# A New Property of Maximally-Flat Lowpass Filter Prototype Coefficients with Application in Dissipative Loss Calculations

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**Abstract**—This paper presents a new property of maximally-flat filter prototype coefficients. The property can be used to relate the summation of all the coefficients to an elegant expression which only includes the first coefficient. This property is then used to calculate the increase in insertion loss of this type of filters in the presence of dissipative losses due to elements/resonators finite quality factors. This presented equation for the excess loss is very convenient and does not require referring to the prototype element value table. The property is also used to show that the group delay of a maximally-flat lowpass filter at $\omega = 0$ rad/sec is only a function of the first element value of the prototype filter. Finally, a commercial circuit simulation tool is used to generate examples to verify the accuracy of the presented analytical equations. Additionally, the results are compared to expressions found in classical literature.

## 1. INTRODUCTION

The frequency spectrum is continuously becoming more crowded with the rapid growth of wireless communications services, satellite applications, and multifunction radar systems. This means that filters will play a more important role in future systems to make sharing of this limited resource possible. The conventional procedure for filter design using the insertion loss method includes the design of a lumped element lowpass filter with a normalized source impedance of $1\,\Omega$ and a cutoff frequency of $\omega_c = 1$ rad/sec, known as the lowpass prototype. Then, impedance and frequency scaling are used to obtain the desired source and load impedances as well as the new corner frequency. Conversion from the lowpass response to the desired bandpass, bandstop, or highpass response, with any arbitrary cutoff frequency or bandwidth, is achieved using frequency transformations. Finally, the filter is implemented using the calculated element values. In the case of RF and microwave filters, this implementation consists of the use of different techniques to convert lumped elements to distributed elements suitable for the technology, which is intended to be used for realizing the filter. However, this procedure does not consider the dissipative loss of the components which is usually modeled as their quality factor (Q). Therefore, both the magnitude and phase of the frequency response of the practical filter deviate from the ideal filter response. This deviation is more severe in the filter passband magnitude response than the phase response (group delay) and results in higher insertion loss, rounded corner(s) at the cutoff point(s), and reduced passband ripple [1, 2].

The design of a filter in the presence of dissipative loss has been investigated from very early stages of filter-design development by the insertion loss method. Dishal presented a design procedure for synthesizing asymmetric narrowband coupled resonator filters using finite-Q resonators [3]. Bode has also provided an expression for the calculation of the increase in insertion loss of a filter when all of its resonators have uniform dissipation (equal Q) [4]. In his expression, the excess loss is a function of the filter group delay and the resonators' dissipation factor. A similar equation was also derived by Matthaei, where the poles and zeros of a lossless filter are shifted by a constant distance along the

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real axis in the complex frequency plane [5]. The excess loss is expressed in terms of the dissipation factor and a constant. The value of the constant has been tabulated for the maximally-flat filter and plotted for the equal-ripple response in [5]. Another formula for calculating the excess loss of a filter with resonators of non-uniform quality factor has been derived by Cohn in terms of the filter's fractional bandwidth, the resonators quality factors, and the prototype coefficients [1].

Despite the fact that filter design and analysis have been revolutionized by recent advances in computer-aided design (CAD) tools and especially full-wave electromagnetic (EM) simulators, analytical expressions for the calculation of excess loss due to finite Q are still of particular interest. Recently, the effect of non-uniform distribution of the quality factor for resonators over the filter passband was studied where the first order approximations for the filter insertion loss and group delay as a function of frequency are provided [2]. This approach is more general, but due to the rigorous mathematical analysis, it is limited to low-order filters in practice.

In this paper an interesting new property of maximally-flat filter prototype coefficients is first derived. This property links the summation of prototype coefficients to the first coefficient. Then, this property is used to simplify Cohn's formula [1] to derive an elegant formula for the calculation of excess loss in maximally-flat filters of any order. It is shown that the excess loss due to the finite Q of the resonators can be calculated without knowing the prototype coefficients. This new property is also used to simplify the equation for the group delay of maximally-flat lowpass filters at $\omega = 0\,\text{rad/s}$. This expression is used to show that the formulas for the excess loss of the maximally-flat filters derived by Bode, Cohn and Matthaei are essentially all the same. At the end, examples are provided in order to compare the accuracy of the above mentioned formulas. Overall, it is envisioned that the property introduced in this paper, along with the derived expressions, will become widely used as a rule of thumb for quick filter calculations and also benefit future filter textbooks.

## 2. THEORETICAL EXPANSION FOR MAXIMALLY-FLAT FILTERS

The transmission and reflection functions of a two-port passive, reciprocal, lossless filter network in general, can be written using the ratio of two polynomials [6]

$$S_{21}(s) = \frac{1}{\varepsilon}\frac{P(s)}{E(s)} \tag{1}$$

$$S_{11}(s) = \frac{1}{\varepsilon_R}\frac{F(s)}{E(s)} \tag{2}$$

where polynomials $P(s)$ and $F(s)$ contain the filter transmission and reflection zeros, respectively and $E(s)$ is a Hurwitz polynomial. In addition, $\varepsilon$ and $\varepsilon_R$ are real constants used to normalize the highest degree coefficients of the polynomials to unity. For a lossless case the conservation of energy must be maintained; giving

$$|S_{11}(s)|^2 + |S_{21}(s)|^2 = 1 \tag{3}$$

Therefore,

$$\frac{1}{\varepsilon_R^2}\left|\frac{F(s)}{E(s)}\right|^2 + \frac{1}{\varepsilon^2}\left|\frac{P(s)}{E(s)}\right|^2 = 1 \tag{4}$$

In the case of maximally-flat lowpass prototype filters with 3.01 dB of attenuation at the corner frequency of $\omega = 1\,\text{rad/s}$, the constant and the polynomials are

$$P(s) = 1 \tag{5a}$$

$$F(s) = s^N \tag{5b}$$

$$\varepsilon_R = 1 \tag{5c}$$

$$\varepsilon = 1 \tag{5d}$$

where $N$ is the order of the filter. $E(s)$ can be found using Eq. (4) and the Hurwitz polynomial property for the lossless case.

Also, the transfer function for a maximally-flat lowpass filter can be written in terms of its poles as [7],

$$S_{21}(s) = \frac{1}{\prod\limits_{k=1}^{N}(s - p_k)} = \frac{1}{s^N - \left(\sum\limits_{k=1}^{N} p_k\right) s^{(N-1)} + \ldots - \prod_{k=1}^{N} p_k} \tag{6}$$

where the poles are defined by

$$p_k = -\sin(\theta_k) + j\cos(\theta_k) \tag{7a}$$

$$\theta_k = (2k - 1)\frac{\pi}{2N} \qquad \text{for } k = 1, 2, 3, ..., N \tag{7b}$$

Then $S_{11}(s)$ can be found from $S_{21}(s)$ using Eq. (3) which is used to construct the input driving point impedance to extract the prototype coefficients $(g_k)$ for the filter [7]. These values are tabulated in literature [5, 7] for different filter orders. However, closed form expressions have been developed [5], which directly give the prototype filter normalized element values (with cutoff frequency of 1 rad/s and unitary terminations) as

$$g_0 = g_{N+1} = 1 \ \Omega \tag{8a}$$

$$g_k = 2\sin\left(\frac{(2k-1)\pi}{2N}\right) = 2\sin(\theta_k) \qquad \text{for } k = 1, 2, 3, ..., N \tag{8b}$$

where ladder lowpass prototypes with the elements definition have been shown in Fig. 1.
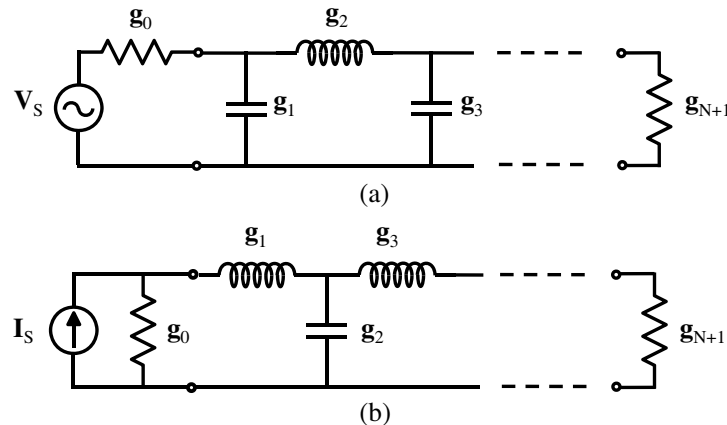


(a)

(b)

**Figure 1.** Ladder lowpass prototype with (a) shunt first element (b) series first element.

Knowing $p_k$, the coefficient of the second term in the denominator of Eq. (6) can be simplified in two different ways. First, using Eq. (7a), it can be written

$$-\sum_{k=1}^{N} p_k = \sum_{k=1}^{N} \sin(\theta_k) - j\sum_{k=1}^{N} \cos(\theta_k) \tag{9}$$

Distribution of $\theta_k$ for both odd and even orders can been seen in Fig. 2 where they are equally spaced on the circumference. For all even orders, the angles are symmetrically distributed about the vertical axis and form supplementary pairs. Therefore, any combination of $\theta_k$ and $\theta_{(N-k+1)}$ has equal sine and opposite cosine contributions. Thus when evaluating the sum in Eq. (9), the imaginary part is canceled and the real part is doubled. This is correct for odd orders as well, with the exception of the middle angle which is always $\pi/2$ rad/s and does not contribute to the imaginary part. Therefore,
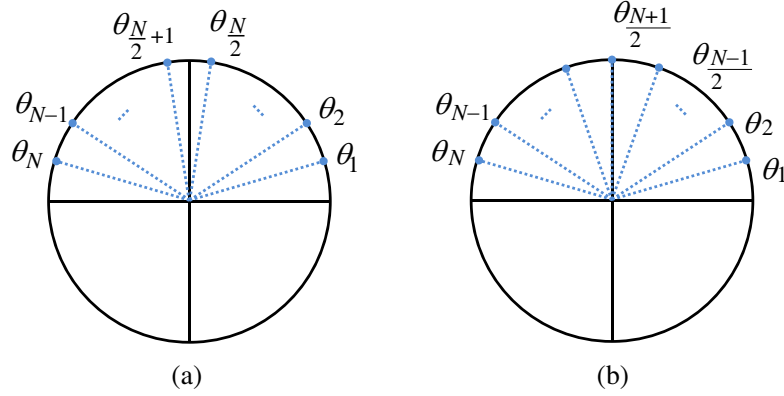
**Figure 2.** Positions of the angles of poles, $\theta_k$, defined in Eq. (8b) for the maximally-flat lowpass prototype on the real unit trigonometric circle, (a) $N$ even, (b) $N$ odd.

Eq. (9) can be written as

$$-\sum_{k=1}^{N} p_k = \begin{cases} \displaystyle\sum_{k=1}^{\frac{N}{2}} 2\sin(\theta_k) & \text{for } N \text{ even} \\ \displaystyle\sum_{k=1}^{\frac{N-1}{2}} 2\sin(\theta_k) + \sin\left(\theta_{\frac{N+1}{2}}\right) & \text{for } N \text{ odd} \end{cases} \tag{10}$$

Applying Eq. (8b) to Eq. (10) will give,

$$-\sum_{k=1}^{N} p_k = \begin{cases} \displaystyle\sum_{k=1}^{\frac{N}{2}} g_k & \text{for } N \text{ even} \\ \displaystyle\sum_{k=1}^{\frac{N-1}{2}} g_k + \frac{1}{2} g_{\left(\frac{N+1}{2}\right)} & \text{for } N \text{ odd} \end{cases} \tag{11}$$

Multiplying Eq. (11) by two and making use of the symmetry property of the prototype filter coefficients ($g_k = g_{N-k}$), we can rewrite it in a more closed form by combining the odd and even cases as

$$-2\sum_{k=1}^{N} p_k = \sum_{k=1}^{N} g_k \qquad \text{for all } N(\text{even and odd}) \tag{12}$$

Now, Eq. (9) is evaluated using the following identities [8],

$$\sum_{k=1}^{N} \sin\left((2k-1)\,x\right) = \sin^2(Nx)\csc(x) \tag{13a}$$

$$\sum_{k=1}^{N} \cos\left((2k-1)x\right) = \frac{1}{2}\sin(2Nx)\csc(x) \tag{13b}$$

by setting $x = \dfrac{\pi}{2N}$, Eqs. (13a) and (13b) can be written as

$$\sum_{k=1}^{N} \sin\left((2k-1)\frac{\pi}{2N}\right) = \csc(\frac{\pi}{2N}) \tag{14a}$$

$$\sum_{k=1}^{N} \cos\left((2k-1)\frac{\pi}{2N}\right) = 0 \tag{14b}$$

or by using Eq. (7b)

$$\sum_{k=1}^{N} \sin(\theta_k) = \csc\left(\frac{\pi}{2N}\right) \tag{15a}$$

$$\sum_{k=1}^{N} \cos(\theta_k) = 0 \tag{15b}$$

Therefore, using Eqs. (15a) and (15b), Eq. (9) is written as

$$-\sum_{k=1}^{N} p_k = \csc\left(\frac{\pi}{2N}\right) = \frac{1}{\sin(\theta_1)} \tag{16}$$

and finally using Eq. (8b), Eq. (16) becomes

$$-2\sum_{k=1}^{N} p_k = \frac{4}{g_1} \tag{17}$$

Comparison between Eqs. (12) and (17) reveals that for the maximally-flat normalized prototype filter,

$$\sum_{k=1}^{N} g_k = \frac{4}{g_1} \tag{18}$$

which is an interesting property that shows for a maximally-flat prototype filter, the summation of all prototype coefficients can be found from the first coefficient.

It is worth mentioning that using the same procedure, it can be shown that Eq. (17) holds for equal-ripple prototype lowpass filter regardless of the ripple value (see Appendix A). Although Eq. (18) does not hold for equal-ripple filters, Eq. (17) by itself is of great importance. It shows that the coefficient of the second term in the denominator of $S_{21}(s)$ for both maximally-flat and equal-ripple filters is only a function of $g_1$. This property has application in direct frequency scaling of coupling matrix for designing filters with predefined coupling values [9]. Deriving similar expressions like Eq. (18) for equal-ripple filters is left for future publication.

A common use of the sum of the lowpass prototype coefficients, that is Eq. (18), is the calculation of a filter insertion loss in the presence of loss. Cohn [1] has shown that dissipative loss in a lowpass (bandpass) multi-coupled prototype filter at $\omega = 0\,\mathrm{rad/s}$ ($\omega = \omega_0$) can be estimated by

$$L_0 = \frac{4.343}{\Delta} \sum_{k=1}^{N} \frac{g_k}{Q_{uk}} \quad \mathrm{dB} \tag{19}$$

where $\Delta$ is the filter fractional bandwidth and $Q_{uk}$ is the unloaded quality factor of the $k$-th resonator. In the case of uniform quality factor for all resonators, which is of considerable practical interest, Eq. (19) is written as

$$L_0 = \frac{4.343}{\Delta Q_u} \sum_{k=1}^{N} g_k \quad \mathrm{dB} \tag{20}$$

Using the new property of maximally-flat filters in Eq. (18), Eq. (20) can be expressed as

$$L_0 = \frac{17.372}{\Delta Q_u g_1} = \frac{8.686}{\Delta Q_u \sin\left(\frac{\pi}{2N}\right)} \quad \mathrm{dB} \tag{21}$$

which eliminates the need for the summation of $g_k$, and the prototype coefficient table. In other words, knowing the unloaded quality factor of resonators and the filter fractional bandwidth, the insertion loss of a maximally-flat filter (at $\omega = 0\,\mathrm{rad/s}$ and $\omega = \omega_0$ in the case of lowpass and bandpass filters, respectively) can be calculated quickly using an ordinary calculator for different filter orders

without referring to any tables or software. For higher order filters where the sine contribution can be approximated by its argument, Eq. (21) can be written as

$$L_0 \approx \frac{5.530N}{\Delta Q_u} \quad \text{dB} \tag{22}$$

Fig. 3 shows the percent error due to using of Eq. (22) instead of Eq. (21) as a function of filter order. It can be seen, for filters with orders higher than four the error is less than 2%. Additionally, based on the Cohn's and Bode's [4] approximate formulas, it was shown in [5] that the group delay of a maximally-flat lowpass filter at $\omega = 0 \, \text{rad/s}$ is expressed as

$$\tau_0 = \frac{1}{2} \sum_{k=1}^{N} g_k \tag{23}$$

which can be simplified using Eq. (18). A comparison of Eqs. (18) and (23) reveals that

$$\tau_0 = \frac{2}{g_1} = \frac{1}{\sin\left(\dfrac{\pi}{2N}\right)} \tag{24}$$

This equation shows that the group delay of a maximally-flat lowpass filter at $\omega = 0 \, \text{rad/s}$ is only a function of the first element value in the prototype filter. A comparison of the group delay evaluated using Eq. (24) and its exact value found from circuit analysis using the admittance method is shown in Fig. 4 which confirms the accuracy of Eq. (24). Evaluation of this equation for different filter orders, gives the constant value tabulated in [5] for the calculation of excess loss due to dissipative resonators. Finally, inserting Eq. (24) into Eq. (21) gives

$$L_0(dB) = \frac{8.686}{\Delta Q_u} \tau_0 \tag{25}$$

which is Bode's formula for the excess loss of maximally-flat filters with finite quality factor resonators [4]. Therefore, comparing the formulas for the excess loss of the maximally-flat filters derived by Bode [4], Cohn [1], and Matthaei [5] to what is presented in this paper shows that they are all the same in essence. However, the equations presented here are more convenient to evaluate than the others.
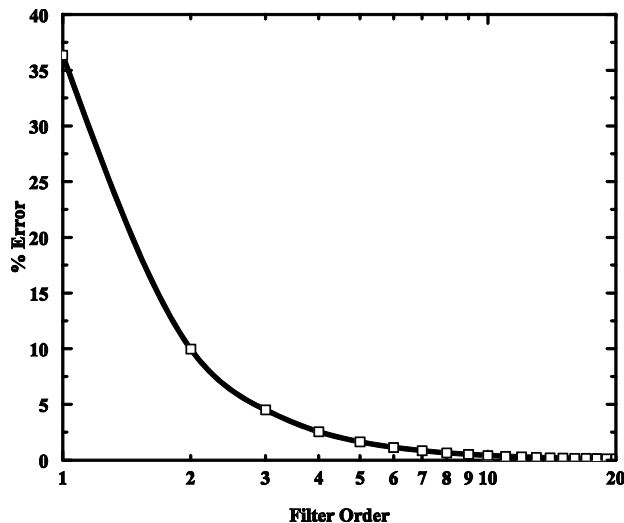


**Figure 3.** Error introduced by using the approximation introduced in Eq. (22) instead of using the full expression in Eq. (21).
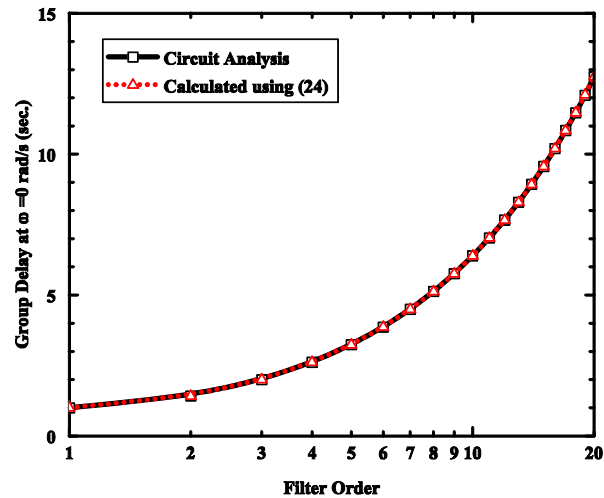
**Figure 4.** Comparison between the group delay found from the circuit analysis and Eq. (24) of a normalized, maximally-flat, lowpass filter at $\omega = 0\,\mathrm{rad/s}$.

## 3. VERIFICATION EXAMPLES

The first step in practical microwave filter design is usually the choice of a proper resonator. Many different resonators in various technologies such as microstrip, stripline, coaxial, waveguide, and evanescent-mode cavities (with a range of Q from about 200 to 10,000) have been investigated and used for a wide range of RF/Microwave applications such as radio communications and radar systems. Since the loss in a coupled resonator filter is inversely proportional to the filter bandwidth and the Q of the resonators, in narrow band applications the loss requirement is the most dominant factor affecting the resonator selection. Therefore, a closed-form formula like Eq. (21) is quite helpful for quick evaluation of the filter loss.

In order to demonstrate the accuracy of the presented equations, a comparison between the values of excess loss calculated using different methods is provided in Table 1. Maximally-flat lowpass filters of orders two to six with two different quality factors have been considered. In this table, $L_A$ is the actual excess loss calculated from circuit analysis using the admittance method while $L_B$, $L_C$, and $L_M$, are excess losses using Bode's, Cohn's, and Matthaei's formulations, respectively. Finally, LN is the excess loss found from the new equation presented in this paper that is Eq. (21). It can be seen that the Matthaei's formula produces the greatest error among all. However, for the higher Q filters, all methods give acceptable values. For the case of low-Q filters, Bode's formula generates the most accurate results for lower order filters, whereas for higher order filters, Cohn's and the new equation give better results. In all cases, Cohn's and the new equation give the same results. However, the new equation is more convenient for quick evaluation by hand than all the other methods. Also, it should be noted that the accuracy of Cohn's formula depends on the accuracy of the group delay calculation. In Table 1 the group delay has been calculated using Eq. (24), which gives an exact value.

In addition, Fig. 5 demonstrates the value of excess loss as a function of Q for second and sixth-order lowpass filters. The results from the circuit analysis, Cohn's formula, and the new equation are shown for comparison. It can be seen that Eq. (21) and Cohn's formula are always the same and for Qs greater than 5, they agree well with the exact results.

Finally, as a last example, the excess loss at the center frequency of two bandpass filters is investigated. Two fifth-order coupled-resonator maximally-flat filters with a center frequency of $3.0\,\mathrm{GHz}$ and a fractional bandwidth of 1% (narrowband) and 20% (wideband) have been considered. The filter is shown in Fig. 6. As soon as the filter center frequency, the resonators quality factor, and the filter fractional bandwidth are chosen the filter can be designed for any arbitrary terminations using the provided formulas in Fig. 6. To calculate the coupling between the resonators, the prototype filter normalized element values are found using Eq. (8b). More detail about designing coupled-resonator filters can be found in [5] and [6]. The simulated frequency response of 20% bandpass filter with

different quality factors is shown in Fig. 7 as a reference. As discussed in Section 1, the rounded corners at the cut-off points and increase in the passband insertion loss are observed when the quality factor is decreased.

**Table 1.** Comparison of the different methods for calculating excess loss in maximally-flat lowpass filters.

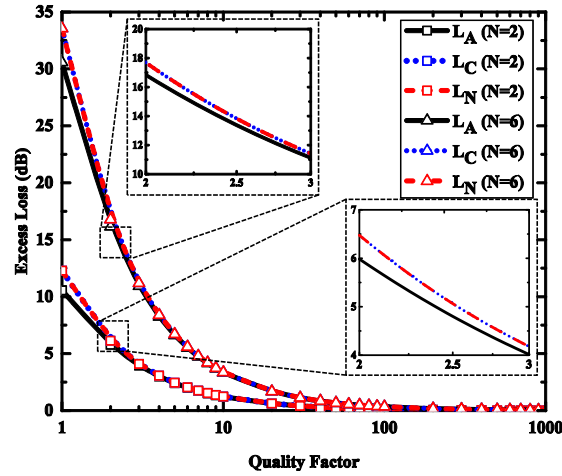| N | Q=10 | | | | | Q=100 | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $L_A$ (dB) | $L_B$ (dB) | $L_C$ (dB) | $L_M$ (dB) | $L_N$ (dB) | $L_A$ (dB) | $L_B$ (dB) | $L_C$ (dB) | $L_M$ (dB) | $L_N$ (dB) |
| 2 | 1.212 | 1.218 | 1.228 | 1.146 | 1.228 | 0.122 | 0.123 | 0.123 | 0.122 | 0.123 |
| 3 | 1.717 | 1.727 | 1.737 | 1.584 | 1.737 | 0.172 | 0.174 | 0.174 | 0.172 | 0.174 |
| 4 | 2.267 | 2.261 | 2.270 | 2.014 | 2.270 | 0.227 | 0.227 | 0.227 | 0.224 | 0.227 |
| 5 | 2.807 | 2.800 | 2.811 | 2.438 | 2.811 | 0.281 | 0.281 | 0.281 | 0.277 | 0.281 |
| 6 | 3.352 | 3.342 | 3.356 | 2.835 | 3.356 | 0.336 | 0.336 | 0.336 | 0.329 | 0.336 |



**Figure 5.** Excess loss for maximally-flat lowpass filter of order two and six as a function of quality factor. $L_A$, $L_C$, and $L_N$ are excess losses from circuit analysis, Cohn's formula, and this work, respectively.
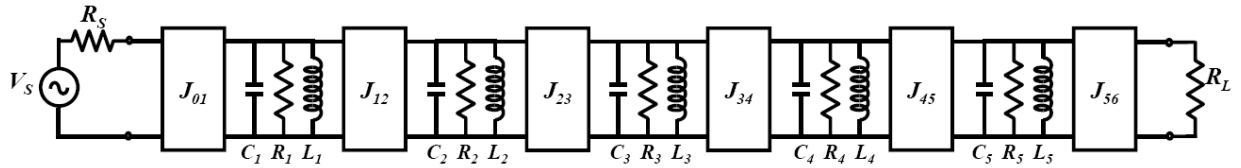


**Inputs:**
$f_{res}$=3 GHz

$Q$=10~10000

$FBW$=1% or 20%

$N$=5

$R_S$=$R_L$=50 $\Omega$

**Filter Parameters:**
$C_i$=arbitrary

$L_i = \frac{1}{(2\pi f_{res})^2 C_i}$

$R_i = \frac{Q}{2\pi f_{res}C_i}$

$G_S = G_L = \frac{1}{50}\ \Omega^{-1}$

$BW = FBW \cdot f_{res}$

from equation (8b):

$g_0$=1.0000

$g_1$=0.6180

$g_2$=1.6180

$g_3$=2.0000

$g_4$=1.6180

$g_5$=0.6180

$m_{ij} = \frac{1}{\sqrt{g_i g_j}}$  $i = 0, 1, 2, ..., 5\,;\ j = i + 1$

$J_{01} = \sqrt{2\pi BW G_s C_1}\, m_{01}$

$J_{12} = 2\pi BW \sqrt{C_1 C_2}\, m_{12}$

$J_{23} = 2\pi BW \sqrt{C_2 C_3}\, m_{23}$

$J_{34} = 2\pi BW \sqrt{C_3 C_4}\, m_{34}$

$J_{45} = 2\pi BW \sqrt{C_4 C_5}\, m_{45}$

$J_{56} = \sqrt{2\pi BW G_L C_5}\, m_{56}$

**Figure 6.** Fifth-order coupled-resonator bandpass filter with shunt resonators and J-inverters.
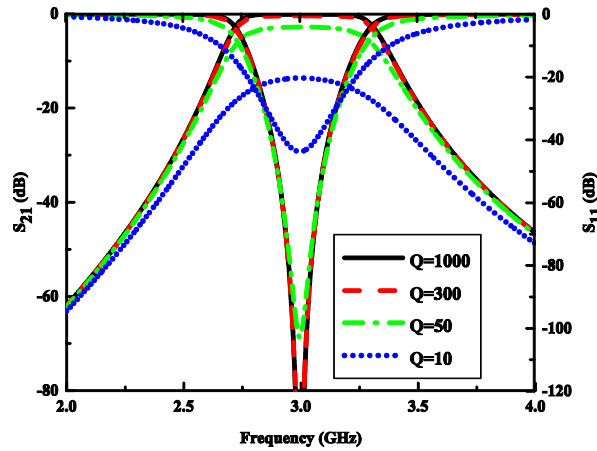
**Figure 7.** Frequency response of a 20% fifth-order maximally-flat bandpass filter with different quality factors.

Figure 8 shows the excess loss of both bandpass filters with respect to the resonator Q. In both cases the results calculated from Eq. (21) are exactly the same as the results found from Cohn's formula. However, for the case of 1% bandpass filter, close results to the circuit analysis are obtained for Qs beyond 500 while for the 20% bandpass filter results found from Eq. (21) are in good agreement with the circuit analysis for all quality factors. The reason behind this can be found from the effect of fractional bandwidth on the excess loss suggested by the denominator of Eq. (21). In other words, in bandpass filters the product of fractional bandwidth and quality factor must be considered as the accuracy criterion for the calculation of excess loss. Therefore, both 1% and 20% bandpass filters will behave the same for the same quality factor-bandwidth product.
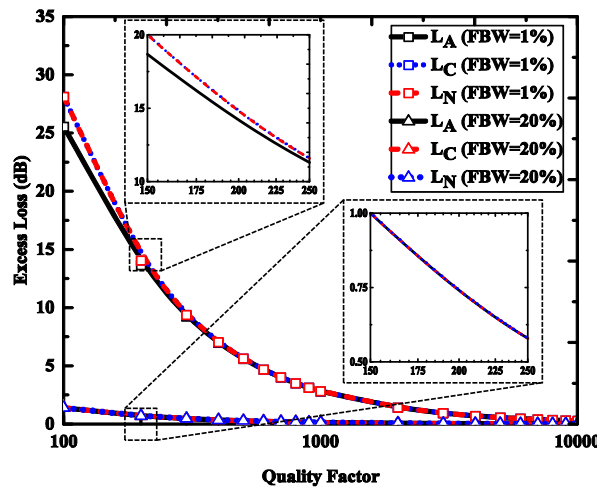


**Figure 8.** Excess loss for a fifth-order maximally-flat bandpass filter with fractional bandwidth of 1% and 20% as a function of quality factor. $L_A$, $L_C$, and $L_N$ are excess losses from circuit analysis, Cohn's formula, and this work, respectively.

To summarize, Eq. (21) is mathematically proved to be equivalent to Cohn's formula and it is accurate for practical usage for both lowpass and bandpass filters. Furthermore, Eq. (22) is demonstrated to be within 2% of Cohn's formula for filters of orders five and greater.

## 4. CONCLUSION

A new property for maximally-flat prototype filters, which simplifies the sum of prototype coefficients to a single term, is presented in this paper. This new derived property is then utilized to calculate the excess dissipative loss due to finite quality factor of resonators in maximally-flat lowpass and bandpass filters at $\omega = 0\,\text{rad/s}$ and $\omega = \omega_0$, respectively. Additionally, the property is used to develop a new equation for the calculation of the filter group delay. The significance of the presented equations is that they are convenient to manipulate quickly by hand, eliminating the need for any coefficients' tables or complicated formulas. A comparison of the proposed equations with those available in past literature is provided to demonstrate the accuracy of the new equations. To the authors' best knowledge, this is the first time that such a property and its application in the calculation of excess loss and group delay of maximally-flat filters have been reported.

## ACKNOWLEDGMENT

## APPENDIX A. EQUAL-RIPPLE FILTERS

Using the same procedure, it can be shown that Eq. (17) holds for equal-ripple prototype lowpass filter regardless of the ripple value. In fact, Eq. (6) still holds for a lowpass equal-ripple filter with ripple constant $\varepsilon$, where its poles are defined by

$$p_k = -\eta \sin(\theta_k) - j\sqrt{1+\eta^2}\cos(\theta_k) \tag{A1}$$

where $\theta_k$ is defined the same as Eq. (7b) and

$$\eta = \sinh\left(\frac{1}{N}\sin^{-1}(\frac{1}{\varepsilon})\right) \tag{A2}$$

Therefore, similar to Eq. (9), we can write

$$-\sum_{k=1}^{N} p_k = \eta \sum_{k=1}^{N} \sin(\theta_k) + j\sqrt{1+\eta^2}\sum_{k=1}^{N}\cos(\theta_k) \tag{A3}$$

Using Eqs. (14a) and (14b), Eq. (A3) is simplified to

$$-\sum_{k=1}^{N} p_k = \eta \csc\left(\frac{\pi}{2N}\right) = \frac{\eta}{\sin(\theta_1)} \tag{A4}$$

On the other hand, the first coefficient of the lowpass prototype has been defined [5] as

$$\mathrm{g}_1 = \frac{2}{\eta}\sin\left(\frac{\pi}{2N}\right) = \frac{2}{\eta}\sin(\theta_1) \tag{A5}$$

Finally, comparing Eqs. (A4) and (A5) gives

$$-\sum_{k=1}^{N} p_k = \frac{2}{\mathrm{g}_1} \tag{A6}$$

which is the same as Eq. (17).

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