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Equivalent Model from Two Layers Stratified Media to Homogeneous Media for Overhead Lines

Zeyneb Belganche[1], Abderrahman Maaouni[2, *], Ahmed Mzerd[1], and Amine Bouziane[2]

Abstract—Overhead power transmission line is influenced by the resistivity of earth return path. The topic is developed in literature by considering a homogeneous and isotropic earth, or verily the soil is more represented by several layers. The scope of this paper is to provide an equivalent homogeneous soil to the two layers stratified soil. The equivalent electromagnetic properties of the soil are calculated using an accurate minimization method. Numerical results presented in this paper, show the efficiency of the proposed model.

1. INTRODUCTION

To analyze in power line carrier and extra low frequency communications, surge problems and so on, impedance and admittance correction terms are important. Consequently, it is mandatory to have an accurate transmission line modelling. Modelling of Transmission line (TL) above an imperfect earth, claims to consider the impact of the imperfect soil on the conductors parameters. The primordial assumptions which must be taken into account, are to consider the conductor cables as thin wires of infinite length above a stratified media with adoption of a quasi-TEM approach. Many approaches have been used and reported by the literature to include earth return path effects in the transmission line impedances, the first approach is the Carson's homogeneous earth model [1], which considers an infinite earth but neglects displacement current and the influence of the imperfect earth on shunt admittance. Or the soil is composed of many layers with different electromagnetic properties (resistivities, permittivities and permeabilities), that is why a lot of models have been developed taking into account the stratified media and also the displacement current. Sunde [2], and Iwamoto [3] for the cases of two-layer earth, developed a stratified earth impedance. Nevertheless, the boundary conditions are not sufficiently general and the propagation of current along a line is neglected in Sunde's solution and Iwamoto's formula.

As a matter of fact, displacement current appears to be added as a correction term, although Wedepohl and Wasley [4] calculated a two-layer earth impedance using double-integral transform. Later Nakagawa et al. [5] have developed an accurate and general solution for the earth-return impedance of an overhead line. He considered the soil consisting of three layers with arbitrary resistivity, permittivity and permeability.

The authors reported an accurate analytical approximation of integrals relating to overhead transmission line. More recently, Papadopoulos et al. [6] have presented a paper about a generalized model for the calculation of the impedances and admittances of an overhead power line above stratified earth. The authors have reported a formulation which purpose is to calculate the series impedances and shunt admittances which present the influence of the imperfect earth.

In his paper, Papadopoulos's impedance and admittance are applied to investigate the effects of a stratified earth on wave propagation. The assumption of quasi-TEM field propagation is the base of the

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* Corresponding author: Abderrahman Maaouni (maaouni@lycos.com).
[1] Physics Department, LPM, Faculty of Sciences, Mohammed V University, Rabat, Morocco. [2] Physics Department, LPT, URAC 13, Faculty of Science, Mohammed V University, Rabat, Morocco.

analysis. The Hertzian Vector approach is used to solve the electromagnetic field equations. The present work demonstrates a simplified methodology to determine the electrical parameters for an equivalent homogeneous soil with resistivity and permittivity which respond to the two layers stratified media.

2. IMPEDANCE AND ADMITTANCE OF OVERHEAD LINES ABOVE A STRATIFIED EARTH

Consider a system of N parallel thin wires located in the air above the stratified media. The j-th ($j = 1, 2, \ldots, N$) wire has a radius of a_j, and is located at a height h_j, and a position $x = x_j$. The air has a conductivity σ_0 equal to zero, permeability μ_0 and permittivity ε_0 equal to those of the free space. The soil is assumed to be composed by two horizontal layers. The first layer has a depth d with μ_1, ε_1 and σ_1 are respectively the permeability, permittivity and the conductivity. The second layer has electrical parameters μ_2, ε_2, σ_2 and is considered to be of infinite depth. Figure 2 illustrates the geometry of the problem.

In accordance with Papadopoulos, the system is governed by the telegrapher's equations:

$$\frac{\partial \mathbf{V}}{\partial x} = -\mathbf{Z}\ \ \mathbf{I} \tag{1}$$

$$\frac{\partial \mathbf{I}}{\partial x} = -\mathbf{Y}\ \ \mathbf{V} \tag{2}$$

V and **I** are the column matrices of voltages and currents at a distance x along the line. The differential equivalent circuit of the line is shown in Figure 1. **Z** and **Y** are square matrices of the per-unit (pu) length impedance and admittance. The admittance matrix is related to the potential coefficient matrix **P** [5, 7], by the relation:

$$\mathbf{Y} = j\omega \mathbf{P}^{-1} \tag{3}$$

The pu length impedance matrix can be expressed [5] at the following forms:

$$Z = Z_c + Z_s + Z_e \tag{4}$$

where

- $\mathbf{Z}_c$ is the internal impedance matrix of the conductor.
- $\mathbf{Z}_s$ is the space impedance matrix.
- $\mathbf{Z}_e$ is the ground return impedance matrix.

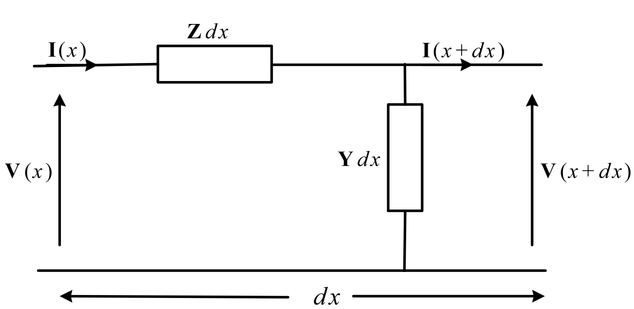

Figure 1. Differential equivalent circuit of transmission line.

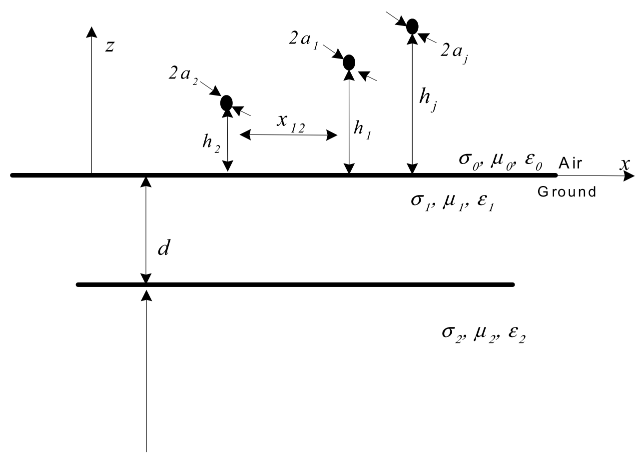

Figure 2. Geometry of the structure of N-wires line over stratified earth.

Papadopoulos's two layer mutual impedance for the system of Figure 2 is given in the following form:

$$Z_{mn}^{(s)} = j\omega\frac{\mu_0}{2\pi}\ln\frac{D_{mn}^{+}}{D_{mn}^{-}} + j\omega\frac{\mu_0}{\pi}\times\int\limits_0^\infty \chi^{(s)}(v)e^{-v(h_m+h_n)}\cos(x_{mn}v)dv \tag{5}$$

where

$$D_{mn}^{\pm} = \sqrt{x_{mn}^2 + (h_m \pm h_n)^2}, \quad x_{mn} = |x_m - x_n|$$

and

$$\chi^{(s)} = \mu_1\frac{s_{12}^{+} + s_{12}^{-}e^{-2\alpha_1 d}}{s_{01}^{+}s_{12}^{+} + s_{01}^{-}s_{12}^{-}e^{-2\alpha_1 d}} \tag{6}$$

$$\alpha_i = \sqrt{v^2 + \gamma_i^2 + k_0^2}, \quad \gamma_i = \sqrt{j\omega\mu_i\left(\sigma_i + j\omega\varepsilon_i\right)}$$

$$s_{ij}^{\pm} = \alpha_i\mu_j \pm \alpha_j\mu_i, \quad i,j = 0,1,2$$

The superscript s is related to the layer structure, and $k_0 = \omega\sqrt{\mu_0\varepsilon_0}$ is the propagation constant in the air.

The first term in (5) is the pu length inductance due to the geometry of the conductor. It is determined by adopting the method of images under the assumption of a perfectly conducting ground. The second term is the pu length mutual ground return impedance. The formula for the pu length mutual impedance between conductors is obtained using the quasi-TEM approximation. This approximation is good only if the conductor radius is small compared to the freespace wavelength, the distance between the conductors, and the heights of the conductors above the earth.

The pu length admittance is given by:

$$Y_{mn}^{(s)} = j\omega\left[\frac{1}{2\pi\varepsilon_0}\ln\frac{D_{mn}^{+}}{D_{mn}^{-}} + \frac{1}{\pi\varepsilon_0}\times\int\limits_0^\infty \left(\chi+\Psi\right)^{(s)} e^{-(h_m+h_n)v}\cos(y_{mn}v)dv\right]^{-1} \tag{7}$$

where

$$\Psi^{(s)} = \left(\mu_0\mu_1\left(\gamma_0^2 - \gamma_1^2\right)\left(s_{12}^{+} + s_{12}^{-}e^{-2\alpha_1 d}\right)\times\left(S_{12}^{+} + S_{12}^{-}e^{-2\alpha_1 d}\right) + 4\mu_0\mu_1^2\mu_2\alpha_1^2\gamma_0^2\times\left(\gamma_1^2 - \gamma_2^2\right)e^{-2\alpha_1 d}\right)\Big/(\delta\Delta) \tag{8}$$

In these formulas:

$$S_{ij}^{\pm} = \mu_i\gamma_j\alpha_i \pm \mu_j\gamma_i\alpha_i, \quad \delta = s_{01}^{+}s_{12}^{+} + s_{01}^{-}s_{12}^{-}e^{-2\alpha_1 d},$$
$$\Delta = S_{01}^{+}S_{12}^{+} + S_{01}^{-}S_{12}^{-}e^{-2\alpha_1 d},$$
$$j = \sqrt{-1},$$
$$\omega = 2\pi f = \text{radian frequency}.$$

In dealing with the electromagnetic fields produced by current-carrying wires, it is convenient to employ the Hertz vector $\mathbf{\Pi}$. For a system of horizontal wires parallel to the earth's surface, the boundary conditions at the air-ground could not be met if the Hertz vector had only an x component. As Sommerfeld pointed out many years ago [8], the situation could be remedied if $\mathbf{\Pi}$ was allowed to have x and z components.

The electric and magnetic fields must satisfy the boundary conditions at each separating surface (the planes $z = 0$ and $z = -d$). Thus the boundary conditions for the components of the Hertzian potentials and their derivatives are the following:

$$\begin{aligned}
\gamma_i^2\Pi_{ix} &= \gamma_{i+1}^2\Pi_{(i+1)x}\\
\frac{1}{\mu_i}\gamma_i^2\frac{\partial\Pi_{ix}}{\partial z} &= \frac{1}{\mu_{i+1}}\gamma_{i+1}^2\frac{\partial\Pi_{(i+1)x}}{\partial z}\\
\frac{\gamma_i^2}{\mu_i}\Pi_{iz} &= \frac{\gamma_{i+1}^2}{\mu_{(i+1)}}\Pi_{(i+1)z}\\
\frac{\partial\Pi_{ix}}{\partial x} + \frac{\partial\Pi_{iz}}{\partial z} &= \frac{\partial\Pi_{(i+1)x}}{\partial x} + \frac{\partial\Pi_{(i+1)z}}{\partial z}
\end{aligned} \tag{9}$$

where the subscript 0 specifies these quantities in the upper half space ($z > 0$). The nature of the boundary conditions in this problem suggests the potential required forms [6, 9]:

$$\begin{aligned}
\Pi_{0x} &= \int_0^\infty \left(C\frac{\lambda}{u_0} e^{-u_0|z-h|} + a_0 e^{-u_0 z} \right) J_0(r\lambda) d\lambda \\
\Pi_{0z} &= \frac{x}{r} \int_0^\infty a_0' e^{-u_0 z} J_1(r\lambda) d\lambda, \quad z > 0 \\
\Pi_{1x} &= \int_0^\infty \left(a_1 e^{u_1 z} + b_1 e^{-u_1 z} \right) J_0(r\lambda) d\lambda \\
\Pi_{1z} &= \frac{x}{r} \int_0^\infty \left(a_1' e^{u_1 z} + b_1' e^{-u_1 z} \right) J_1(r\lambda) d\lambda, \quad -d < z < 0 \\
\Pi_{2x} &= \int_0^\infty a_2 e^{u_2 z} J_0(r\lambda) d\lambda \\
\Pi_{2z} &= \frac{x}{r} \int_0^\infty a_2' e^{u_2 z} J_1(r\lambda) d\lambda, \quad z < -d
\end{aligned} \tag{10}$$

where C is proportional to the dipole moment Ids. $J_k(\cdot)$ is the Bessel function of the first kind of order k, $u_k = \sqrt{\lambda^2 + \gamma_k^2}$, where $k = 0, 1, 2$, $r = \sqrt{x^2 + y^2}$.

This completes the formal integral solution for electromagnetic fields and consequently leads to the pu length line parameters by integrating along the conductors.

3. EQUIVALENT CONDUCTIVITY AND PERMITTIVITY OF A TWO LAYER STRATIFIED SOIL

For an horizontal wire structures located above an homogeneous dissipative earth, the soil effect is given by the following Sommerfeld-type Fourier integrals [11]:

$$J(D_{mn}^+) = \int_0^\infty \frac{e^{-u_0'(h_m+h_n)}}{u_0' + u_g'} \cos(v x_{mn}) dv \tag{11}$$

$$G(D_{mn}^+) = \int_0^\infty \frac{e^{-u_0'(h_m+h_n)}}{n_0^2 u_0' + u_g'} \cos(v x_{mn}) dv \tag{12}$$

where $u_0' = \sqrt{\tau_0^2 + v^2}$, $u_g' = \sqrt{\tau_g^2 + v^2}$. τ and τ_g are the transverse propagation constant in the air and in the ground respectively. $n_0 = \sqrt{\varepsilon_{rg} + j\frac{\sigma_g}{\omega\varepsilon_0}}$ is the complex refractive index of the soil with electrical parameters $\varepsilon_g = \varepsilon_{rg}\varepsilon_0$, μ_0, and σ_g. The evaluation of the integrals such as (11) and (12) requires considerable effort [10]. However, within the framework of the quasi-TEM approximation, the transverse propagation constant in the air approaches zero, and the pu length line parameters for this system of N parallel wires can be expressed as:

$$Z_{mn}^{(h)} = j\omega \frac{\mu_0}{2\pi} \ln \frac{D_{mn}^+}{D_{mn}^-} + j\omega \frac{\mu_0}{\pi} \times \int_0^\infty \chi^{(h)} e^{-v(h_m+h_n)} \cos(x_{mn} v) dv \tag{13}$$

and

$$Y_{mn}^{(h)} = j\omega \left[\frac{1}{2\pi\varepsilon_0} \ln \frac{D_{mn}^+}{D_{mn}^-} + \frac{1}{\pi\varepsilon_0} \times \int_0^\infty \Psi^{(h)} e^{-(h_m+h_n)v} \cos(x_{mn} v) dv \right]^{-1} \tag{14}$$

The superscript h denotes the homogeneous earth. $\chi^{(h)}$ and $\Psi^{(h)}$ are defined by:

$$\chi^{(h)}(v) = \frac{1}{v + \sqrt{v^2 + k_0^2\left(1 - n_0^2\right)}} \tag{15}$$

$$\Psi^{(h)}(v) = \frac{1}{n_0^2 v + \sqrt{v^2 + k_0^2\left(1 - n_0^2\right)}} \tag{16}$$

From the analysis of the contribution of soil in terms of the pu length line parameters, it follows that the behavior of homogeneous and stratified structures are equivalent when the values of integrals are almost similar. The electrical parameters of the equivalent homogeneous ground model to the layered structure can therefore be deduced mathematically in terms of the following error minimization:

$$\min_{\sigma_g,\varepsilon_g}\left(\left|\chi^{(s)} - \chi^{(h)}\right|^2 + \left|(\chi+\Psi)^{(s)} - \Psi^{(h)}\right|^2\right) \quad \text{Subject to} \quad \sigma_g > 0, \quad \mu_g = \mu_0, \quad \varepsilon_g > 0, \quad \forall v \tag{17}$$

Minimizing the square error between the integrands on the semi-infinite interval is closely related to the effective length of the integration interval which depends on the frequency. To remedy this frequency sensitivity in the determination of the electrical parameters of the equivalent homogeneous ground model, the following change of variables $v(t) = \frac{1-t}{t}$ is adopted so as to change the interval $0 \le v \le \infty$ into the interval $0 \le t \le 1$. By dividing the interval into $N_{\max}$ subintervals of the same length, the minimization problem (17) can be rewritten correctly as follows:

$$\min_{\sigma_g,\varepsilon_g}\left(\sum_{i=1}^{N_{\max}+1}\left\{\left|\chi^{(s)}(v_i) - \chi^{(h)}(v_i)\right|^2 + \left|(\chi+\Psi)^{(s)}(v_i) - \Psi^{(h)}(v_i)\right|^2\right\}\right)$$

$$\text{Subject to} \quad \sigma_g > 0, \quad \mu_g = \mu_0, \quad \varepsilon_g > 0 \tag{18}$$

where $v_i = v(t_i)$, and $t_{i,\,(i=1,\,2,\,\ldots,\,N_{\max}+1)}$, are equally spaced samples of the interval $0 \le t \le 1$. The Nelder-Mead method [12] is used in this paper, to allow the determination of values of conductivity and permittivity corresponding to an equivalent earth of only homogeneous layer, referred to an equivalent soil with conductivity and permittivity, which has a behavior similar to a soil of two layers. The frequency is set at the value of 500 kHz.

4. NUMERICAL RESULTS

The geometry of the layered structure is shown in Figure 2. In this first example, the thickness of the first layer is $d = 2.69$ m. Its conductivity and relative permittivity are respectively $\sigma_1 = 3.666\times10^{-3}$ S/m and $\varepsilon_{r1} = 10$. The second layer is characterized by the electrical parameters $\sigma_2 = 6.884\times10^{-3}$ S/m and $\varepsilon_{r2} = 12$. The lower half space ($z < 0$) is assumed to have the free space permeability. The minimization process leads to the equivalent homogeneous earth parameters $\sigma_g = 3.3208\times10^{-3}$ S/m and $\varepsilon_{rg} = 7.87$. The number of samples in the interval is set to the value of $N_{\max} = 40$. Figure 3 shows the behavior of integrals representing losses in the stratified earth in comparison with their counterparts in the equivalent homogeneous soil model, as function of frequency. These integrals are defined by

$$J^{(s,h)} = \int_0^\infty \chi^{(s,h)} e^{-v\beta}\cos(\alpha v)dv$$

$$G^{(s)} = \int_0^\infty (\chi+\Psi)^{(s)} e^{-v\beta}\cos(\alpha v)dv$$

$$G^{(h)} = \int_0^\infty \Psi^{(h)} e^{-v\beta}\cos(\alpha v)dv$$

Analytical and accurate approximations of integrals $J^{(h)}$, and $G^{(h)}$ [13] representing losses in the homogeneous earth are adopted in this paper. Moreover, since the integrands of $J^{(s)}$ and $G^{(s)}$ do not contain 'critical factors' such as singularities and highly oscillating factors for the frequency range concerning the quasi-TEM approach, usual numerical integration methods, such as Gaussian quadrature and Clenshaw-Curtis quadrature may be employed. However, to overcome the subdivision of the semi-infinite interval into a sequence of intervals of finite length, and the sum of the integrals over the elements of the sequence until the convergence, a procedure based on a change of variable reducing an integral over an infinite range to one over a finite range is utilized. The details of the procedure and the analytical expressions for homogeneous soil integrals are reported in the Appendix.

Parameters related to the geometry of the conductors are $\beta = 10\,\mathrm{m}$ and $\alpha = 8\,\mathrm{m}$. As illustrated in the Figure 3, the equivalent model is able to estimate accurately the effect of the stratified earth on the multiwire structure.

In this second example, the effect of the ratio ($\zeta = \frac{\sigma_1}{\sigma_2}$) of the electrical conductivities of the two layers on the result of the equivalent model is discussed. The thickness of the first layer is $d = 4\,\mathrm{m}$. The parameters for conductors are $\alpha = 0\,\mathrm{m}$, $\beta = 8\,\mathrm{m}$. The values $\zeta = 0.5$ and $\zeta = 2$ which respectively correspond to $(\sigma_1, \sigma_2) = (0.1, 0.2)\,\mathrm{S/m}$ and $(\sigma_2, \sigma_1) = (0.2, 0.1)\,\mathrm{S/m}$ are used. The two layers admit electrical permittivities $\varepsilon_{r1} = 8$ and $\varepsilon_{r2} = 4$. Figures 4(a), (b), (e), (f) show the integral G function of frequency for different values of ζ. The contribution of this integral, representing displacement current losses in the earth, is greater when the conductivity of the first layer is lower. The curves related to stratified earth, and those obtained by the equivalent model are superposed. It is interesting to note that any reduction in the displacement current losses in the earth is accompanied by an increase in the conduction losses and vice versa. This is illustrated in Figures 4(c), (d), (g), (h). These figures give the results of the equivalent homogeneous earth model (dashed curves) and those concerning the stratified earth, which are in good agreement. Note that the results of the equivalent model estimate either by lower values or higher values those related to stratified earth, and this according to the value of ζ.

Now, considering the effect of permeability layers on the accuracy of the proposed equivalent homogeneous earth model. The geometrical parameters are set to values $\alpha = 0\,\mathrm{m}$ and $\beta = 8\,\mathrm{m}$. The

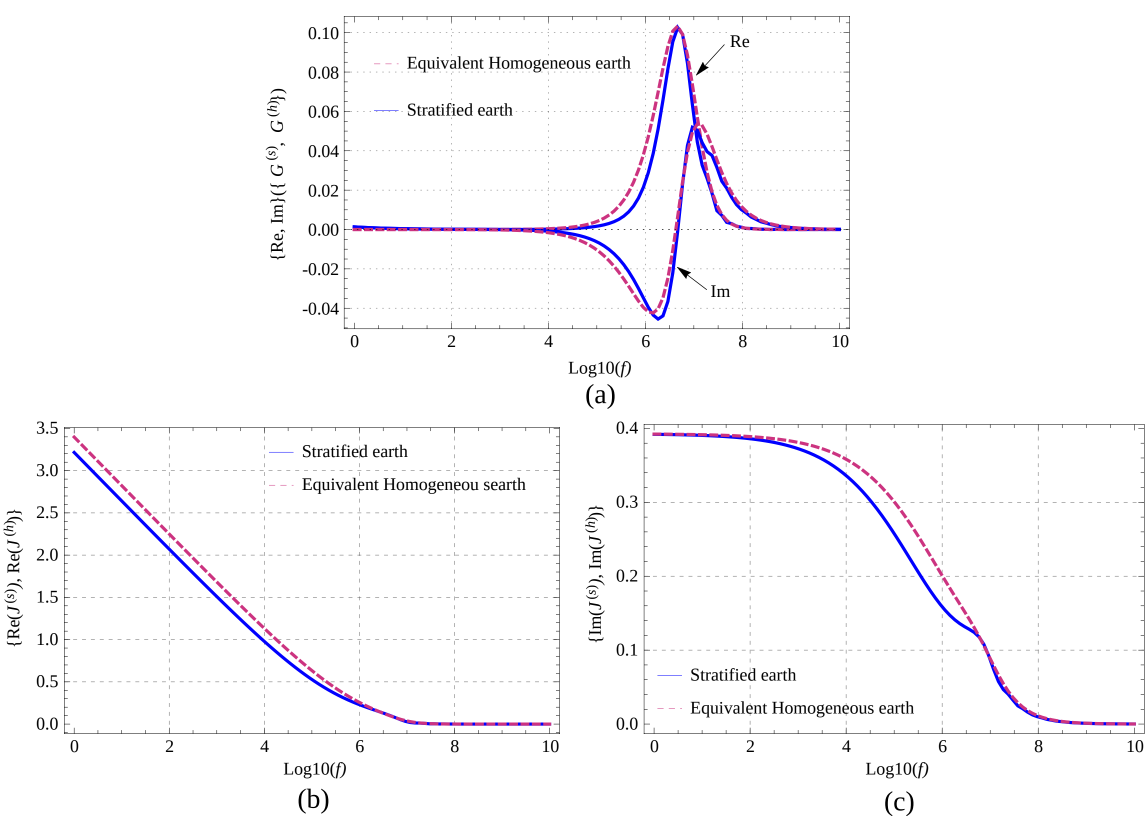

Figure 3. Variation as a function of a frequency of (a) $G^{(s,h)}$, (b) $\mathrm{Re}(J^{(s,h)})$, and (c) $\mathrm{Im}(J^{(h,s)})$. The superscripts h and s are for homogeneous and stratified earth.

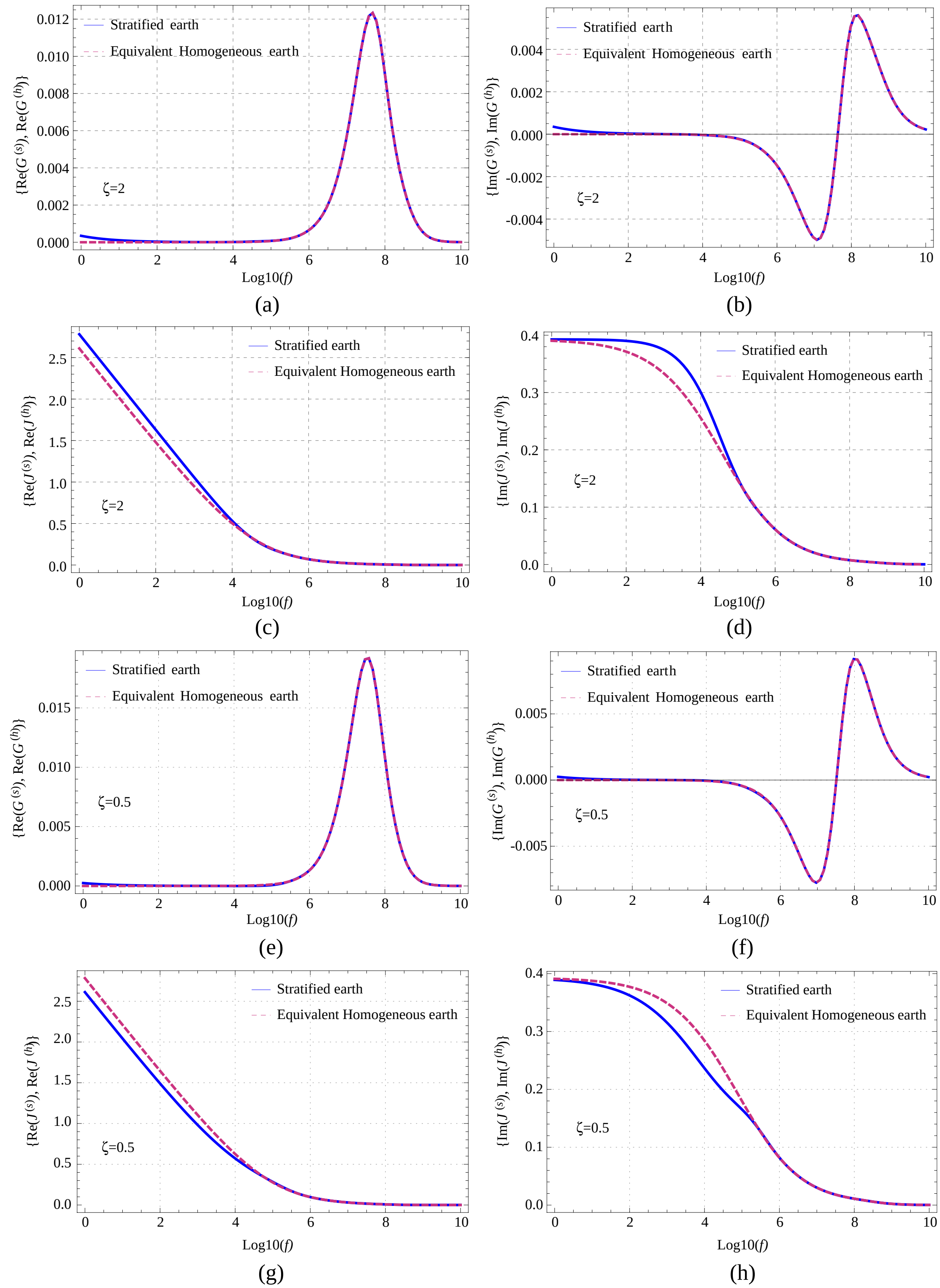

Figure 4. $G^{(h,s)}$ and $J^{(h,s)}$ as function of frequency for different values of ζ.

depth of the first layer is $d = 1\,\mathrm{m}$. The electrical properties of the two layers are $(\sigma_1, \sigma_2) = (0.2, 0.1)\,\mathrm{S/m}$, and $(\varepsilon_{r1}, \varepsilon_{r2}) = (8, 4)$. The curves (Figures 5(a), (b), (c), (d)) show the variation of the integrals $G^{(h,s)}$ and $J^{(h,s)}$ as a function of the frequency for $\mu_{r1} = 1.2$, and $\mu_{r2} = 1$. The variation of these integrals as a function of frequency for $\mu_{r1} = 1$, and $\mu_{r2} = 1.2$ is represented in (Figures 5(e), (f), (g), (h)).

From the analysis of curves in Figure 5, particularly those relating to the integral J, it appears that the proposed model is more accurate when the permeability of the layer 1 is larger than that

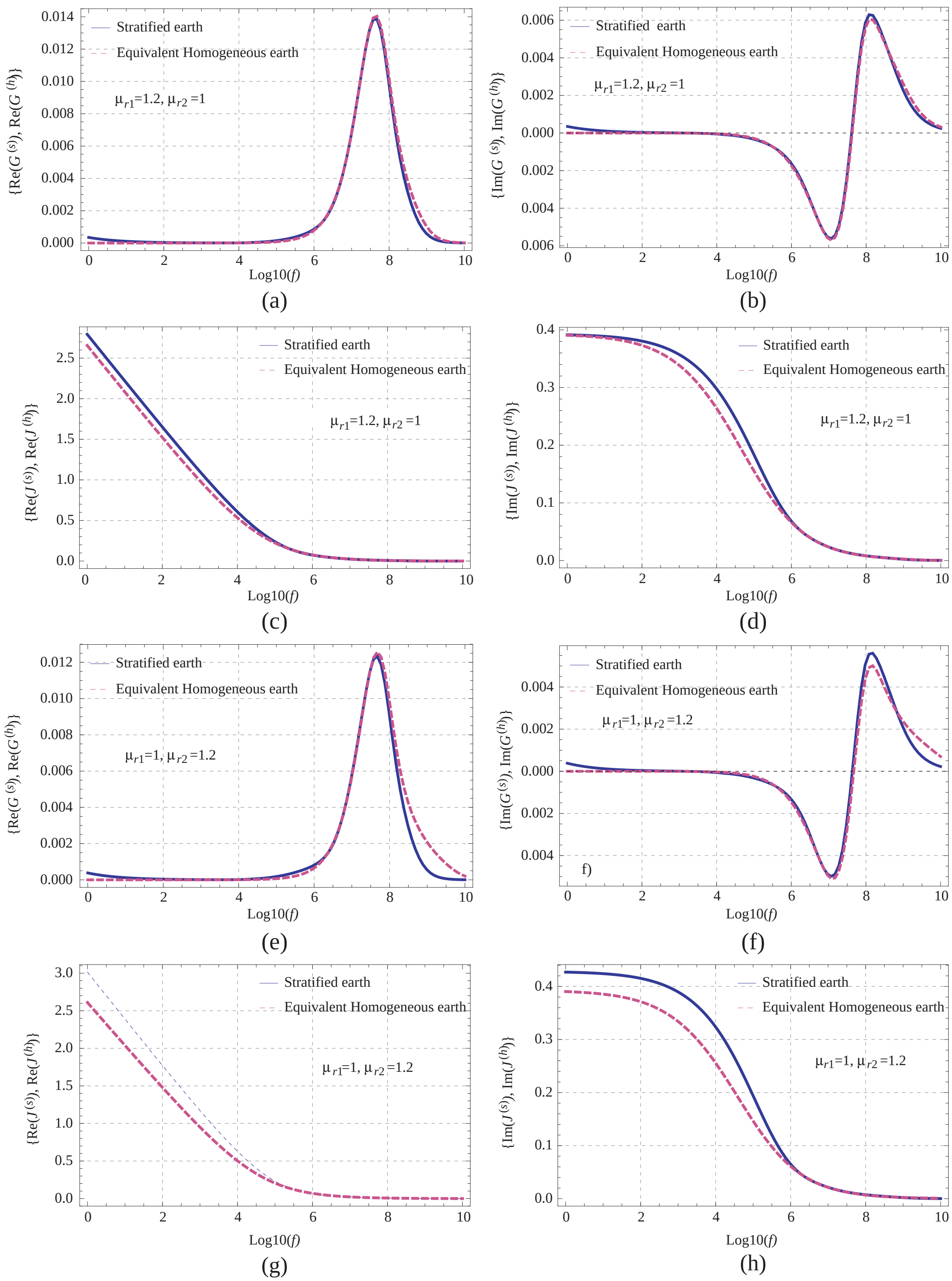

Figure 5. $G^{(h,s)}$ and $J^{(h,s)}$ as function of frequency for different values of permeability μ_{ri}, $i = 1, 2$.

of the layer 2. This discrepancy between the results of $J^{(h)}$ and $J^{(s)}$ relates to the fact that the equivalent model assumes a soil permeability μ_0. For more accuracy when the relative permeabilities of the layers are different from the unit, it is necessary to develop a model of a homogeneous soil that takes into account the permeability. The minimization process will therefore focus on three parameters (ε_{rg}, σ_g, μ_{rg}) instead of two (ε_{rg}, σ_g) as is the case here.

5. CONCLUSION

An accurate method for determining the electrical parameters of an homogeneous earth equivalent to a two-layer stratified earth, in an overhead lines structure is presented in this paper. The method is based on minimizing the quadratic error between the integrands of the integrals representing conduction and the displacement current losses in the stratified earth and in the equivalent homogeneous earth. A suitable change of variables is adopted to bypass dependency on the frequency of the effective lengths of the integration intervals. This has led to a better definition of minimization problem and gave results in good agreement.

APPENDIX A.

Accurate analytical expressions for $J^{(h)}$ and $G^{(h)}$ have been developed in [13]. Only the result is quoted here

$$J^{(h)} = \int_0^\infty \chi^{(h)} e^{-v\beta} \cos(\alpha v) dv = \int_0^\infty \frac{1}{v + \sqrt{v^2 + k_0^2(1-n_0^2)}} e^{-v\beta} \cos(\alpha v) dv \simeq \frac{1}{2} \ln\left(\frac{\rho_J^*}{\rho^*}\right)$$

where

$$\rho_J^* = \sqrt{\alpha^2 + \left(\beta + \frac{2}{k_0\sqrt{1-n_0^2}}\right)^2}, \quad \rho^* = \sqrt{\alpha^2+\beta^2}$$

For the integral $G^{(h)}$, we have

$$\begin{aligned}G^{(h)} &= \int_0^\infty \Psi^{(h)}(v) e^{-v\beta} \cos(\alpha v) dv = \int_0^\infty \frac{1}{n_0^2 v + \sqrt{v^2 + k_0^2(1-n_0^2)}} e^{-v\beta} \cos(\alpha v) dv \\ &= \frac{n_0^2}{4\left(n_0^4-1\right)}\left(Q(bz) + Q\left(b\bar{z}\right)\right) - \frac{1}{4b\left(n_0^4-1\right)}\left(P(b,z) + P\left(b,\bar{z}\right) - P(-b,z)\right. \\ &\quad \left. -P(-b,\bar{z})) - n_0^2 b\left(Q(-bz) + Q(-b\bar{z})\right)\right)\end{aligned}$$

where $Q(z) = \exp(-z)E_1(z)$. The exponential integral is defined by $E_1(z) = \int_z^\infty \exp(-t)/t \ dt$. $z = k_0(\alpha + j\beta)$, $\bar{z}$ is the complex conjugate of z, and $b = j/\sqrt{1+n_0^2}$.

$P(b,z)$ is defined by

$$P(b,z) = -\frac{1-n_0^2}{2b}\left(\ln\left(1 + \frac{2}{z\sqrt{1-n_0^2}}\right) + Q\left(bz + \frac{2b}{\sqrt{1-n_0^2}}\right)\right) + \frac{1}{z} + bQ(bz)\left(1 + \frac{1-n_0^2}{2b^2}\right)$$

APPENDIX B.

As mentioned in Section 3, the substitution $v = (1-t)/t$ changes the interval $0 \le v < \infty$ into the interval $0 \le t \le 1$. The integrals $J^{(s)}$ and $G^{(s)}$ can be expressed as

$$\int_0^\infty F(v) dv = \int_0^1 F\left(\frac{1-t}{t}\right) \frac{1}{t^2} dt$$

where $F(v) = \chi^{(s)}(v)e^{-v\beta}\cos(\alpha v)$ or $F(v) = (\chi + \Psi)^{(s)}e^{-v\beta}\cos(\alpha v)$. Then, we apply the following IMT transformation [14] which alleviates the singularity at the end point $t = 0$

$$\phi_0(t) = \exp\left(-\frac{1}{t} - \frac{1}{1-t}\right)$$

$$\psi_0(t) = \frac{1}{K}\int_0^t \phi_0(t)dt, \quad K = \int_0^1 \phi_0(t)dt \simeq 0.00702985840$$

Adopting this transformation, we get

$$\int_0^\infty F(v)dv = \frac{1}{K}\int_0^1 \phi_0(t)F\left(\frac{1-\psi_0(t)}{\psi_0(t)}\right)\frac{1}{\psi_0(t)^2}dt$$

Finally, the Gauss-Legendre quadrature [14] leads to the following expression

$$\int_0^\infty F(v)dv = \frac{1}{2K}\sum_{i=1}^n w_i\phi_0\left(\frac{x_i+1}{2}\right)F\left(\frac{1-\psi_0(\frac{x_i+1}{2})}{\psi_0\left(\frac{x_i+1}{2}\right)}\right)\frac{1}{\psi_0\left(\frac{x_i+1}{2}\right)^2}$$

where $w_i = \frac{2}{(1-x_i^2)[P_n'(x_i)]^2}$, and x_i are n zeroes of the nth-degree Legendre polynomial $P_n(x)$.

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