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On Inhomogeneous Metamaterials Media: A New Alternative Method for Analysis of Electromagnetic Fields Propagation

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Abstract—The analysis of waves propagation in homogeneous anisotropic media constitutes a classical topic in every field of science and has been preferentially discussed using locally plane waves. Specific physical quantities and their behaviour laws are really what make the difference. Although the use of Fourier transform enables an approach formally analogous to that of plane waves in linear evolution equations, its application to constitutive equations of inhomogeneous media involves cumbersome convolution products that mask the solution. This paper proposes a polar representation (amplitude and phase) of electromagnetic fields, that appears to be more suitable and provides two sets of equations that can be easily decoupled, reducing the problem to the superposition of two simpler ones. The procedure is based upon the following steps: *a)* The identification of dispersion equation with Hamilton-Jacobi equation yields the evolution laws of rays and/or wave-fronts. *b)* From the knowledge of tensor $\bar{\bar{\varepsilon}}(\bar{r})$ at any point $\bar{r}$ of the wave front (or the ray), the use of the intrinsic character (conjugation relations) of fields, introduced by the authors in a previous work, together with ray velocity or phase gradient (found in the first step) the remaining fields are immediately obtained.

1. INTRODUCTION

Only few decades ago, electromagnetic characterization of a new material required experimental studies in order to find a model of its equivalent circuit. Continuing advances in numerical methods have provided not only exact solutions for real problems with given boundary conditions and constitutive laws, but the design of new metamaterials whose optical properties may be controlled.

In this sense, Transformation Optics is a promising recent field, dealing with the design of metamaterials to control and manipulate the trajectories of light [1–3]. One of the most important objectives of this discipline is the achievement of macroscopic invisibility-cloaking structures [4–6]. Nevertheless, in Transformation Optics *impedance matched to vacuum* media are required, where the relative permittivity tensor is equal to the relative permeability one ($\bar{\bar{\varepsilon}} = \bar{\bar{\mu}}$) at every point of space, when referred to the same vector basis ($\bar{\bar{\varepsilon}}(\bar{r}) = \bar{\bar{\mu}}(\bar{r})$). Moreover, since these tensors are functions of the position vector, $\bar{r}$, media are also inhomogeneous. Eigenvalues of $\bar{\bar{\varepsilon}}$, at every point, are named ε_1, ε_2 and ε_3. We are then concerned with those artificial media (metamaterials) that obey the following linear constitutive equations when subjected to external actions:

$$\bar{D} = \varepsilon_0 \, \bar{\bar{\varepsilon}}(\bar{r}) \, \bar{E}; \qquad \bar{B} = \mu_0 \, \bar{\bar{\varepsilon}}(\bar{r}) \, \bar{H} \tag{1}$$

The approximate plane waves propagation model constitutes an excellent calculation tool, in such a way that for homogeneous media the accuracy of its results is so high that no other model is needed.

Although description of physical quantities in terms of its Fourier transform enables an approach for inhomogeneous media formally analogous to that of plane waves, its application to constitutive laws require (according to the convolution theorem: "the Fourier transform of a product of functions is

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equal to the convolution of their Fourier transforms" [7]) to replace matrix products by cumbersome convolution matrix products in dispersion equation.

For that reason, a simplified model of wave fields with slowly varying amplitude and rapidly changing phase (appropriate for the geometric limit $\lambda_0 \to 0$) are traditionally adopted to continue using plane waves solutions [8]. Also, a wave front can be considered a moving surface of discontinuity travelling through a continuous medium, similar to a shock wave [9].

In one of his lessons, Jacobi asserted that "the main difficulty in integrating a given differential equation lies in introducing convenient variables, which there is no rule for finding. Therefore, we must travel the reverse path and after finding a suitable substitution, look for problems to which it can be successfully applied" [10]. This sentence taken by Arnold [11] fits perfectly to our aims. Thus, in this paper we propose a polar representation of electromagnetic fields and we look for the conditions that the proposed functional form must fulfill under the hypotheses of smooth eikonal and propagation in metamaterials, in order to be an exact solution. The procedure yields two sets of equations that can be easily decoupled, reducing the problem to the superposition of two simpler ones. A detailed analysis of the proposed method is shown below.

2. LAWS OF EVOLUTION

Here we propose the representation of fields in polar form (amplitude and phase), because, in our opinion, it appears to be the most suitable for analysing propagation of electromagnetic waves in inhomogeneous anisotropic media with $\bar{\bar{\varepsilon}} = \bar{\bar{\mu}}$. Thus,

$$\bar{E} = \hat{E}(\bar{r}, t)\, e^{i\,S} \;; \quad \bar{B} = \hat{B}(\bar{r}, t)\, e^{i\,S} \;; \quad \bar{D} = \hat{D}(\bar{r}, t)\, e^{i\,S} \;; \quad \bar{H} = \hat{H}(\bar{r}, t)\, e^{i\,S} \tag{2}$$

where amplitude $\hat{E}$, $\hat{B}$ $\hat{D}$ and $\hat{H}$ denote any vector functions of position and time and S is the eikonal or phase, any real function of position and time, too.

Substitution of these fields into Maxwell equations for source-free media

$$\nabla \times \bar{E} + \frac{\partial \bar{B}}{\partial t} = 0 \;; \quad \nabla \times \bar{H} - \frac{\partial \bar{D}}{\partial t} = 0 \tag{3}$$

and separating the real and imaginary parts, we find that:

$$\nabla \times \hat{E} + \frac{\partial \hat{B}}{\partial t} = 0 \;; \quad \nabla \times \hat{H} - \frac{\partial \hat{D}}{\partial t} = 0 \tag{4}$$

$$\nabla S \times \hat{E} + \frac{\partial S}{\partial t}\, \hat{B} = 0 \;; \quad \nabla S \times \hat{H} - \frac{\partial S}{\partial t}\, \hat{D} = 0 \tag{5}$$

We have obtained two sets of equations equivalent to (3): a set (4) depends on the fields amplitudes only, and the other one (5) depends on eikonal $S^{\dagger}$ and on amplitudes; assuming the validity of (1) for any amplitude of electromagnetic fields, it is possible to decouple the problem of propagation in an anisotropic and inhomogeneous media into two simpler and amenable problems:

a) Maxwell equations for amplitudes $\hat{E}$ and $\hat{H}$.

b) Wavefront evolution, that becomes formally identical to equations of propagation of local plane waves from which eikonal equation and its law of evolution are obtained.

Throughout the paper it will be shown that from the knowledge of tensor $\bar{\bar{\varepsilon}}$ (relative permittivity/permeability) at any point $\bar{r}$ of the wave front (or the ray), the use of direct and reciprocal bases associated with vector fields [6, 12] together with ray velocity or phase gradient (found in the first step) the remaining fields may be obtained immediately. In fact, for a given electromagnetic energy density, not all the fields are allowed: only those whose directions form reciprocal bases with respect to relativepermittivity/permeability tensor are possible.

2.1. Background

From the analysis of Equations (4) and (5), many relevantant results (some of them, although trivial, are included in order to attain a self-contained theory) concerning propagation of electromagnetic waves in these media arise. In next subsections, their derivations are shown.

† For monochromatic quasi-plane waves: $\bar{k} = \nabla S$; $\partial S/\partial t = -\omega$

2.1.1. Energy Evolution

Following the usual steps, we obtain an analogue to the Poynting's theorem in terms of fields amplitudes. Indeed,

$$\nabla\cdot\left(\hat{E}\times\hat{H}\right)+\frac{\partial\hat{W}}{\partial t}=0 \tag{6}$$

where $\hat{W}$ is the corresponding electromagnetic energy density for linear media [13] given by

$$\hat{W}=\frac{1}{2}\left(\hat{E}\cdot\hat{D}+\hat{H}\cdot\hat{B}\right) \tag{7}$$

that, written in the form of an *equation of continuity*, leads to the following expression for the velocity of propagation of energy (ray velocity):

$$\hat{v}=\frac{\hat{E}\times\hat{H}}{\hat{W}} \tag{8}$$

2.1.2. Evolution of Phase S

Multiplying the first equation in (5) scalarly by $\hat{H}$, and the second one by $\hat{E}$ and adding, and taking into account (8), one has that:

$$\hat{v}\cdot\nabla S+\frac{\partial S}{\partial t}=0\quad\Longrightarrow\quad\frac{D\,S}{D\,t}=0 \tag{9}$$

according to the definition of *material derivative* [14]. This means that S behaves as a *material surface*, namely the surface that consists of the same set of material points for all time. Then S is a wavefront, that geometrically accounts for the discontinuity surface of field amplitude [9].

2.1.3. Evolution of Electromagnetic Momentum

Defining the corresponding electromagnetic momentum density as the vector product of fields $\hat{D}$ and $\hat{B}$ [15], and taking into account Equation (5), we obtain:

$$\hat{D}\cdot\nabla S=\hat{B}\cdot\nabla S=0 \tag{10}$$

that accounts for the condition that $\hat{D}$ and $\hat{B}$ are on the plane tangent to the wave front. Then, according to (10), one can write:

$$\hat{D}\times\left(\nabla\,S\times\hat{E}\right)+\hat{B}\times\left(\nabla\,S\times\hat{H}\right)=-2\,\frac{\partial\,S}{\partial\,t}\left(\hat{D}\times\hat{B}\right)\;\Longrightarrow\;\nabla S=-\frac{\partial S}{\partial t}\,\frac{\left(\hat{D}\times\hat{B}\right)}{\hat{W}} \tag{11}$$

Equation (11) enables us to identify the evolution direction of electromagnetic momentum density $(\hat{D}\times\hat{B})$ with the normal to the wave front (∇S).

2.1.4. Correspondence between Ray and Wave-Front

We start from the mathematical property [14] that says: For all vector $\bar{a}$, $\bar{b}$ and an invertible tensor $\bar{\bar{A}}$:

$$\left(\bar{\bar{A}}\cdot\bar{a}\right)\times\left(\bar{\bar{A}}\cdot\bar{b}\right)=\left(\det\bar{\bar{A}}\right)\;\bar{\bar{A}}^{-1}\cdot\left(\bar{a}\times\bar{b}\right) \tag{12}$$

Applying this property to the product $\hat{D}\times\hat{B}$, assuming constitutive equations given in Equation (1), and taking into account Equations (8) and (11), one has:

$$\hat{D}\times\hat{B}=\frac{1}{c^2}\left(\bar{\bar{\varepsilon}}\cdot\hat{E}\right)\times\left(\bar{\bar{\varepsilon}}\cdot\hat{H}\right)=I_3\,\bar{\bar{\varepsilon}}^{-1}\cdot\left(\hat{E}\times\hat{H}\right)/c^2 \tag{13}$$

where $I_3=\varepsilon_1\,\varepsilon_2\,\varepsilon_3$ is the third invariant of tensor(determinant) of $\bar{\bar{\varepsilon}}$. Then:

$$\nabla S=-\frac{I_3}{c^2}\,\frac{\partial S}{\partial t}\,\bar{\bar{\varepsilon}}^{-1}\cdot\hat{v}\quad\text{or}\quad\hat{v}=-\frac{c^2}{I_3}\,\frac{\bar{\bar{\varepsilon}}\cdot\nabla S}{\frac{\partial S}{\partial t}} \tag{14}$$

expression that puts into correspondence the ray velocity $(\hat{v})$ with the normal to the wave-front (∇S).

2.1.5. Dispersion Equation Versus Hamilton-Jacobi Equation

Bearing in mind the mathematical property (12), expressions in (5) lead to:

$$\left(\nabla S\times\hat{H}\right)\times\left(\hat{E}\times\nabla S\right)=\left(\left(\hat{E}\times\hat{H}\right)\cdot\nabla S\right)\nabla S=\left(\frac{\partial S}{\partial t}\right)^2\left(\hat{D}\times\hat{B}\right)=I_3\varepsilon_0\mu_0\left(\frac{\partial S}{\partial t}\right)^2\bar{\bar{\varepsilon}}^{-1}\cdot\left(\hat{E}\times\hat{H}\right)\tag{15}$$

Then,

$$\left(\left(\hat{E}\times\hat{H}\right)\cdot\nabla S\right)\nabla S=\frac{I_3}{c^2}\left(\frac{\partial S}{\partial t}\right)^2\bar{\bar{\varepsilon}}^{-1}\cdot\left(\hat{E}\times\hat{H}\right)\tag{16}$$

valid for any value of the corresponding Poynting's vector $(\hat{E}\times\hat{H})$. Naming $\bar{\bar{\varepsilon}}^{-1}=\bar{\bar{\tau}}$ and writing the ith-component:

$$\left(\frac{I_3}{c^2}\left(\frac{\partial S}{\partial t}\right)^2\tau_{ij}-\frac{\partial S}{\partial x_i}\frac{\partial S}{\partial x_j}\right)\left(\hat{E}\times\hat{H}\right)_j=0\quad\text{valid for all }\left(\hat{E}\times\hat{H}\right)\tag{17}$$

Nontrivial solutions can be obtained if and only if the determinant of the coefficients vanishes

$$\left|\frac{I_3}{c^2}\left(\frac{\partial S}{\partial t}\right)^2\tau_{ij}-\frac{\partial S}{\partial x_i}\frac{\partial S}{\partial x_j}\right|=0\tag{18}$$

If tensor τ_{ij} is expressed in principal directions $\{\bar{\imath}_1,\bar{\imath}_2,\bar{\imath}_3\}$,

$$\tau_{ij}=\begin{pmatrix}\frac{1}{\varepsilon_1}&0&0\\0&\frac{1}{\varepsilon_2}&0\\0&0&\frac{1}{\varepsilon_3}\end{pmatrix}_{(\bar{\imath}_1,\bar{\imath}_2,\bar{\imath}_3)}\tag{19}$$

vanishing of the determinant leads to the compact expression:

$$\frac{I_3}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-\nabla S\cdot\bar{\bar{\varepsilon}}\cdot\nabla S=0\tag{20}$$

Equation (20) constitutes the *eikonal equation* [16, 17] as well as the *dispersion equation* for propagation of waves in an inhomogeneous anisotropic media with $\bar{\bar{\varepsilon}}=\bar{\bar{\mu}}$ and may be written as:

$$\frac{\partial S}{\partial t}\pm c\sqrt{\nabla S\cdot\frac{\bar{\bar{\varepsilon}}}{I_3}\cdot\nabla S}=0\tag{21}$$

Identifying $\bar{p}$ with ∇S and comparing with the Hamilton-Jacobi equation:

$$\frac{\partial S}{\partial t}+H\left(\frac{\partial S}{\partial q},q\right)=0\tag{22}$$

it follows that the Hamiltonian of the equivalent mechanical problem can be expressed in terms of momentum and vector position as

$$H(\bar{r},\bar{p})=c\sqrt{\nabla S\cdot\frac{\bar{\bar{\varepsilon}}}{I_3}\cdot\nabla S}=c\sqrt{\bar{p}\cdot\frac{\bar{\bar{\varepsilon}}}{I_3}\cdot\bar{p}}\tag{23}$$

and from Hamilton equations

$$\nabla_{\bar{p}}H=\frac{c}{\sqrt{I_3}}\frac{\bar{\bar{\varepsilon}}\cdot\bar{p}}{\sqrt{\bar{p}\cdot\bar{\bar{\varepsilon}}\cdot\bar{p}}}=\frac{\partial H}{\partial\nabla S}=\hat{v}\tag{24}$$

$$\nabla_{\bar{r}}H=-\frac{\partial}{\partial t}(\nabla S)=-\dot{\bar{p}}\tag{25}$$

Then, $\bar{p}=\nabla S$ and $\bar{r}$ (where $\bar{v}=d\bar{r}/dt$) are conjugate variables and consequently they provide equations of evolution of the wave front: the problem is decoupled into two independent and amenable problems: 1) wave front propagation and 2) fields amplitudes evolution.

Taking into account Equation (20), it is immediate to show that expression of $\hat{v}$ given by Equation (14) coincides with that of (24).

2.2. Geometrical Structure of Fields

We see from Equation (5) that vectors $\hat{D}$ and $\hat{B}$ are orthogonal to ∇S and that vector $\hat{E}$ is orthogonal to $\hat{B}$. Moreover, Equations (8), (11) and (14) also show relevant geometrical relations between electromagnetic fields. Naming tensor $\bar{\bar{\varepsilon}}^{-1} = \bar{\bar{\tau}}$, from Equations (1) and (14) one can write

$$\bar{\bar{\tau}} \cdot \hat{D} = \varepsilon_0 \, \hat{E}; \quad \bar{\bar{\tau}} \cdot \hat{B} = \mu_0 \, \hat{H}; \quad \bar{\bar{\tau}} \cdot \hat{v} = -\frac{c^2}{I_3} \frac{\nabla S}{\frac{\partial S}{\partial t}} \tag{26}$$

Following the procedure given in [6], two conjugate vector bases can be introduced. We choose a direct local basis $(P, \bar{e}_i / \{i = 1, 2, 3\})$ consisting of three unit vectors:

$$\bar{e}_1 = \frac{\hat{D}}{|\hat{D}|}; \quad \bar{e}_2 = \frac{\hat{B}}{|\hat{B}|}; \quad \bar{e}_3 = \frac{\hat{v}}{|\hat{v}|}; \tag{27}$$

We can also define a reciprocal local basis $(P, \bar{e}^j / \{j = 1, 2, 3\})$ as:

$$\bar{e}^j = \frac{\bar{\bar{\tau}} \cdot \bar{e}_j}{\tau_j} \quad j \in (1, 2, 3) \quad \rightarrow \ \bar{e}_i \cdot \bar{e}^j = \delta_i^j \tag{28}$$

where $\tau_j = \bar{e}_j \cdot \bar{\bar{\tau}} \cdot \bar{e}_j$. It is immediate to find that:

$$\bar{e}^1 = \frac{\varepsilon_0 \, \hat{E}}{|\hat{D}| \, \tau_1}; \quad \bar{e}^2 = \frac{\mu_0 \, \hat{H}}{|\hat{B}| \, \tau_2}; \quad \bar{e}^3 = -\frac{c^2 \, \nabla S}{I_3 \, |\hat{v}| \, \frac{\partial S}{\partial t} \, \tau_3} \tag{29}$$

Therefore, directions of fields $\hat{D}$, $\hat{B}$ and $\hat{v}$ are a triad of conjugate directions with respect to tensor $\bar{\bar{\tau}} = \bar{\bar{\varepsilon}}^{-1}$ and constitutes a local basis, whose reciprocal one is formed by directions of $\hat{E}$, $\hat{H}$ fields and ∇S. It must be pointed out that fields have only a component (covariant or contravariant) in these bases: e.g., $\hat{D} = D^1 \, \bar{e}_1$; $\hat{E} = E_1 \, \bar{e}^1$ and so on. There is an intrinsic geometrical structure associated with propagation of electromagnetic waves in transformation media. Thus, the Euclidean space is endowed, at every point P, with two dual bases defined from physical properties of the medium (see Fig. 1).

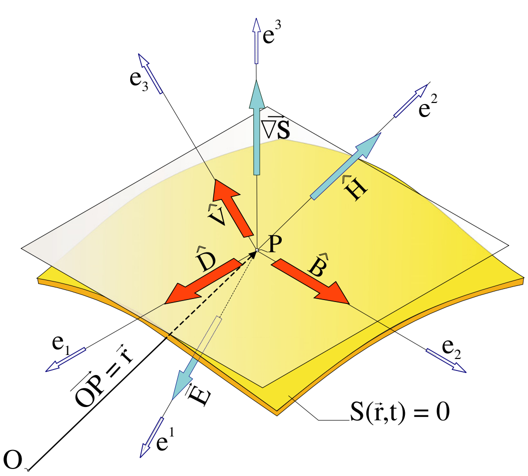

Figure 1. Local reciprocal vector bases $\bar{e}_i$ and $\bar{e}^i$ associated with the structure of electromagnetic waves propagating in an inhomogeneous anisotropic media with $\bar{\bar{\varepsilon}} = \bar{\bar{\mu}}$.

Evaluating τ_i, one gets

$$\tau_1 = \frac{\varepsilon_0 \, \hat{E} \cdot \hat{D}}{|\hat{D}|^2} = \frac{\hat{W}}{\hat{W}_0}; \quad \tau_2 = \frac{\mu_0 \, \hat{H} \cdot \hat{B}}{|\hat{B}|^2} = \frac{\hat{W}}{\hat{W}_0}; \quad \tau_3 = -\frac{c^2 \, \nabla S \cdot \hat{v}}{I_3 \, \frac{\partial S}{\partial t} \, |\hat{v}|^2} = \frac{n_r^2}{I_3} \tag{30}$$

where n_r is the ray index, defined as $n_r = c / |\hat{v}|$ [13] and then,

$$\bar{e}^1 = \frac{\varepsilon_0 \, \hat{W}_0}{|\hat{D}| \, \hat{W}} \, \hat{E}; \quad \bar{e}^2 = \frac{\mu_0 \, \hat{W}_0}{|\hat{B}| \, \hat{W}} \, \hat{H}; \quad \bar{e}^3 = -\frac{|\hat{v}| \, \nabla S}{\frac{\partial S}{\partial t}} \tag{31}$$

where $\tau_1 = \tau_2$ are the ratio between the electromagnetic energy density in the medium, $\hat{W}$, and the electromagnetic energy density $\hat{W}_0$, carried by the wave if propagation were in vacuum.

According to continuum mechanics [18, 19], the bound vector to an unit vector (e.g., $\bar{e}_1$) by $\bar{\bar{\tau}}$ tensor is defined as $\bar{\tau}_1 = \bar{\bar{\tau}} \cdot \bar{e}_1$. Then $\tau_1 = \bar{e}_1 \cdot \bar{\bar{\tau}} \cdot \bar{e}_1$ represents the normal intrinsic component of bound vector $\bar{\tau}_i$ and its graphical representation is named Cauchy's quadric. Assuming, without loss of generality, that eigenvalues of $\bar{\bar{\tau}}$ are all positive, Cauchy's quadric becomes an ellipsoid, that is known in optics as *ellipsoid of wave normals* [13]. Since the normal to Cauchy's quadric at every point [18] is collinear with bound vector $\bar{\tau}_u$ for any direction $\bar{u}$, conjugacy of electromagnetic vectors enable to determine the wave structure easily. For instance, the orthogonality of vectors $\bar{e}^1$ and $\bar{e}_3$ (and from Cauchy's theorem [18], $\bar{e}^3$ and $\bar{e}_1$) implies that $\bar{e}_1$ and $\bar{e}_3$ are conjugate directions. In order to visualize these concepts, a section of Cauchy's quadric of revolution is shown (see Fig. 2) and conjugate vectors lie along conjugate diameters of the elliptic section [20, 21].

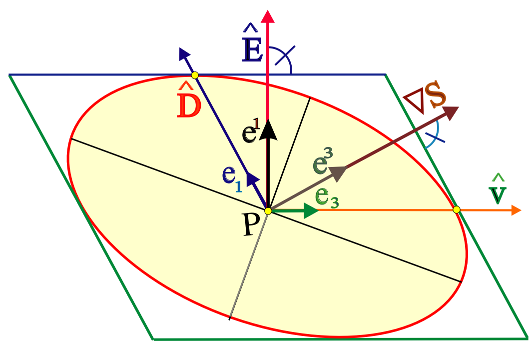

Figure 2. Conjugacy of vector fields. Section of the Cauchy's quadric associated with tensor $\bar{\bar{\varepsilon}} = \bar{\bar{\mu}}$. The orthogonality between vectors $\bar{e}_1$ and $\bar{e}^3$ implies that vectors $\bar{e}_1$ and $\bar{e}_3$ are directed along conjugate directions.

3. REFRACTION AT A BOUNDARY SURFACE

In the frame of optical-mechanical analogies, when mechanical energy is a constant, Maupertuis' principle [22] from mechanics ($\Delta \int_{\bar{r}_0}^{\bar{r}} \bar{p} \cdot d\,\bar{r} = 0$) leads for monochromatic waves to Fermat's principle given by $\Delta \int n\,ds = 0$.

In a previous paper [23] we introduced an alternative dual eikonal equation, in terms of ξ, in momentum space (duality is understood in the sense of Young), where $\nabla_p \xi$ and $\partial\,\xi/\partial t$ play the role of $\bar{r}$ and $\omega(\bar{r}, \bar{p})$, respectively. This duality provides an useful and powerful tool in determining particular solutions of eikonal equation in certain cases. Associated with dual eikonal equation, a dual Maupertuis' principle in momentum space can be written as:

$$\Delta\chi = \Delta \int_{\bar{p}_0}^{\bar{p}} \bar{r} \cdot d\,\bar{p} = 0 \tag{32}$$

that, when applied to a discontinuity surface Σ separating two media, Snell's law yields immediately. Therefore, refraction is completely analogous to the imposition of an impulsive constraint in mechanics and, reciprocally, if principle (32) is applied to a discontinuity surface, coplanarity of $\bar{p}_1$, $\bar{p}_2$ and unit normal to the boundary $\bar{n}$ arises in agreement with the Snell's law of refraction.

4. CONCLUSIONS

The use of a phase-amplitude description for fields has allowed us to reobtain some laws of evolution, in which phases decouple from amplitudes and unavoidable convolution products, inherent to the application of integral transforms, vanish. The two sets of obtained equations are easily interpretable:

(i) In the framework of transformation theory in classical mechanics, phase evolution (dispersion equation) is identified with Hamilton-Jacobi equation and from Hamilton equations (for the associated Hamiltonian), ray evolution is determined.

(ii) Its geometry is intrinsically related to the physics of the problem. Indeed, local reciprocal bases lie along field vectors, Poynting's vector and electromagnetic momentum and the associated space metrics is related to local relative permittivity/permeability tensor of the medium, $\bar{\bar{\varepsilon}}$. Therefore, from the knowledge of the ray trajectory, the wave front can be immediately determined and vice versa.

It must be noticed that polar representation of fields provides and exact solution of the equations, in decoupling the problem and reducing it to another equivalent one, that is the superposition of two known problems.

On the other hand, in searching solutions of section *(i)*, the specific functional dependency of tensor $\bar{\bar{\varepsilon}}$ is sometimes greatly facilitated if dual dispersion equation is used [23]. According to the cited procedure, refraction at a boundary surface can be identified with the imposition of a mechanical impulsive constraint and the facilities of dual Fermat's principle in the discussion is suggested.

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REFERENCES

1. Leonhardt, U. and T. G. Philbin, *Geometry and Light: The Science of Invisibility*, Dover, Mineola, 2010.
2. Urzhumov, Y. A., N. B. Kundtz, D. R. Smith, and J. B. Pendry, "Cross-section comparison of cloaks designed by transformation optical and optical conformal mapping approaches," *Journal of Optics*, Vol. 13, 024002, 2011.
3. Pendry, J. B., D. Schurig, and D. R. Smith, "Controlling electromagnetic fields," *Science*, Vol. 312, No. 5781, 1780–1782, 2006.
4. Halimeh, J. C. and M. Wegener, "Time-of-flight imaging of invisibility cloaks," *Opt. Express*, Vol. 20, No. 1, 63–74, 2012.
5. Halimeh, J. C. and M. Wegener, "Photorealistic rendering of unidirectional free-space invisibility cloaks," *Opt. Express*, Vol. 21, 9457–9472, 2013.
6. Bellver-Cebreros, C. and M. Rodriguez-Danta, "An alternative model for wave propagation in anisotropic impedance-matched metamaterials," *Progress In Electromagnetics Research*, Vol. 141, 149–160, and References Therein, 2013.
7. Melrose, D. B. and R. C. McPhedran, *Electromagnetic Processes in Dispersive Media*, Cambridge University Press, Cambridge, 1991.
8. Schurig, D., J. B. Pendry, and D. R. Smith, "Calculation of material properties and ray tracing in transformation media," *Opt. Express*, Vol. 14, No. 21, 9794–9804, 2006.
9. Bellver-Cebreros, C. and M. Rodriguez-Danta, "Eikonal equation from continuum mechanics and analogy between equilibrium of a string and geometrical light rays," *Am. J. Phys.*, Vol. 69, 360–367, 2001.
10. Jacobi, C. G., *Vorlesungen Fiber Dynamik*, A. Clebsch, Berlin, 1866, Reprinted by Chelsea, 1969.
11. Arnold, V. I., *Mathematical Methods of Classical Mechanics*, 2nd edition, (Section 47), Springer-Verlag, 1989.
12. Bellver-Cebreros, C. and M. Rodriguez-Danta, "Conjugacy and reciprocal bases in anisotropic dielectric media," *Optica Pura Y Aplicada*, Vol. 44, No. 3, 571–580, 2011.
13. Born, M. and E. Wolf, *Principles of Optics*, Pergamon Press, New York, 1999.

14. Chandrasekharaiah, D. L. and L. Debnath, *Continuum Mechanics*, Page 80, Academic Press, San Diego, 1994.
15. Jackson, J. D., *Classical Electrodynamics*, Page 80, John Wiley and Sons, Hoboken, New Jersey, 1994.
16. Landau, L. D. and E. M. Lifshitz, *The Classical theory of Fields*, 3rd edition, Vol. 2, Pergamon Press, Oxford, 1971.
17. Landau, L. D., E. M. Lifshitz, and L. P. Pitaevskii, *Electrodynamics of Continuous Media*, 2nd edition, Vol. 8, Butterworth-Heinemann, Oxford, 1984.
18. Fung, Y. C., *Foundations of Solid Mechanics*, Prentice Hall, Englewood Cliffs, New Jersey, 1965.
19. Bellver-Cebreros, C. and M. Rodriguez-Danta, "Analogies between the torque-free motion of a rigid body about a fixed point and light propagation in anisotropic media," *Eur. J. Phys.*, Vol. 30, 285–294, 2009.
20. Salmon, G., *A Treatise on Conic Sections*, AMS Chelsea Publishing, Providence, Rhode Island, 1960.
21. Zwikker, C., *The Advanced Geometry of Plane Curves and Their Applications*, Dover Publications, Mineola, NY, 2005.
22. Landau, L. D. and E. M. Lifshitz, *Mechanics*, Pergamon Press, 1976.
23. Bellver-Cebreros, C., M. Rodriguez-Danta, and E. Gomez-Gonzalez, "Angle eikonal from reciprocal action and other optical-mechanical analogues," *Optik*, Vol. 111, No 6, 261–268, and References Therein, 2000.