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Reflection and Transmission of an Electromagnetic Wave due to Fractal Slab Sandwiched between Ordinary Material

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Abstract—This paper presents an analytical solution to study the reflection and transmission of an electromagnetic wave impinged upon a multilayered structure. The structure is composed of a fractal slab sandwiched by ordinary material on either side. Modified Maxwell equations for fractional dimension space are used to represent the fields in a fractal slab. The electromagnetic characteristics of the structure are studied for different dimensions (D) and numerical results are presented for both the classical (D is integer) and fractal (D is non-integer) slabs. This study provides foundations for investigating the waveguides filled with fractal media and electromagnetic waves propagation in multilayered structures at fractional boundaries.

1. INTRODUCTION

In optics, microelectronic and engineering, multilayered structures have been of great interest to the researchers. In applications, such as lens designing, resonators, fibre optics, antireflection coatings etc., properties of multilayered structures have been exploited [1, 2]. Multilayered structures consisting of dielectric slabs or metallic sheets have been used in industry over the glass and plastic substrate to operate on microwave frequency and/or optical frequency to produce shielding effect from electromagnetic interference [3]. However, increase in losses and thickness are the serious limitations associated with the use of metallic multilayered structures. Therefore, composite structures are used as an alternate solution. Metamaterials (MTMs) are artificial composite structures made to acquire unusual electromagnetic properties that do not exist in materials found in nature [4]. In recent years, due to increasing interest of researchers in metamaterials, multilayered structures have gained considerable attention in the the field of electromagnetics. The applications of multilayered structures made up of metamaterials are polarization rotators, cloaks, electromagnetic tunneling, radomes and filter designing [5–7].

When an electromagnetic (EM) field interacts with material, its behaviour mainly depends on permeability (μ), permittivity (ϵ), and conductivity (σ) of the material. Multilayered structures are made of alternating layers of material with different constitutive parameters [8–10]. Smirnova, et al., suggested to employ multilayer graphene structure to overcome the difficulty to excite the graphene surface plasmon modes, due to its deep sub-wavelength nature [11]. Liu, et al., designed microwave filter using ϵ-negative material [12]. Sabah, et al., proposed filters made up of double positive and double negative MTMs and further concluded that chiral mirrors act as Bragg reflectors [10, 13]. Study on multilayered structures carried out by researchers can be easily found in literature [5–13].

The slabs (layers) constituting multilayered structures have always been considered ordinary dielectric [5–13]. If a fractal slabs were introduced instead of dielectric slabs, how would it affect the

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characteristics of these structures? Therefore, the aforementioned question and the analysis presented in [29] about the quasi fractional slab provides motivation for this work.

In 1982, Mandelbrot described the irregular and complex structures and geometries by introducing the concept of "fractals" [14]. This concept laid foundations to categorize objects such as Menger sponge, Serpinski triangle, porous media etc., as fractal geometries [15]. In this way, Mandelbrot was able to present a solution to differentiate between pure geometries and fractal geometries. It is not easy to use euclidean geometry to model every object in this universe. Therefore, fractional calculus is used to model these object in fractal geometry. Fractional calculus has been applied to various fields of sciences and is used by many scientists and researchers to find solutions of complex problems. Fractional calculus in physics and engineering has found applications in electrical, control and diffusive systems and mechanics [17]. Therefore, the fractional calculus study provided a way to analyze an electromagnetic wave propagation in fractal media [18]. Engheta presented the fractional solution to the Helmholtz wave equation followed by analysis on the role of fractional calculus in electromagnetics [19, 20]. In 1998, he also introduced the fractional curl operator in electromagnetics [21]. Naqvi et al. further extended the work using fractional calculus and provided important results in the field of electromagnetics [22–24]. Later on, Zubair et al. presented solutions to spherical wave, cylindrical wave, differential electromagnetic (EM) wave and general plane wave in D-dimension fractional space [25]. Extending these concepts electromagnetic wave interaction with fractal interfaces have been studied by various researchers [26–28].

In the presented work, the effect of dimension of the slabs on the reflectance and transmittance of the multilayered structure have been studied. In Section 2, structure layout, important parameters and model are discussed. In Section 3, the expressions for the transmission and reflection as a function of dimension, incident angle, slab width and frequency are derived. Transfer Matrix Method (TMM) is used to find the solutions. Numerical results to these expressions have then presented in Section 4 followed by the conclusion in Section 5.

2. DESIGN AND MODEL

A planar stratified structure used in this paper is composed of three layers, as shown in Figure 1. The structure is placed in cartesian coordinate system. The slabs are infinite in length. A quasi fractional space, slab F is sandwiched between homogenous, isotropic and non dispersive medium, slabs A. Index of refraction, wave number, and width for slab A are denoted by n_A, k_A, and d_A, respectively and for slab F, are denoted by n_F, k_F, and d_F, respectively. Constitutive parameters associated with slab A are (μ_A, ϵ_A) and slab F are (μ_F, ϵ_F). The structure is placed in air having index of refraction, $n_\circ = \sqrt{\mu_\circ \epsilon_\circ}$. Intrinsic impedance of the slabs for non-magnetic material is given by $\eta_i = \frac{\eta_\circ}{n_i}$, where $i = (A, F, \circ)$ and $\eta_\circ$ is the free space impedance. When wave travels in a slab from one point to another, the optical width is the geometric length d multiplied by n (refractive index of the slab). Therefore, optical width of slab A and slab F are $|n_A|d_A$ and $|n_F|d_F$, respectively. Transfer Matrix Method (TMM) is used to find the relation for reflected fields and transmitted fields. TMM helps in avoiding tedious calculations by solving matrices for periodic multilayered structures. In TMM, matching matrices relate fields on one side of the interface (I_i where $i = 1, AF, FA, 2$) to the other side and propagation matrices include the phase change occurred while propagating through optical width of slab. Combining matching matrices and propagation matrices yields transition matrix, which gives the expression for incident and reflected fields with transmitted fields. Note that fractionality of slab F is taken only in single direction i.e., z-axis and $e^{j\omega t}$ is the time dependency considered in this paper and is kept suppressed.

3. FORMULATION

A plane wave of magnitude E_i impinges on a three layered structure at an interface I_1 (located at $z = 0$) with an incident angle of θ_i from air. It is partially reflected with the magnitude E_r at a reflection angle θ_r and partially transmitted through the structure. The magnitude of transmitted field is E_t and transmission angle is θ_t. The electric and magnetic field expressions for incident, reflected

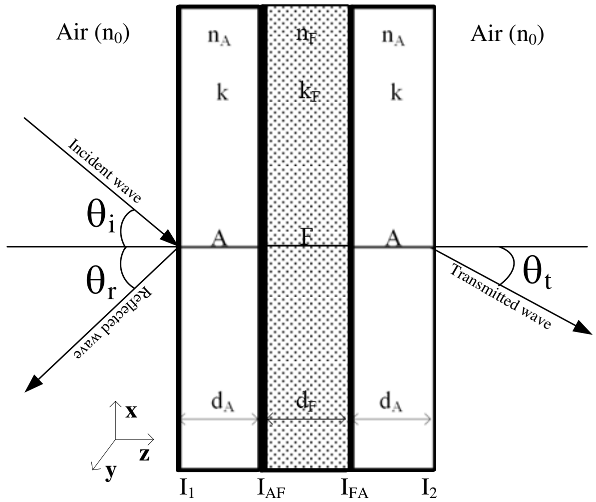


Figure 1. Three layered structure with fractal slab sandwiched.

and transmitted wave in air and slab A are [9],

$$\mathbf{E}_i = [E_{i\|}(\hat{x}\cos_{\theta_i} + \hat{z}\sin_{\theta_i}) + E_{i\perp}\hat{y}]e^{-jk_\circ(\hat{z}\cos_{\theta_i} - \hat{x}\sin_{\theta_i})}, \tag{1}$$

$$\mathbf{E}_r = [E_{r\|}(\hat{x}\cos_{\theta_r} - \hat{z}\sin_{\theta_r}) + E_{r\perp}\hat{y}]e^{jk_\circ(\hat{r}\cos_{\theta_r} + \hat{x}\sin_{\theta_r})}, \tag{2}$$

$$\mathbf{E}_t = [E_{t\|}(\hat{x}\cos_{\theta_t} + \hat{z}\sin_{\theta_t}) + E_{t\perp}\hat{y}]e^{-jk_\circ(\hat{z}\cos_{\theta_t} - \hat{x}\sin_{\theta_t})}, \tag{3}$$

$$\mathbf{H}_i = \frac{1}{\eta}[E_{i\|}\hat{y} - E_{i\perp}(\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)]e^{-jk_\circ(\hat{z}\cos_{\theta_i} - \hat{x}\sin_{\theta_i})}, \tag{4}$$

$$\mathbf{H}_r = \frac{1}{\eta}[E_{r\perp}(\hat{x}\cos\theta_r - \hat{z}\sin\theta_r - E_{r\|}\hat{y})]e^{jk_\circ(\hat{r}\cos_{\theta_r} + \hat{x}\sin_{\theta_r})}, \tag{5}$$

$$\mathbf{H}_t = \frac{1}{\eta}[E_{t\|}\hat{y} - E_{t\perp}(\hat{x}\cos\theta_t + \hat{z}\sin\theta_t)]e^{-jk_\circ(\hat{z}\cos_{\theta_t} - \hat{x}\sin_{\theta_t})}, \tag{6}$$

In above equations, the incident, reflected and transmitted fields are represented by the subscripts i, r, and t, respectively. Subscripts $\perp$ and $\|$ denote the perpendicular and parallel components of corresponding field, respectively. η and $k_\circ$ are the intrinsic impedance and wave number.

The wave incident on slab A from air, after propagating through optical path length of slab A, interacts with slab F at interface I_{AB}. Slab F is composed of fractal medium. Its dimension is represented by D ($2 \leq D \leq 3$). Electric and magnetic fields for fractal structures are computed using modified Maxwell equations [25]. Therefore, incident, reflected and transmitted fields in slab F can be written as,

$$\mathbf{E}_i = [E_{i\|}(\hat{x}\cos\theta_i + \hat{z}\sin\theta_i) + E_{i\perp}\hat{y}]e^{-jk_F(-x\sin\theta_i)}(k_F z\cos\theta_i)^n \left[H_n^{(2)}(k_F z\cos\theta_i)\right], \tag{7}$$

$$\mathbf{E}_r = [E_{r\|}(\hat{x}\cos\theta_r - \hat{z}\sin\theta_r) + E_{r\perp}\hat{y}]e^{-jk_F(-x\sin\theta_r)}(k_F z\cos\theta_r)^n \left[H_n^{(1)}(k_F z\cos\theta_r)\right], \tag{8}$$

$$\mathbf{E}_t = [E_{t\|}(\hat{x}\cos\theta_t + \hat{z}\sin\theta_t) + E_{t\perp}\hat{y}]e^{-jk_F(-x\sin\theta_t)}(k_F z\cos\theta_t)^n \left[H_n^{(2)}(k_F z\cos\theta_t)\right], \tag{9}$$

$$\mathbf{H}_i = \frac{1}{\eta}[E_{i\|}\hat{y} - E_{i\perp}(\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)]e^{-jk_F(-x\sin\theta_i)}(k_F z\cos\theta_i)^{nh} \left[H_{nh}^{(2)}(k_F z\cos\theta_i)\right], \tag{10}$$

$$\mathbf{H}_r = \frac{1}{\eta}[-E_{r\|}\hat{y} + E_{r\perp}(\hat{x}\cos\theta_r - \hat{z}\sin\theta_r)]e^{-jk_F(-x\sin\theta_r)}(k_F z\cos\theta_r)^{nh} \left[H_{nh}^{(1)}(k_F z\cos\theta_r)\right], \tag{11}$$

$$\mathbf{H}_t = \frac{1}{\eta}[E_{t\|}\hat{y} - E_{t\perp}(\hat{x}\cos\theta_t + \hat{z}\sin\theta_t)]e^{-jk_F(-x\sin\theta_t)}(k_F z\cos\theta_t)^{nh} \left[H_{nh}^{(2)}(k_F z\cos\theta_t)\right], \tag{12}$$

where, $\hat{x}$, $\hat{y}$ and $\hat{z}$ are the unit vectors. Similarly, subscripts i, r and t represent the incident, reflected and transmitted wave, respectively in fractal slab with $\|$ (parallel) and $\perp$ (perpendicular) components. The wave travelling in $+z$ axis (forward propagating wave) is denoted by Hankel function of the second kind and wave travelling in $-z$ axis (backward propagating wave) is expressed by Hankel function of the first kind. Order of the Hankel function is given by subscripts n or nh. Moreover, $n = \frac{|3-D|}{2}$ and $nh = \frac{|D-1|}{2}$ where D is the dimension. Wave number $(k_F) = \omega\sqrt{\mu\epsilon}$ and index of refraction (n_F) are assumed to be almost equal to that of integer spaces.

In order to find the relation between the fields on either side of the structure and study the effect of frequency, angle and dimension on the fields, boundary conditions are applied at each interface $(I_1, I_{AF}, I_{FA}, I_2)$ of the structure. I_1, I_{AF}, I_{FA}, and I_2 represent the air-dielectric, dielectric-fractal, fractal-dielectric, and dielectric-air interface, respectively. Applying boundary conditions on each interface yields matching matrices $(M_1, M_{AF}, M_{FA}, M_2)$. To compute M_{AF}, Eq. (19) represent equation in matrix form.

$$\begin{bmatrix} \cos\theta_i & 0 & \cos\theta_r & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -n_A\cos\theta_i & 0 & n_A\cos\theta_r \\ n_A & 0 & -n_A & 0 \end{bmatrix} \begin{bmatrix} E^A_{i\|} \\ E^A_{i\perp} \\ E^A_{r\|} \\ E^A_{r\perp} \end{bmatrix} = \begin{bmatrix} H_{2e}\cos\theta_i & 0 & H_{1e}\cos\theta_r & 0 \\ 0 & H_{2e} & 0 & H_{1e} \\ 0 & -n_B H_{2h}\cos\theta_i & 0 & n_B H_{1h}\cos\theta_r \\ n_B H_{2h} & 0 & -n_B F H_{1h} & 0 \end{bmatrix} \begin{bmatrix} E^B_{i\|} \\ E^B_{i\perp} \\ E^B_{r\|} \\ E^B_{r\perp} \end{bmatrix}, \tag{13}$$

where,

$$H_{1e} = (k_F z\cos\theta_r)^n \left[H^{(1)}_n(k_F z\cos\theta_r)\right], \tag{14}$$

$$H_{2e} = (k_F z\cos\theta_i)^n \left[H^{(2)}_n(k_F z\cos\theta_i)\right], \tag{15}$$

$$H_{1h} = (k_F z\cos\theta_r)^{nh} \left[H^{(1)}_{nh}(k_F z\cos\theta_r)\right], \tag{16}$$

$$H_{2h} = (k_F z\cos\theta_i)^{nh} \left[H^{(2)}_{nh}(k_F z\cos\theta_i)\right], \tag{17}$$

$[M_{AF}]$ in Eq. (18) can be obtained by substituting Eq. (14)–(17) in Eq. (13) by applying matrix operations,

$$\begin{bmatrix} E^A_{i\|} \\ E^A_{i\perp} \\ E^A_{r\|} \\ E^A_{r\perp} \end{bmatrix} = [M_{AF}] \begin{bmatrix} E^B_{i\|} \\ E^B_{i\perp} \\ E^B_{r\|} \\ E^B_{r\perp} \end{bmatrix}. \tag{18}$$

$$[M_{AF}] = \begin{bmatrix} \cos\theta_i & 0 & \cos\theta_r & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -n_A\cos\theta_i & 0 & n_A\cos\theta_r \\ n_A & 0 & -n_A & 0 \end{bmatrix}^{-1} \begin{bmatrix} H_{2e}\cos\theta_i & 0 & H_{1e}\cos\theta_r & 0 \\ 0 & H_{2e} & 0 & H_{1e} \\ 0 & -n_B H_{2h}\cos\theta_i & 0 & n_B H_{1h}\cos\theta_r \\ n_B H_{2h} & 0 & -n_B F H_{1h} & 0 \end{bmatrix}. \tag{19}$$

Similar procedure is followed for matching matrices $[M_1]$, $[M_{FA}]$ and $[M_2]$. All the matching matrices constituted are of 4×4 size except $[M_2]$ which is 4×2. After computing matching matrices, propagation matrix for each slab is constructed. It is a 4×4 diagonal matrix which includes path difference due to change in refractive indices. Eq. (20) shows a propagation matrix for slab A,

$$P_A = \begin{bmatrix} e^{-jk_\circ d_A} & 0 & 0 & 0 \\ 0 & e^{-jk_\circ d_A} & 0 & 0 \\ 0 & 0 & e^{jk_\circ d_A} & 0 \\ 0 & 0 & 0 & e^{jk_\circ d_A} \end{bmatrix} \tag{20}$$

A 4×2 transition matrix is constructed using propagation matrices and matching matrices as shown in Eq. (22). It relates the incident and reflected fields with transmitted field. In this equation, m is a positive integer which represents structure periodicity. The equation holds true for structure composed of any odd number of slabs.

$$\begin{bmatrix} E_{i\parallel} \\ E_{i\perp} \\ E_{r\parallel} \\ E_{r\perp} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \begin{bmatrix} E_{t\parallel} \\ E_{t\perp} \end{bmatrix} \tag{21}$$

$$T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} = [\mathrm{M}_1][\mathrm{P}_A][\mathrm{T}_1]^m[\mathrm{M}_2] \tag{22}$$

$$T_1 = [\mathrm{M}_{AB}][\mathrm{P}_B][\mathrm{M}_{BA}][\mathrm{P}_A] \tag{23}$$

The mathematical expressions derived in this section are valid for non-integer values of dimensions. Inserting integer value of dimension ($D = 1, 2, 3$) verifies classical results. In Eq. (21), incident, reflected and transmitted fields are function of frequency (f), incident angle (θ_i), dimension (D), and optical width. Numerical results for three layered are presented in Section 4.

4. RESULTS

In this section, numerical results for three layered structure composed of dielectric-fractal-dielectric interface are presented. The structure presented is periodic i.e., AFA and can be extended to odd number of slabs. Optical width of all the slabs is $\lambda_\circ/4$, where $\lambda_\circ$ is the wavelength at operational frequency 1 THz. Firstly, effect of dimension on the characteristic of reflection and transmission from the structure when the frequency is varied is shown in Figure 2. In this figure only parallel polarized wave is considered ($E_{i\parallel} \neq 0, E_{i\perp} = 0$). In Figure 2(a), it can be seen that reflected power at (0.8, 1.4, 1.65, 2.3, 2.6, 3.2 and 3.85) THz increases as the dimension of the sandwiched slab increases from $2.0D$ to $2.5D$. The structure behaves like a narrow band filter for $2.0D$, $2.3D$ and $2.5D$ with maximum bandwidth of 0.2 THz at centre frequency 3.2 THz for wave passing through $2.5D$ medium. Figure 2(b) shows the transmitted field behaviour. It contains sharp narrowband peaks with approximately 100% power (anti-reflective) at even harmonics for wave passing through $2.0D$, $2.3D$ and $2.5D$. Secondly, the effect of dimension on the characteristic of reflection and transmission from the structure when the incident angle is varied is shown in Figure 3. Figure 3(a), shows that the reflection power is

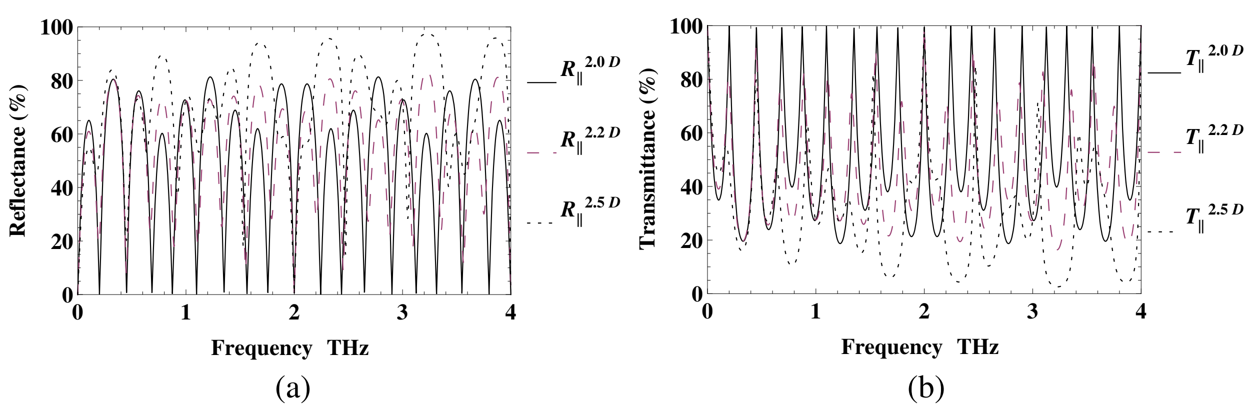
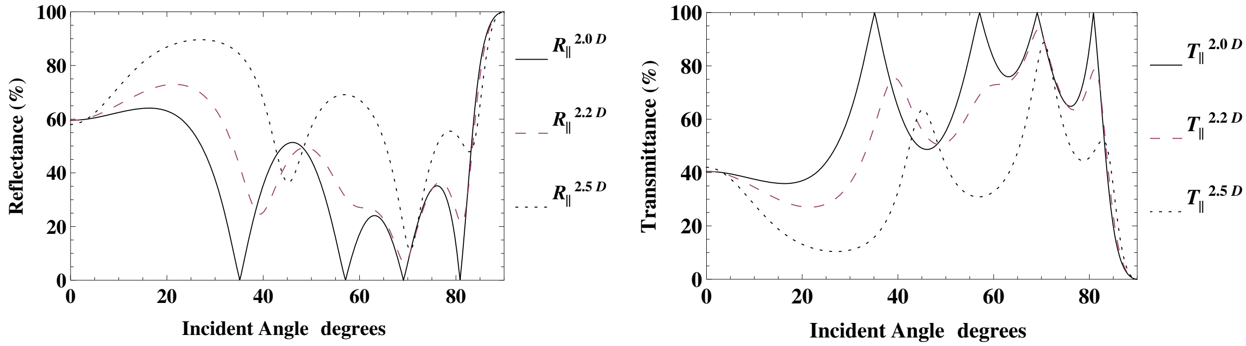


Figure 2. Reflectance and transmittance versus frequency for three layered structure with $n_A = 2.5$, $n_F = 2.0$, $|n_A|d_A = |n_F|d_F = \lambda_\circ/4$, and $\theta_i = 0°$.



(a) (b)

Figure 3. Reflectance and transmittance versus incident angle for three layered structure with $n_A = 2.5$, $n_F = 2.0$ $|n_A|d_A = |n_F|d_F = \lambda_0/4$, and $f/f_0 = 1.5$.

always greater when $2.5D$ and $2.3D$ medium are sandwiched except between (43°–50°) and (40°–50°), respectively. The minimum reflection power when $2.0D$ medium is sandwiched is zero at (35°, 57°, 69° and 81°). The minimum reflection power when $2.3D$ and $2.5D$ medium are sandwiched between ordinary material are 5% and 10% at 69.5° and 70.5°, respectively. Figure 3(b), shows the transmit power as a function of incident angle and dimension.

5. CONCLUSION

In this paper, analysis and solution to fractal structure sandwiched between ordinary material are presented using modified Maxwell equations for fractional space. Field equations inside fractal slab follow the fractional space equations. Reflection and transmission coefficients are obtained as a function of dimension, frequency, and angle of incidence for periodic multilayered structures by applying boundary conditions on all the interfaces of structure and Maxwell's curl equations. The reflection and transmission coefficients for parallel polarized wave are presented. These results show interesting frequency and incident angle analysis when dimension changes. The study explores ways to investigate waveguides and slabs filled with fractal medium and provide analysis for wave propagation at fractal boundaries.

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