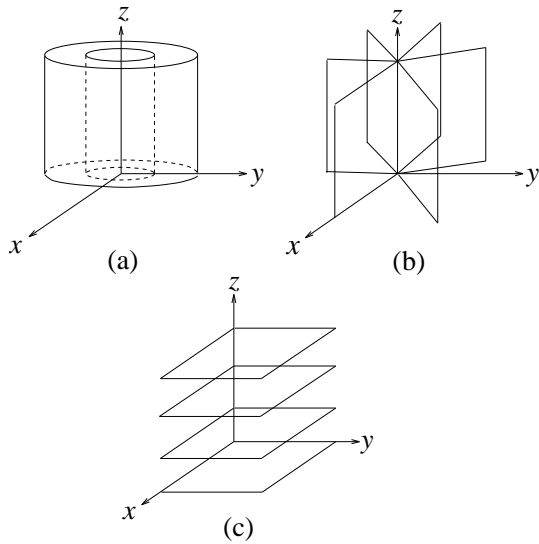
$$\begin{aligned} d\rho &\leftrightarrow \hat{\boldsymbol{\rho}} \\ \rho d\phi &\leftrightarrow \hat{\boldsymbol{\phi}} \\ dz &\leftrightarrow \hat{\boldsymbol{z}} \end{aligned}$$

Figure 15 shows the pictures of the differentials of the cylindrical coordinate system. The two-forms can be obtained by superimposing these surfaces. Tubes of  $dz \wedge d\rho$ , for example, are square donut-shaped and point in the  $\phi$  direction.

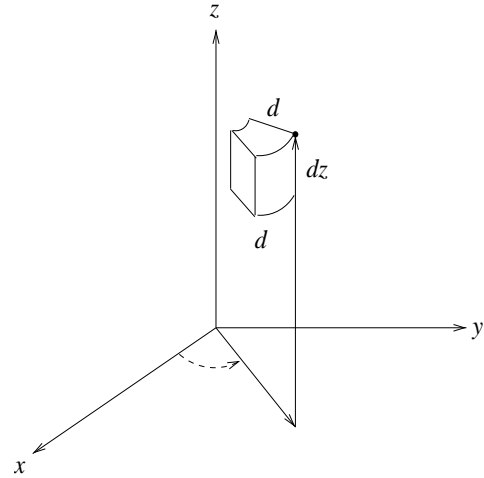
In curvilinear coordinates, differentials must be converted to unit differentials before the star operator is applied. The star operator in cylindrical coordinates acts as follows:

$$\begin{aligned} \star d\rho &= \rho d\phi \wedge dz \\ \star \rho d\phi &= dz \wedge d\rho \\ \star dz &= d\rho \wedge \rho d\phi \end{aligned}$$

so that  $\rho$  must be included with  $d\phi$  before the star operator can be applied to make it a two-form with unit differentials. Also,  $\star 1 = \rho d\rho d\phi dz$ . As with the rectangular coordinate system,  $\star\star = 1$ , so that this same table can be used to convert two-forms to one-forms.



**Figure 15.** Surfaces of  $d\rho$ ,  $d\phi$  scaled by  $3/\pi$  and  $dz$ .



**Figure 16.** Unit differentials in cylindrical coordinates represented as faces of a differential volume.

### 7.2. Spherical Coordinates

In the spherical coordinate system, a point in space is specified by the radial distance from the origin  $r = \sqrt{x^2 + y^2 + z^2}$ , angle from the  $+x$  axis in the  $x-y$  plane  $\phi$ , and angle from the  $z$  axis  $\theta$ . The differentials of the spherical coordinate system are  $dr$ ,  $d\theta$  and  $d\phi$ . To convert forms into unit vectors, the angular differentials must be made into unit differentials  $r d\theta$  and  $r \sin \theta d\phi$ . One-forms correspond to vectors by the rules

$$\begin{aligned} dr &\leftrightarrow \hat{r} \\ r d\theta &\leftrightarrow \hat{\theta} \\ r \sin \theta d\phi &\leftrightarrow \hat{\phi} \end{aligned}$$

Figure 17 shows the pictures of the differentials of the spherical coordinate system.

As in the cylindrical coordinate system, differentials of the spherical coordinate system must be converted to length elements before the star operator is applied. The star operator acts on one-forms and two-forms as follows:

$$\begin{aligned} \star dr &= r d\theta \wedge r \sin \theta d\phi \\ \star r d\theta &= r \sin \theta d\phi \wedge dr \\ \star r \sin \theta d\phi &= dr \wedge r d\theta \end{aligned}$$

Again,  $\star\star = 1$ , so that this table can be used to convert two-forms to one-forms. The star operator applied to one is  $r^2 \sin \theta dr d\theta d\phi$ .

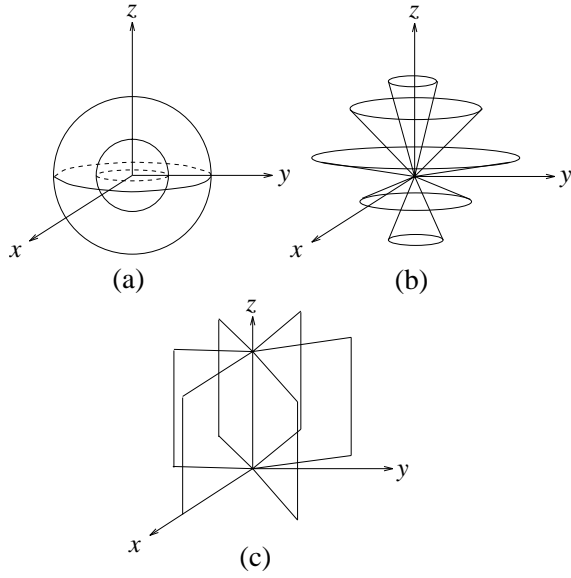
### 7.3. Exterior Derivative in Curvilinear Coordinates

Regardless of the particular coordinate system, the form of the exterior derivative operator remains the same. If the coordinates are  $(u, v, w)$ , the exterior derivative operator is

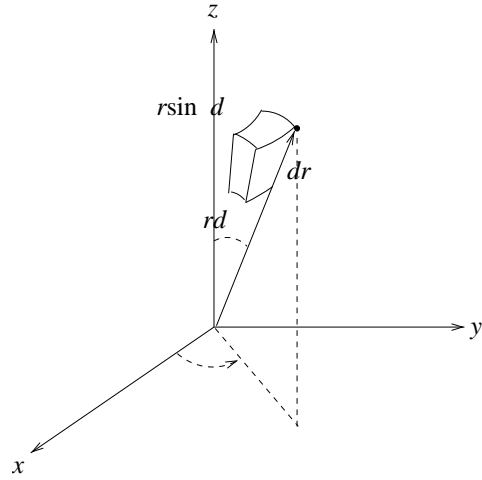
$$d = \left( \frac{\partial}{\partial u} du + \frac{\partial}{\partial v} dv + \frac{\partial}{\partial w} dw \right) \wedge \tag{42}$$

In cylindrical coordinates,

$$d = \left( \frac{\partial}{\partial \rho} d\rho + \frac{\partial}{\partial \phi} d\phi + \frac{\partial}{\partial z} dz \right) \wedge \tag{43}$$



**Figure 17.** Surfaces of  $dr$ ,  $d\theta$  scaled by  $10/\pi$  and  $d\phi$  scaled by  $3/\pi$ .



**Figure 18.** Unit differentials in spherical coordinates represented as faces of a differential volume.

which is the same as for rectangular coordinates but with the coordinates  $\rho, \phi, z$  in the place of  $x, y, z$ . Note that the factor  $\rho$  associated with  $d\phi$  must be present when converting to vectors or applying the star operator, but is not found in the exterior derivative operator.

The exterior derivative in spherical coordinates is

$$d = \left( \frac{\partial}{\partial r} dr + \frac{\partial}{\partial \theta} d\theta + \frac{\partial}{\partial \phi} d\phi \right) \wedge. \tag{44}$$

From these expressions, we see that computing exterior derivatives in a curvilinear coordinate system is no different from computing in rectangular coordinates.

### 8. MAXWELL'S EQUATIONS IN LOCAL FORM

Applying Stokes' theorem to the integral form of Maxwell's Equations (23) to (26) we obtain the differential representation of Maxwell's equations:

$$d\mathcal{H} = \frac{\partial}{\partial t} \mathcal{D} + \mathcal{J}, \quad \text{Ampère's Law} \tag{45}$$

$$d\mathcal{E} = -\frac{\partial}{\partial t} \mathcal{B}, \quad \text{Faraday's Law} \tag{46}$$

$$d\mathcal{B} = 0, \quad \text{Magnetic Flux Continuity} \tag{47}$$

$$d\mathcal{D} = \mathcal{Q}. \quad \text{Gauss' Law} \tag{48}$$

Figure 19 shows the graphical representation of Maxwell's equations after Deschamps [51]. We have two groups of equations, namely Faraday's law and magnetic flux continuity on the left-hand side of the diagram and Ampère's law and Gauss' law on the right-hand side. Within each group exterior differentiation is metric-independent and increases the order of the respective  $p$ -forms. Both groups are related via the constitutive relations depending on the material properties and the coordinate metrics and either incrementing or decrementing the order of the respective  $p$ -forms.



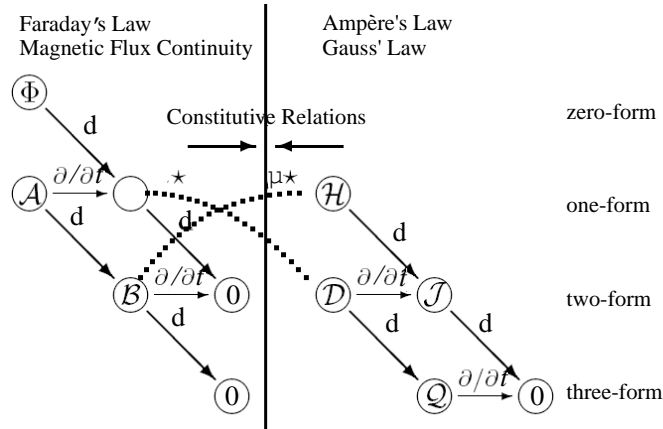


Figure 19. Graphical representation of Maxwell's equations.

### 9. BOUNDARY CONDITIONS

If a magnetic field changes abruptly along some boundary surface, Maxwell's laws require that an electric current flow along the boundary to account for the step in field intensity. Similarly, Maxwell's laws restrict the possible discontinuity in the electric field at a boundary. In this section, we derive expressions for these boundary conditions on  $\mathcal{E}$ ,  $\mathcal{H}$ ,  $\mathcal{D}$ , and  $\mathcal{B}$ .

#### 9.1. Field Intensity

In this section, we derive boundary conditions for the electric and magnetic field intensity one-forms  $\mathcal{E}$  and  $\mathcal{H}$ . As in Figure 20, we denote the magnetic field on one side of a boundary as  $\mathcal{H}_1$ , and the field on the other side as  $\mathcal{H}_2$ . If we choose an Amperian contour with one side just above and the other just below the boundary, as shown in Figure 20, the left hand side of Ampere's law becomes

$$\oint_C \mathcal{H} = \int_P (\mathcal{H}_1 - \mathcal{H}_2) \tag{49}$$

in the limit as the width of the contour goes to zero and the sides of the contour meet each other along a common path  $P$  on the boundary. The right hand side is equal to the surface current flowing across the path  $P$ , so that

$$\int_P (\mathcal{H}_1 - \mathcal{H}_2) = \int_P \mathcal{J}_s \tag{50}$$

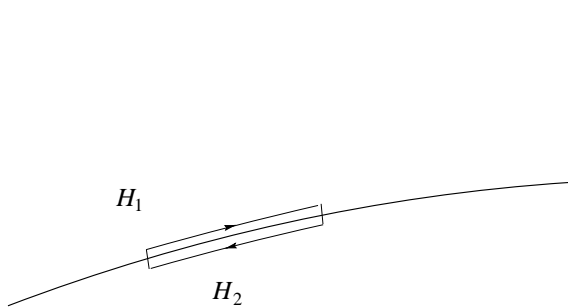


Figure 20. A discontinuity in the magnetic field above and below a boundary. The surface current flowing on the boundary can be found using an Amperian contour with infinitesimal width.

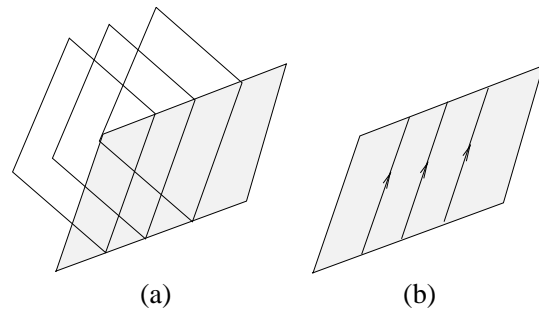


Figure 21. (a) The one-form  $\mathcal{H}_1 - \mathcal{H}_2$ . (b) The one-form  $\mathcal{J}_s$ , represented by lines on the boundary. Current flows along the lines.

where  $J\mathcal{J}_s$  is a one-form representing the surface current density on the boundary.

Since (50) holds for any path  $P$  on the boundary, the integrands must be equal on the boundary. We thus have that

$$\mathcal{J}_s = (\mathcal{H}_1 - \mathcal{H}_2)|_B \quad (51)$$

where the right hand side is the restriction of the magnetic field discontinuity to the boundary. The one-form  $\mathcal{J}_s$  is represented graphically by the lines along which the one-form  $\mathcal{H}_1 - \mathcal{H}_2$  intersects the boundary, as shown in Figure 21. Current flows along these lines. If the surfaces of  $\mathcal{H}_1 - \mathcal{H}_2$  are parallel to the boundary, then the surfaces do not intersect, and the restriction is zero. Thus, (51) represents the tangential component of the magnetic field discontinuity. The direction of current flow along these lines can be obtained using the right hand rule: if the right hand is on the boundary and the fingers point in the direction of  $\mathcal{H}_1 - \mathcal{H}_2$ , then the thumb points in the direction of current flow.

In order to compute the restriction mathematically, we employ an expression of the form  $z = f(x, y)$  to represent the boundary, and replace all occurrences of the variable  $z$  in  $\mathcal{H}_1 - \mathcal{H}_2$  with the function  $f(x, y)$ , so that

$$\begin{aligned} \mathcal{J}_s &= [\mathcal{H}_1(x, y, z) - \mathcal{H}_2(x, y, z)]|_{z=f(x, y)} \\ &= [H_{1x}(x, y, f) - H_{2x}(x, y, f)]dx + [H_{1y}(x, y, f) - H_{2y}(x, y, f)]dy + [H_{1z}(x, y, f) - H_{2z}(x, y, f)]df \\ &= \left[ H_{1x} - H_{2x} + \frac{\partial f}{\partial x}(H_{1z} - H_{2z}) \right] dx + \left[ H_{1y} - H_{2y} + \frac{\partial f}{\partial y}(H_{1z} - H_{2z}) \right] dy \end{aligned}$$

If part of the boundary is parallel to the  $x-y$  plane, then the boundary must be expressed as  $x = g(y, z)$  or  $y = h(x, z)$ .

In a similar manner, we can show that the electric field satisfies the boundary condition

$$(\mathcal{E}_1 - \mathcal{E}_2)|_B = 0 \quad (52)$$

This condition requires that the tangential component of the electric field above and below a boundary must be equal at the boundary.

## 9.2. Flux Density

From Gauss's law, it can be shown that the electric flux density satisfies the boundary condition

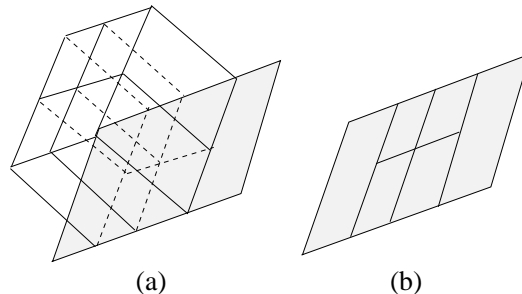
$$(\mathcal{D}_1 - \mathcal{D}_2)|_B = \mathcal{Q}_s \quad (53)$$

where  $\mathcal{Q}_s$  is a two-form representing the density of electric surface charge on the boundary. This two-form is represented graphically as boxes which are the intersection of the tubes of  $\mathcal{D}_1 - \mathcal{D}_2$  with the boundary, as in Figure 22. If the tubes of the magnetic flux discontinuity are parallel to the boundary, then the tubes do not intersect and the restriction is zero. The left-hand side of (53) is the component of the jump in flux which is normal to the boundary.

The magnetic flux density satisfies the boundary condition

$$(\mathcal{B}_1 - \mathcal{B}_2)|_B = 0 \quad (54)$$

so that the normal components of the magnetic flux above and below a boundary must be equal.



**Figure 22.** (a) The two-form  $\mathcal{D}_1 - \mathcal{D}_2$ . (b) The two-form  $\mathcal{Q}_s$ , represented by boxes on the boundary.

We collect all of the boundary conditions for reference:

$$\begin{aligned} (\mathcal{E}_1 - \mathcal{E}_2)|_B &= 0 \\ (\mathcal{H}_1 - \mathcal{H}_2)|_B &= \mathcal{J}_s \\ (\mathcal{D}_1 - \mathcal{D}_2)|_B &= \mathcal{Q}_s \\ (\mathcal{B}_1 - \mathcal{B}_2)|_B &= 0 \end{aligned}$$

The first two involve the tangential component of the field intensity, and the second pair involve the normal component of flux density. These conditions are simply convenient restatements of Maxwell’s laws for fields at a boundary.

### 10. ENERGY AND POWER

The electric and the magnetic energy densities are represented by the three-forms

$$\mathcal{W}_e = \frac{1}{2} \mathcal{E} \wedge \mathcal{D} = \frac{1}{2} (E_x D_x + E_y D_y + E_z D_z) dx \wedge dy \wedge dz, \tag{55a}$$

$$\mathcal{W}_m = \frac{1}{2} \mathcal{H} \wedge \mathcal{B} = \frac{1}{2} (H_x B_x + H_y B_y + H_z B_z) dx \wedge dy \wedge dz. \tag{55b}$$

Figure 23 visualizes the exterior product of the field one-form  $\mathcal{E}$  and the flux density two-form  $\mathcal{D}$ . The resulting energy density three-forms  $\mathcal{W}_e$  and  $\mathcal{W}_m$  are visualized by the subdivision of the space into cells as shown in Figure 23.

Multiplying Ampère’s law from the left with  $-\mathcal{E}$  and Faraday’s law from the right with  $\mathcal{H}$ , we obtain

$$-\mathcal{E} \wedge |d\mathcal{H} = \frac{\partial}{\partial t} \mathcal{D} + \mathcal{J}, \tag{56}$$

$$d\mathcal{E} = -\frac{\partial}{\partial t} \mathcal{B} \wedge \mathcal{H}. \tag{57}$$

This yields

$$d(\mathcal{E} \wedge \mathcal{H}) = -\mathcal{E} \wedge \frac{\partial}{\partial t} \mathcal{D} - \mathcal{H} \wedge \frac{\partial}{\partial t} \mathcal{B} - \mathcal{E} \wedge \mathcal{J}. \tag{58}$$

This equation can be brought into the form

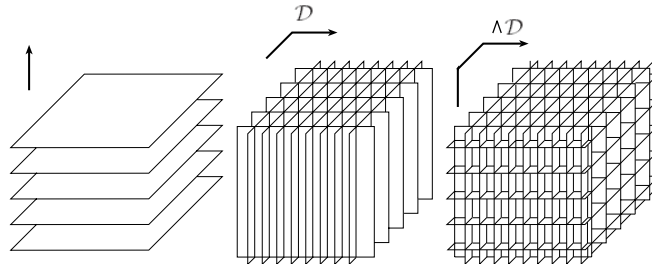
$$d(\mathcal{E} \wedge \mathcal{H}) = -\frac{\partial}{\partial t} \left( \frac{1}{2} \mathcal{E} \wedge \mathcal{D} + \frac{1}{2} \mathcal{H} \wedge \mathcal{B} \right) - \mathcal{E} \wedge \mathcal{J}. \tag{59}$$

The power loss density  $p_L(\mathbf{x}, t)$  with the corresponding differential form

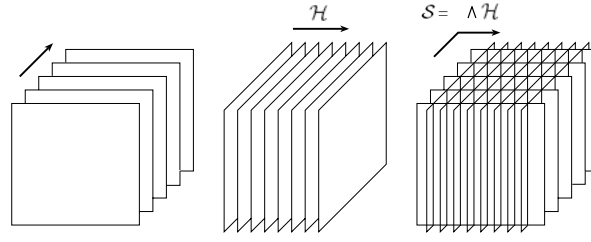
$$\mathcal{P}_L = p_L(\mathbf{x}, t) dx \wedge dy \wedge dz \tag{60}$$

is given by

$$\mathcal{P}_L = \mathcal{E} \wedge \sigma \star \mathcal{E}. \tag{61}$$



**Figure 23.** The exterior product of the field form  $\mathcal{E}$  and the flux density form  $\mathcal{D}$ .



**Figure 24.** The Poynting form  $\mathcal{S}$  as the product of the field forms  $\mathcal{E}$  and  $\mathcal{H}$ .

Due to the impressed current density  $\mathcal{J}_0$ , a power per unit of volume

$$\mathcal{P}_0 = -\mathcal{E} \wedge \mathcal{J}_0 \quad (62)$$

is added to the electromagnetic field. Introducing the Poynting differential form

$$\mathcal{S} = \mathcal{E} \wedge \mathcal{H} \quad (63)$$

and inserting (55a), (55b), (61) and (62) into (59) yields the differential form of Poynting's theorem

$$d\mathcal{S} = -\frac{\partial}{\partial t}\mathcal{W}_e - \frac{\partial}{\partial t}\mathcal{W}_m - \mathcal{P}_L + \mathcal{P}_0. \quad (64)$$

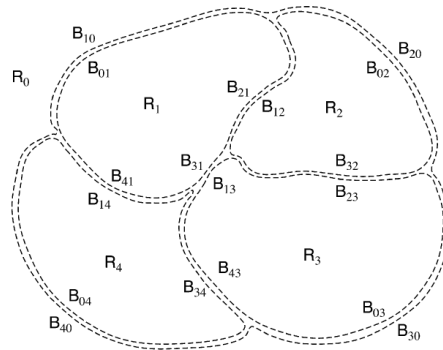
Figure 24 visualizes the Poynting two-form as the exterior product of the electric and magnetic field one-forms  $\mathcal{E}$  and  $\mathcal{H}$ . The potential planes of the electric and magnetic fields together form the tubes of the Poynting form. The distance of the electric and magnetic potential planes exhibit the dimensions V and A respectively. The cross sectional areas of the flux tubes have the dimension VA. The power flows through these Poynting flux tubes.

Integrating (64) over a volume  $V$  and transforming the integral over  $\mathcal{S}$  into a surface integral over the boundary  $\partial V$ , we obtain the integral form of Poynting's Theorem

$$\oint_{\partial V} \mathcal{S} = \int_V \mathcal{P}_0 - \frac{d}{dt} \int_V \mathcal{W}_e - \frac{d}{dt} \int_V \mathcal{W}_m - \int_V \mathcal{P}_L. \quad (65)$$

## 11. TELLEGEN'S THEOREM

Figure 25 shows the segmentation of an electromagnetic structure into different regions  $R_l$  separated by boundaries  $B_{lk}$ . The regions  $R_l$  may contain any electromagnetic substructure. In a network analogy the two-dimensional manifold of all boundary surfaces  $B_{lk}$  represents the connection circuit, whereas the subdomains  $V_l$  are representing the circuit elements. Tellegen's theorem states fundamental relations between voltages and currents in a network and is of considerable versatility and generality



**Figure 25.** Segmentation of a closed structure.

in network theory [52]. The field form of Tellegen's theorem may be derived directly from Maxwell's equations [50, 53] and is given by

$$\int_{\partial V} \mathcal{E}'(\mathbf{x}, t') \wedge \mathcal{H}''(\mathbf{x}, t'') = 0. \quad (66)$$

The integration is performed over both sides of all boundary surfaces. Also the integration over finite volumes filled with ideal electric or magnetic conductors gives no contribution to these integrals. The prime ' and double prime '' denote the case of a different choice of sources and a different choice of materials filling the subdomains. Also the time argument may be different in both cases.

## 12. THE ELECTROMAGNETIC POTENTIALS

The Maxwell's Equations (45)–(48) are a system of twelve coupled scalar partial differential equations. The introduction of electromagnetic potentials allows a systematic solution of the Maxwell's equations [54–56]. We are distinguishing between scalar potentials and vector potentials. After solution of the wave equation for a potential, all field quantities may be derived from this potential.

Due to (47), i.e.,  $d\mathcal{B} = 0$  the magnetic flux density is free of divergence. Therefore  $\mathcal{B}$  may be represented as the exterior derivative of an one-form  $\mathcal{A}$ :

$$\mathcal{B} = d\mathcal{A}. \quad (67)$$

The corresponding vector field  $\mathbf{A}$  is called the magnetic vector potential. Inserting (67) into the second Maxwell's Equation (46) yields

$$d\left(\mathcal{E} + \frac{\partial}{\partial t}\mathcal{A}\right) = 0. \quad (68)$$

According to Poincaré's lemma, the exterior derivative of the one-form inside the brackets vanishes, we may express this one-form as the exterior derivative of the scalar potential  $\Phi$  and obtain

$$\mathcal{E} = -d\Phi - \frac{\partial}{\partial t}\mathcal{A}. \quad (69)$$

The negative sign of  $\Phi$  has been chosen due to the physical convention in defining potentials. Whereas in electrostatics the electric field may be computed from a scalar potential  $\Phi$ , in the case of rapidly varying electromagnetic fields, we also need the vector potential  $\mathbf{A}$ . The potentials  $\mathbf{A}$  and  $\Phi$  are not defined in an unambiguous way. Adding the gradient of a scalar function  $\Psi$  to the vector potential  $\mathbf{A}$  does not influence the magnetic induction  $\mathbf{B}$ . The electric field  $\mathbf{E}$  also remains unchanged, if  $\mathbf{A}$  and  $\Phi$  together are transformed in the following way:

$$\mathcal{A}_1 = \mathcal{A} + d\Psi, \quad (70)$$

$$\Phi_1 = \Phi - \frac{\partial \Psi}{\partial t}. \quad (71)$$

This transformation is called a gauge transformation. The one-form  $\mathcal{A}$  may be defined in an unambiguous way, if we are prescribing its exterior derivative.

Inserting (67) and (69) into the first Maxwell's Equation (45) yields

$$\star d \star d\mathcal{A} + \mu\varepsilon \frac{\partial^2}{\partial t^2}\mathcal{A} + \mu\sigma \frac{\partial}{\partial t}\mathcal{A} + \mu d\left(\varepsilon \frac{\partial \Phi}{\partial t} + \sigma\Phi\right) = \star\mu\mathcal{J}_0. \quad (72)$$

Inserting (69) and (21) into (48) yields

$$\star d \star d\Phi + \star d \star \frac{\partial}{\partial t}\mathcal{A} = -\frac{1}{\varepsilon} \star \mathcal{Q}. \quad (73)$$

Since we may choose the exterior derivative of  $\star\mathcal{A}$  arbitrarily, we can make use of this option in order to decouple the differential equations for  $\mathcal{A}$  and  $\Phi$ . We impose the so-called Lorentz condition given by

$$\star d \star \mathcal{A} + \mu\left(\varepsilon \frac{\partial}{\partial t}\Phi + \sigma\Phi\right) = 0. \quad (74)$$

Together with (72) and (73) we obtain the equations

$$(d \star d \star - \star d \star d) \mathcal{A} - \mu \varepsilon \frac{\partial^2}{\partial t^2} \mathcal{A} - \mu \sigma \frac{\partial}{\partial t} \mathcal{A} = - \star \mu \mathcal{J}_0 \quad (75)$$

$$\star d \star d \Phi - \mu \varepsilon \frac{\partial^2}{\partial t^2} \Phi - \mu \sigma \frac{\partial}{\partial t} \Phi = - \frac{1}{\varepsilon} \star \mathcal{Q}. \quad (76)$$

We define the covariant derivative by the operator

$$\tilde{d}\mathcal{U} = (-1)^{\deg \mathcal{U} + 1} \star d \star \mathcal{U}. \quad (77)$$

We introduce the Laplace operator  $\Delta$  defined by

$$\Delta = \tilde{d}d + d\tilde{d}. \quad (78)$$

Applying the Laplace operator to a zero-form  $\Phi$  and an one-form  $\mathcal{A}$  respectively yields

$$\Delta \Phi = \star d \star d \Phi \quad (79)$$

$$\Delta \mathcal{A} = (d \star d \star - \star d \star d) \mathcal{A}. \quad (80)$$

With the Laplace operator  $\Delta$  we can write (75) and (76) as

$$\Delta \mathcal{A} - \mu \varepsilon \frac{\partial^2}{\partial t^2} \mathcal{A} - \mu \sigma \frac{\partial}{\partial t} \mathcal{A} = - \star \mu \mathcal{J}_0, \quad (81)$$

$$\Delta \Phi - \mu \varepsilon \frac{\partial^2}{\partial t^2} \Phi - \mu \sigma \frac{\partial}{\partial t} \Phi = - \frac{1}{\varepsilon} \star \mathcal{Q}. \quad (82)$$

The field intensities  $\mathcal{E}$  and  $\mathcal{H}$  derived from  $\mathcal{A}$  and  $\Phi$  satisfy the four Maxwell's Equations (45) to (48). The Equations (81) and (82) are called wave equations, since their solutions describe propagating waves. Equation (81) is a vector wave equation, whereas (82) is a scalar wave equation.

It is possible to derive both potentials  $\mathcal{A}(\mathbf{x}, t)$  and  $\Phi(\mathbf{x}, t)$  from one vector, the so-called electric Hertz vector  $\mathbf{\Pi}_e(\mathbf{x}, t)$ . We introduce the electric Hertz differential form

$$\Pi_e = \Pi_{ex} dx + \Pi_{ey} dy + \Pi_{ez} dz. \quad (83)$$

The Lorentz condition (74) is fulfilled, if  $\mathcal{A}$  and  $\Phi$  are derived from the Hertz form  $\Pi_e$  via

$$\mathcal{A} = \mu \varepsilon \frac{\partial}{\partial t} \Pi_e + \mu \sigma \Pi_e, \quad (84)$$

$$\Phi = -\tilde{d}\Pi_e. \quad (85)$$

Inserting (84) into (81), we obtain

$$\mu \left( \varepsilon \frac{\partial}{\partial t} + \sigma \right) \left( \Delta \Pi_e - \mu \varepsilon \frac{\partial^2}{\partial t^2} \Pi_e - \mu \sigma \frac{\partial}{\partial t} \Pi_e \right) = -\mu \star \mathcal{J}_0. \quad (86)$$

For  $\mathcal{J}_0 = 0$ , i.e., without impressed current sources, we obtain the homogeneous wave equation

$$\Delta \Pi_e - \mu \varepsilon \frac{\partial^2}{\partial t^2} \Pi_e - \mu \sigma \frac{\partial}{\partial t} \Pi_e = 0. \quad (87)$$

The field intensities  $\mathcal{E}$  and  $\mathcal{H}$  follow from (22), (67), (69), (84) and (85):

$$\mathcal{E} = d\tilde{d}\Pi_e - \mu \varepsilon \frac{\partial^2}{\partial t^2} \Pi_e - \mu \sigma \frac{\partial}{\partial t} \Pi_e, \quad (88)$$

$$\mathcal{H} = \star d \left( \varepsilon \frac{\partial}{\partial t} \Pi_e + \sigma \Pi_e \right). \quad (89)$$

Subtracting from (88) the wave Equation (87) we obtain

$$\mathcal{E} = -\tilde{d}d\Pi_e \quad \text{for } \mathcal{J}_0 = 0. \quad (90)$$

Let us now consider the lossless case with impressed current sources. In this case it is helpful to use the impressed electric polarization  $\underline{M}_{e0}(\mathbf{x}, t)$  instead of the impressed current density  $\underline{J}_0(\mathbf{x}, t)$ . The corresponding differential form is

$$\underline{M}_{e0} = M_{ex} dy \wedge dz + M_{ey} dz \wedge dx + M_{ez} dx \wedge dy. \quad (91)$$

The impressed electric polarization form  $\underline{M}_{e0}$  is related to an impressed electric current  $\underline{J}_0$  via

$$\frac{\partial}{\partial t} \underline{M}_{e0} = \underline{J}_0. \quad (92)$$

By this way it follows from (86)

$$\frac{\partial}{\partial t} \left( \Delta \Pi_e - \mu \varepsilon \frac{\partial^2}{\partial t^2} \Pi_e \right) = -\frac{1}{\varepsilon} \star \frac{\partial}{\partial t} \underline{M}_{e0} \quad \text{for } \sigma = 0 \quad (93)$$

by integration over  $t$  we obtain

$$\Delta \Pi_e - \mu \varepsilon \frac{\partial^2}{\partial t^2} \Pi_e = -\frac{1}{\varepsilon} \star \underline{M}_{e0} \quad \text{for } \sigma = 0. \quad (94)$$

Since the source of the Hertz vector field is an impressed electric polarization, the Hertz vector also is called the electric polarization potential. From the solution of (94) we obtain  $\underline{\mathcal{E}}$  and  $\underline{\mathcal{H}}$  via (88) and (89). From (88) and (94) we obtain

$$\underline{\mathcal{E}} = -\tilde{d}\Pi_e - \frac{1}{\varepsilon} \star \underline{M}_{e0}. \quad (95)$$

In the general case  $\underline{J}_0 \neq 0$  and  $\sigma \neq 0$  we obtain an equation containing time derivatives up to third order. This difficulty can be avoided by using the frequency domain representation.

### 13. TIME-HARMONIC ELECTROMAGNETIC FIELDS

For the description of time-harmonic electromagnetic fields the introduction of phasors is useful. We can describe a time-harmonic electric field by the phasor differential form  $\underline{\mathcal{E}}_0(\mathbf{x})$  that yields the time dependent differential form

$$\underline{\mathcal{E}}(\mathbf{x}, t) = \Re \{ \underline{\mathcal{E}}_0(\mathbf{x}) e^{j\omega t} \}. \quad (96)$$

For time-harmonic electromagnetic fields we can write the Maxwell's equations as

$$d\underline{\mathcal{H}} = j\omega (\underline{\varepsilon} \star \underline{\mathcal{E}} + \underline{M}_{e0}), \quad (97)$$

$$d\underline{\mathcal{E}} = -j\omega (\underline{\mu} \star \underline{\mathcal{H}} + \underline{M}_{m0}), \quad (98)$$

where  $\underline{\varepsilon}$  is the complex permittivity,  $\underline{\mu}$  is the complex permeability and the two-forms  $\underline{M}_{e0}$  and  $\underline{M}_{m0}$  represent the impressed electric and magnetic polarizations. Impressed electric polarization  $\underline{M}_{e0}$  and equivalent impressed electric current  $\underline{J}_0$  are related by

$$\underline{J}_0 = j\omega \underline{M}_{e0}. \quad (99)$$

To compute the electromagnetic field we first compute electric Hertz form  $\underline{\Pi}_e$  and/or the magnetic Hertz form  $\underline{\Pi}_m$  which satisfy the Helmholtz equation:

$$\Delta \underline{\Pi}_e + \omega^2 \underline{\mu} \underline{\varepsilon} \underline{\Pi}_e = -\frac{1}{\underline{\varepsilon}} \star \underline{M}_{e0}, \quad (100)$$

$$\Delta \underline{\Pi}_m + \omega^2 \underline{\mu} \underline{\varepsilon} \underline{\Pi}_m = -\frac{1}{\underline{\mu}} \star \underline{M}_{m0}. \quad (101)$$

The electric and magnetic field forms  $\underline{\mathcal{E}}$  and  $\underline{\mathcal{H}}$  can be derived from the electric and magnetic Hertz forms:

$$\underline{\mathcal{E}} = d\tilde{d}\underline{\Pi}_e + \omega^2 \underline{\mu} \underline{\varepsilon} \underline{\Pi}_e - j\omega \underline{\mu} \star d\underline{\Pi}_m, \quad (102)$$

$$\underline{\mathcal{H}} = j\omega \underline{\varepsilon} \star d\underline{\Pi}_e + d\tilde{d}\underline{\Pi}_m + \omega^2 \underline{\mu} \underline{\varepsilon} \underline{\Pi}_m. \quad (103)$$

## 14. THE GREEN'S FUNCTION

The solution of the vector field problem for a unit point-like vector source is given by the dyadic Green's function with the components  $\underline{G}_{ij}$  relating the  $i$ th component of the excited field at a point  $\mathbf{x}$  to the  $j$ th component of the exciting source at a point  $\mathbf{x}'$  [57, 58]. We define the Green's double one-form

$$\begin{aligned} \underline{\mathcal{G}} = & \underline{G}_{11}dx dx' + \underline{G}_{12}dx dy' + \underline{G}_{13}dx dz' + \underline{G}_{21}dy dx' + \underline{G}_{22}dy dy' + \underline{G}_{23}dy dz' \\ & + \underline{G}_{31}dz dx' + \underline{G}_{32}dz dy' + \underline{G}_{33}dz dz'. \end{aligned} \quad (104)$$

Double one-forms are differential forms representing dyadics [33, 59]. Unprimed differentials  $dx_i$  and primed differentials  $dx'_j$  commute, i.e., they may be interchanged without changing the sign. With the Green's function we can solve the Helmholtz equation for any source distribution by considering the solution as a continuous superposition of pointlike sources. The Helmholtz Equation (100) for a point-like source at  $\mathbf{x}'$  is

$$\Delta \underline{\mathcal{G}}(\mathbf{x}, \mathbf{x}') + k^2 \underline{\mathcal{G}}(\mathbf{x}, \mathbf{x}') = -\frac{1}{\varepsilon} \mathcal{I}(\mathbf{x}, \mathbf{x}'). \quad (105)$$

The Laplace operator only acts upon  $\mathbf{x}$  and not on  $\mathbf{x}'$  since in the Helmholtz equation  $\mathbf{x}$  is the space variable of the field where  $\mathbf{x}'$  is the fixed location of the source. The identity kernel is given by

$$\mathcal{I}(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') (dx dx' + dy dy' + dz dz'). \quad (106)$$

With the identity kernel we can map any one-form  $\mathcal{U}$  and any two-form  $\mathcal{V}$  from the source space to the observation space, i.e., the respective form is mapped in itself and the primed differentials are replaced by unprimed differentials. We obtain

$$\int' \mathcal{I}(\mathbf{x}, \mathbf{x}') \wedge \star \mathcal{U}(\mathbf{x}') = \mathcal{U}(\mathbf{x}), \quad (107a)$$

$$\star \int' \mathcal{I}(\mathbf{x}, \mathbf{x}') \wedge \mathcal{V}(\mathbf{x}') = \mathcal{V}(\mathbf{x}). \quad (107b)$$

The prime  $'$  at the integral symbol denotes that the integration is performed over the primed coordinates only.

For a source embedded in homogeneous isotropic space the solution of (105) is given by

$$\underline{\mathcal{G}}(\mathbf{x}, \mathbf{x}') = \frac{e^{-jk|\mathbf{x}-\mathbf{x}'|}}{4\pi\varepsilon|\mathbf{x}-\mathbf{x}'|} (dx dx' + dy dy' + dz dz'). \quad (108)$$

For details see [26]. Multiplying the Helmholtz Equation (105) from the right with the impressed electric polarization two-form  $\underline{\mathcal{M}}_{e0}(\mathbf{x}')$ , integrating over  $dx' \wedge dy' \wedge dz'$  and applying (107b) we obtain after comparison with (100) the solution

$$\underline{\Pi}_e(\mathbf{x}) = \int' \underline{\mathcal{G}}(\mathbf{x}, \mathbf{x}') \wedge \underline{\mathcal{M}}_{e0}(\mathbf{x}'). \quad (109)$$

This integration has been performed over the primed coordinates of the source distribution. For the integration the primed are the variables and the unprimed coordinates denoting the location of the point of observation are the fixed parameters.

## 15. APERIODIC SPHERICAL WAVES

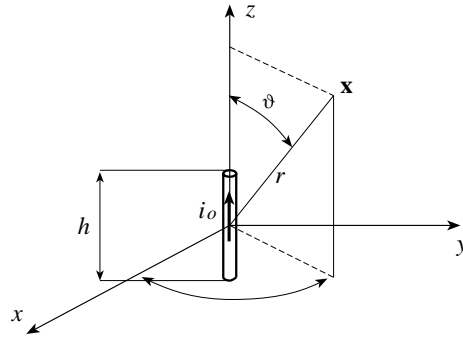
As an example we discuss the impulsive spherical wave emitted from a Hertzian dipole under impulsive excitation, following the treatment in [26]. The Hertzian dipole is a wire of length  $h$  with uniform current  $i_0(t)$  impressed (Figure 26). In time domain the impressed polarization  $m_{e0}(t)$  and the impressed current  $i_0(t)$  are related via

$$i_0(t) = \frac{d}{dt} m_{e0}(t). \quad (110)$$

Since the current in the dipole is flowing in  $z$ -direction the impressed electric polarization is

$$\underline{\mathcal{M}}_{e0}(\mathbf{x}) = \underline{M}_{e0z}(\mathbf{x}) dx \wedge dy. \quad (111)$$





**Figure 26.** Hertzian dipole of length  $h$ .

Inserting this and (108) into (109) yields an electric Hertz form with a  $z$  component only,

$$\underline{\Pi}_e(\mathbf{x}) = \underline{\Pi}_{ez}(\mathbf{x})dz, \tag{112}$$

with

$$\underline{\Pi}_{ez}(\mathbf{x}) = \int' \frac{e^{-jk|\mathbf{x}-\mathbf{x}'|}}{4\pi\epsilon|\mathbf{x}-\mathbf{x}'|} \underline{M}_{e0z}(\mathbf{x}')dx' \wedge dy' \wedge dz'. \tag{113}$$

For  $h$  small compared with wavelength this yields

$$\underline{\Pi}_{ez}(\mathbf{x}) = \frac{e^{-jkr}}{4\pi\epsilon r} \int' \underline{M}_{e0z}(\mathbf{x}')dx' \wedge dy' \wedge dz' \tag{114}$$

with  $r = |\mathbf{x}|$ . This yields in time-domain

$$\Pi_{ez}(\mathbf{x}, t) = \frac{1}{4\pi\epsilon r} \int' M_{e0z}(\mathbf{x}', t - \frac{r}{c}) dx' \wedge dy' \wedge dz'. \tag{115}$$

Integrating the electric polarization form  $\mathcal{M}_{e0}$  over the volume  $V$  of the Hertzian dipole yields

$$\int' \underline{M}_{e0z}(\mathbf{x}')dx' \wedge dy' \wedge dz' = hm_{e0}(t) = h \int_0^t i_0(t_1) \tag{116}$$

where  $i(t)$  is the current through the Hertzian dipole and  $m_{e0}(t)$  the polarization due to this current. The time-dependent electric Hertz form for the Hertzian dipole oriented in  $z$  direction is

$$\Pi_e(\mathbf{x}, t) = \frac{h}{4\pi\epsilon_0 r} m_{e0} \left( t - \frac{r}{c} \right) dz. \tag{117}$$

Using (89) and (90) and considering that  $M_{e0}(\mathbf{x}, t)$  vanishes outside the conductor, we can compute  $\mathcal{E}(\mathbf{x}, t)$  and  $\mathcal{H}(\mathbf{x}, t)$ :

$$\mathcal{H}(\mathbf{x}, t) = \star d\epsilon \frac{\partial}{\partial t} \Pi_e(\mathbf{x}, t), \tag{118a}$$

$$\mathcal{E}(\mathbf{x}, t) = -\tilde{d}d\Pi_e(\mathbf{x}, t). \tag{118b}$$

Using (118a) yields

$$\mathcal{H} = \frac{h}{4\pi} \left[ \frac{1}{r^2} m'_{e0} \left( t - \frac{r}{c} \right) + \frac{1}{cr} m''_{e0} \left( t - \frac{r}{c} \right) \right] \sin\vartheta \cdot r \sin\vartheta d\varphi. \tag{119}$$

The magnetic field only exhibits a  $\varphi$  component

$$H_\varphi = \frac{h}{4\pi} \left[ \frac{1}{r^2} m'_{e0} \left( t - \frac{r}{c} \right) + \frac{1}{cr} m''_{e0} \left( t - \frac{r}{c} \right) \right] \sin\vartheta. \tag{120}$$

The electric field form is

$$\mathcal{E} = \frac{h}{4\pi\epsilon_0} \left\{ \left[ \frac{1}{r^3} m_{e0} \left( t - \frac{r}{c} \right) + \frac{1}{cr^2} m'_{e0} \left( t - \frac{r}{c} \right) + \frac{1}{c^2 r} m''_{e0} \left( t - \frac{r}{c} \right) \right] \sin \vartheta r d\vartheta \right. \\ \left. + 2 \left[ \frac{1}{r^3} m_{e0} \left( t - \frac{r}{c} \right) + \frac{1}{cr^2} m'_{e0} \left( t - \frac{r}{c} \right) \right] \cos \vartheta dr \right\}. \quad (121)$$

The electric field exhibits the  $\vartheta$ - and  $r$ -components

$$E_\vartheta = \frac{h}{2\pi\epsilon_0} \left[ \frac{1}{r^3} m_{e0} \left( t - \frac{r}{c} \right) + \frac{1}{cr^2} m'_{e0} \left( t - \frac{r}{c} \right) + \frac{1}{c^2 r} m''_{e0} \left( t - \frac{r}{c} \right) \right] \cos \vartheta, \quad (122)$$

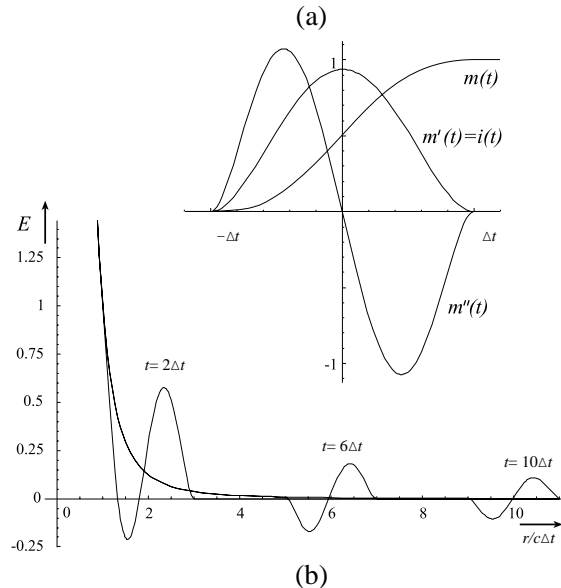
$$E_r = \frac{h}{4\pi\epsilon_0} \left[ \frac{1}{r^3} m_{e0} \left( t - \frac{r}{c} \right) + \frac{1}{cr^2} m'_{e0} \left( t - \frac{r}{c} \right) \right] \sin \vartheta. \quad (123)$$

As an example we consider a wave pulse emitted from a Hertzian dipole excited by a current pulse. In Figure 27(a) the dipole current pulse  $i(t) = m'(t)$  of width  $2\Delta t$ , its integral over time  $m(t)$  and its time derivative  $m''(t)$  are depicted. Figure 27(b) shows the time evolution of  $E_\vartheta(r, 0, 0, t)$ . The wave front of width  $2\Delta t$  mainly depends on  $m'(t)$  and  $m''(t)$ . In the far-field region, defined by  $r \gg c\Delta t$  the terms proportional to  $1/r$  in  $E_\vartheta$  and  $H_\varphi$  exhibit the double pulse shape specified by  $m''(t)$ . The energy connected with this term is constrained within the shell of width  $2c\Delta t$  at the wave-front and transported into the infinity. This is the radiated part of the field. The electric and magnetic far-field time waveforms  $E_\vartheta$  and  $H_\varphi$  of the wave pulse are proportional to the time derivative of the driving current  $i(t)$  of the dipole. The near field parts of the electric and magnetic field proportional to  $m'(t - r/c)/r^2$  also are confined to the wave front shell of width  $2\Delta t$ . This part of the wave front is carrying the electromagnetic energy for building up the near-field. It leaves behind the wave-front an electric field proportional to  $m(t - r/c)/r^3$ . This field behind the wave-front corresponds to the electrostatic field excited by a static dipole. Figure 28 shows the electric field in a meridional plane.

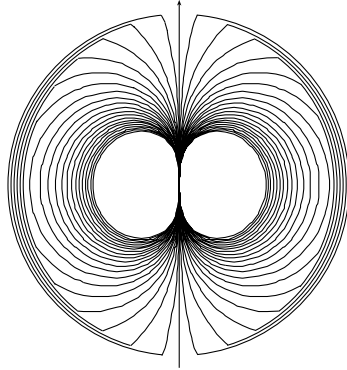
In the far-field we obtain the approximate differential forms

$$\mathcal{E}(r, \vartheta, t) = \frac{\mu_0 h}{4\pi} \frac{m''_{e0} \left( t - \frac{r}{c} \right)}{r} \sin \vartheta r d\vartheta, \quad (124)$$

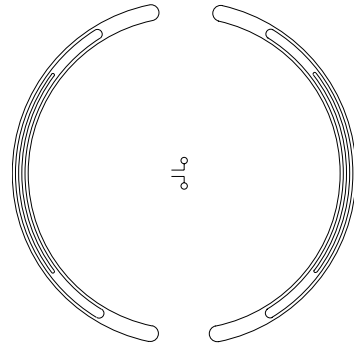
$$\mathcal{H}(r, \vartheta, t) = \frac{h}{4\pi c} \frac{m''_{e0} \left( t - \frac{r}{c} \right)}{r} \sin \vartheta r \sin \vartheta d\varphi \quad (125)$$



**Figure 27.** Wave pulse: (a) pulse waveforms, (b) radial dependence of the wave pulse.



**Figure 28.** Near field of the Hertzian dipole under pulse excitation.



**Figure 29.** Far field of the Hertzian dipole under pulse excitation.

and the corresponding field components

$$E_{\vartheta}(r, \vartheta, t) = \frac{\mu_0 h}{4\pi} \frac{m''_{e0}}{r} \left(t - \frac{r}{c}\right) \sin \vartheta, \tag{126}$$

$$H_{\varphi}(r, \vartheta, t) = \frac{h}{4\pi c} \frac{m''_{e0}}{r} \left(t - \frac{r}{c}\right) \sin \vartheta. \tag{127}$$

From this it follows that the ratio of electric and magnetic field in the far-field is given by the wave impedance

$$Z_{F0} = \sqrt{\frac{\mu_0}{\epsilon_0}}. \tag{128}$$

The far-field is depicted in Figure 29.

From (63), (124) and (125) we obtain the Poynting form  $\mathcal{S}$  for the far-field

$$\mathcal{S}(r, \vartheta, t) = \frac{1}{2} \mathcal{E} \wedge \mathcal{H} = \frac{Z_{F0} h^2}{32\pi^2 c^2} m''_{e0}{}^2 \left(t - \frac{r}{c}\right) \sin^3 \vartheta d\vartheta \wedge d\varphi. \tag{129}$$

In the far-field the complex Poynting vector exhibits only a radial component

$$S_r(r, \vartheta, t) = \frac{Z_{F0} h^2}{32\pi^2 r^2 c^2} m''_{e0}{}^2 \left(t - \frac{r}{c}\right) \sin^2 \vartheta. \tag{130}$$

The power  $P(r, t)$  radiated from the Hertzian dipole through a spherical surface with radius  $r$  in the far-field is obtained by integrating  $\mathcal{S}$  over this surface

$$P(r, t) = \int_{\vartheta=0}^{\pi} \int_{\varphi=0}^{2\pi} \mathcal{S}(r, \vartheta, t). \tag{131}$$

We obtain from (130) and (131)

$$P(r, t) = \frac{Z_{F0} h^2}{12\pi c^2} m''_{e0}{}^2 \left(t - \frac{r}{c}\right). \tag{132}$$

## 16. CONCLUSION

Advantages over conventional vector calculus makes the exterior differential forms an ideal framework for teaching and understanding Maxwell's equations and the principles of electromagnetics. It yields a clear and easy representation of the theory and throws light upon the physics behind the formalism. Axial and polar vectors as well as scalars and pseudoscalars are clearly distinguished. Rules for computation follow in a most natural way from the notation. The translation of formulae from the differential form notation to conventional vector notation not only is easy but also supports understanding of conventional vector notation. Most importantly for the student of electromagnetic theory, differential forms provide a way to visualize fields and sources as potential surfaces and tubes of flux or flow that is not available with vector notation or other mathematical frameworks for working with Maxwell's equations of electromagnetism.

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