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Fast Direct Solution of Composite Conducting-Dielectric Arrays Using Sherman-Morrison-Woodbury Algorithm

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Abstract—In this paper, the Sherman-Morrison-Woodbury (SMW) Formula-based algorithm (SMWA) is used to enable the fast direct solution of conducting-dielectric arrays. To speed up the direct solution of the matrix equation, the dense impedance matrix is transformed into a product of several block diagonal matrices via the SMW formula. In the grouping process, the situation that the elements of an array simultaneously belong to two different subgroups at peer level is avoided in order to promote the efficiency. The SMWA conducts the calculation with a respectable reduction in the computational time as well as memory.

1. INTRODUCTION

This paper focuses on the electromagnetic scattering from composite conducting-dielectric arrays, which is widely used in microwave engineering and antenna designs [1, 2]. To perform accurate numerical analysis, the method of moments (MoM) [3] is a good choice. Compared with the surface integral equation (SIE), the volume-surface integral equation (VSIE) [4] is more advantageous for analyzing targets including inhomogeneous anisotropic dielectrics.

In the MoM, the VSIE is transformed into a dense matrix equation. Iterative solvers require $O(N^2)$ storage and computational time for a matrix-vector product (MVP) at iterations, where N is the number of unknowns. Thus, to reduce the complexity of MVP, many fast iterative solvers have been proposed such as the multilevel fast multipole algorithm (MLFMA) [5], adaptive integral method (AIM) [6], pre-corrected fast Fourier transform (P-FFT) [7], and multilevel fast adaptive cross approximation (MLFACA) [8]. Nevertheless, these fast iterative solvers usually involve a large quantity of iteration steps when solving the VISE.

In order to avoid the convergence problem, direct solvers such as LU decomposition can be employed. However, the conventional direct methods are very time-consuming with computational complexity of $O(N^3)$. To mitigate this problem, in this paper, the fast direct method [9–13] based on the Sherman-Morrison-Woodbury (SMW) formula [14, 15] is employed. The fast direct solver is termed as SMW algorithm (SMWA) for the sake of brevity. Firstly, the SMWA hierarchically divides the MoM dense impedance matrix based on the binary tree. Then, all the off-diagonal submatrices are compressed by the adaptive cross approximation (ACA) [8, 16]. Finally, the matrix is transformed into a product of several block diagonal matrices via the SMW formula, so that the solution of the matrix equation can be efficiently calculated through these block diagonal matrices.

The article is organized as follows. In Section 2, the VSIE is established to analyze the composite conducting-dielectric arrays. In Section 3, the SWMA algorithm is presented in detail to illustrate how this method speeds up the calculation. In Section 4, numerical results are given to verify the efficiency and accuracy of the algorithm.

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2. BASIC PRINCIPLE OF VSIE

In this section, the coupled volume-surface integral equation (VSIE) [4, 17, 18] is presented to calculate the electromagnetic scattering from composite conducting-dielectric arrays. Let S denote all the conducting surfaces and V denote the dielectric volumes, the boundary conditions of the VSIE can be expressed as [18]

$$\left(\mathbf{E^i}(\mathbf{r}) + \mathbf{E^s}(\mathbf{r})\right)|_{\tan} = 0, \quad \mathbf{r} \in S \tag{1}$$

$$\mathbf{E}(\mathbf{r}) = \bar{\bar{\varepsilon}}^{-1}(\mathbf{r}) \cdot \mathbf{D}(\mathbf{r}) = \mathbf{E^i}(\mathbf{r}) + \mathbf{E^s}(\mathbf{r}), \quad \mathbf{r} \in V \tag{2}$$

where $\mathbf{E^i}(\mathbf{r})$ is the incident filed and $\mathbf{E^s}(\mathbf{r})$ is the scattering filed. $\bar{\bar{\varepsilon}}(\mathbf{r})$ is the permittivity tensor. $\mathbf{D}(\mathbf{r})$ is the electric flux density. $\mathbf{E^s}(\mathbf{r})$ can be expressed as the sum of the scattering filed of dielectric volumes $\mathbf{E^s_v}(\mathbf{r})$ and the conducting surfaces $\mathbf{E^s_s}(\mathbf{r})$.

$$\mathbf{E^s_s}(\mathbf{r}) = -j\omega\mathbf{A_s}(\mathbf{r}) - \nabla\phi_s(\mathbf{r}), \tag{3}$$

$$\mathbf{E^s_v}(\mathbf{r}) = -j\omega\mathbf{A_v}(\mathbf{r}) - \nabla\phi_{\mathbf{v}}(\mathbf{r}), \tag{4}$$

where $\mathbf{A}_u(\mathbf{r})$ and $\phi_u(\mathbf{r})$ for $u = s, v$ represent the vector potential functions and scalar potential functions, respectively. In order to solve Eqs. (1) and (2), a set of Rao-Wilton-Glisson (RWG) [19] and Schaubert-Wilton-Glisson (SWG) [20] basis functions are used to discretize the conducting region and the dielectric region, respectively. The unknown surface and volume current can be represented by [18]

$$\mathbf{J}_s(\mathbf{r}) = \sum_{n=1}^{N_s} I_{sn}\mathbf{f}_{sn}(\mathbf{r}), \tag{5}$$

$$\mathbf{J}_v(\mathbf{r}) = \sum_{n=1}^{N_v} I_{vn}\bar{\bar{\kappa}}_{vn} \cdot \mathbf{f}_{vn}(\mathbf{r}), \tag{6}$$

where $\mathbf{f}_{sn}(\mathbf{r})$ is the n-th RWG basis function and $\mathbf{f}_{vn}(\mathbf{r})$ is n-th SWG basis function, N_s is the number of RWG basis functions and N_v is the number of SWG basis functions. The contrast ratio $\bar{\bar{\kappa}}(\mathbf{r})$ is defined as $\bar{\bar{\kappa}}(\mathbf{r}) = \bar{\bar{\mathbf{I}}} - \bar{\bar{\varepsilon}}_{\mathbf{r}}^{\mathbf{-1}}(\mathbf{r})$ [17], in which $\bar{\bar{\varepsilon}}_r(\mathbf{r})$ is the relative permittivity tensor of the electric anisotropic media, and $\bar{\bar{\mathbf{I}}}$ is the unit tensor. $\bar{\bar{\kappa}}_{vn}$ is the contrast ratio in the n-th SWG basis function. I_{sn} and I_{vn} are the unknown coefficients. Substituting Eqs. (3)–(6) into Eqs. (1) and (2), we have the MoM impedance equation, by using the Galerkin's method, as

$$\sum_{n=1}^{N_s} I_{sn}Z^{ss}_{mn} + \sum_{n=1}^{N_v} I_{vn}Z^{vv}_{mn} = \left\langle \mathbf{f}_{sm}(\mathbf{r}), \mathbf{E^i}(\mathbf{r}) \right\rangle, \tag{7}$$

$$\sum_{n=1}^{N_s} I_{sn}Z^{ss}_{mn} + \sum_{n=1}^{N_v} I_{vn}Z^{vv}_{mn} = \left\langle \mathbf{f}_{vm}(\mathbf{r}), \mathbf{E^i}(\mathbf{r}) \right\rangle, \tag{8}$$

where

$$Z^{ss}_{mn} = j\omega \left\langle \mathbf{f}_{sm}(\mathbf{r}), \mathbf{A}_{sn}(\mathbf{r}) \right\rangle + \left\langle \mathbf{f}_{sm}(\mathbf{r}), \nabla\varphi_{sn}(\mathbf{r}) \right\rangle, \tag{9}$$

$$Z^{sv}_{mn} = j\omega \left\langle \mathbf{f}_{sm}(\mathbf{r}), \mathbf{A}_{vn}(\mathbf{r}) \right\rangle + \left\langle \mathbf{f}_{sm}(\mathbf{r}), \nabla\varphi_{vn}(\mathbf{r}) \right\rangle, \tag{10}$$

$$Z^{vs}_{mn} = j\omega \left\langle \mathbf{f}_{vm}(\mathbf{r}), \mathbf{A}_{sn}(\mathbf{r}) \right\rangle + \left\langle \mathbf{f}_{vm}(\mathbf{r}), \nabla\varphi_{sn}(\mathbf{r}) \right\rangle, \tag{11}$$

$$Z^{vv}_{mn} = j\omega \left\langle \mathbf{f}_{vm}(\mathbf{r}), \mathbf{A}_{vn}(\mathbf{r}) \right\rangle + \left\langle \mathbf{f}_{vm}(\mathbf{r}), \nabla\varphi_{vn}(\mathbf{r}) \right\rangle + \frac{1}{j\omega} \left\langle \mathbf{f}_{vm}(\mathbf{r}), \bar{\bar{\varepsilon}}^{-1}(\mathbf{r}) \cdot \mathbf{f}_{vn}(\mathbf{r}) \right\rangle. \tag{12}$$

The aforementioned equations set can be written in the matrix form

$$\begin{pmatrix} \mathbf{Z}_{ss} & \mathbf{Z}_{sv} \\ \mathbf{Z}_{vs} & \mathbf{Z}_{vv} \end{pmatrix} \begin{pmatrix} \mathbf{I}_s \\ \mathbf{I}_v \end{pmatrix} = \begin{pmatrix} \mathbf{V}_s \\ \mathbf{V}_v \end{pmatrix}. \tag{13}$$

In the description above, $\mathbf{Z}_{sv}$ and $\mathbf{Z}_{vs}$ denote the interaction between metallic surface and dielectric body. $\mathbf{Z}_{vs}$ is the impedance matrix with the source point in dielectric body and the field point in conducting surface. $\mathbf{Z}_{sv}$ is the complement. $\mathbf{V}_s$ and $\mathbf{V}_v$ are the voltage vector. It is worth mentioning that metallic parts and dielectric media are permitted to touch.

3. FAST SOLUTION VIA SMWA

The SMWA is an efficient direct solver proposed in [9, 10]. In our recent work, it has been employed to accelerate the direct solution of the SIE [11, 12] and the volume integral equation (VIE) [13]. In this paper, the SWMA is extended to solve the VSIE. To implement the SMWA [9, 10], the entire target needs multi-level grouping first. The array elements are divided into 2^L groups by the L-level binary tree according to the positions of their barycenters. For instance, the first division divides all array elements into 2 groups, ensuring that the number of basis functions in each group are approximately equal. The next division divides each of the previous 2 subgroups into 2 groups, respectively, and so on. In the grouping process, we avoid the situation that the elements of an array simultaneously belong to two different subgroups at peer level. Therefore, the effective rank of impedance matrix associated with adjacent groups is relatively small.

Once the impedance matrix is constructed, our goal is to calculate its inverse. In the SMWA, the impedance matrix $\mathbf{Z}$ is transformed into the product of $L+1$ block diagonal matrices.

$$\mathbf{Z} \approx \mathbf{Z}_L \mathbf{Z}_{L-1} \ldots \mathbf{Z}_1 \mathbf{Z}_0, \tag{14}$$

where $\mathbf{Z}_l$ $(0 \leq l \leq L)$ is the block diagonal matrix of the product of $\mathbf{Z}_{l+1}^{-1} \ldots \mathbf{Z}_L^{-1}\mathbf{Z}$ at the l-th level. As a result, the solution is transformed into solving the inverse of every block diagonal matrix as

$$\mathbf{Z}^{-1} \approx \mathbf{Z}_0^{-1}\mathbf{Z}_1^{-1} \ldots \mathbf{Z}_{L-1}^{-1}\mathbf{Z}_L^{-1}. \tag{15}$$

For the fact that these diagonal matrices present in special forms, their inverses can be efficiently calculated according to the SMW formula [14, 15].

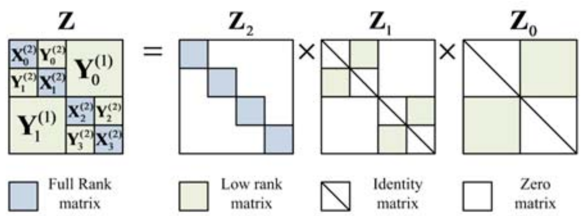

Figure 1. The pictorial representations of $\mathbf{Z}$ and its compression results

To clearly describe the acceleration progress, the implementation of a simple 2-level SMWA is presented as follows. As shown in Fig. 1, the impedance matrix Z is divided by a 2-level binary tree. The diagonal blocks "$\mathbf{X}''$ are the self-impedance matrices of each group at the finest level, and the rest mutual-impedance matrices "$\mathbf{Y}''$ at different levels are at low rank and can be compressed by ACA [8, 16] as

$$\mathbf{Y}_i^{(l)} \approx \mathbf{U}_i^{(l)}\mathbf{V}_i^{(l)}. \tag{16}$$

The impendence matrix after the ACA compression can be written as

$$\mathbf{Z} = \begin{bmatrix} \begin{bmatrix} \mathbf{X}_0^{(2)} & \mathbf{U}_0^{(2)}\mathbf{V}_0^{(2)} \\ \mathbf{U}_1^{(2)}\mathbf{V}_1^{(2)} & \mathbf{X}_1^{(2)} \end{bmatrix} & \mathbf{U}_0^{(1)}\mathbf{V}_0^{(1)} \\ \mathbf{U}_1^{(1)}\mathbf{V}_1^{(1)} & \begin{bmatrix} \mathbf{X}_2^{(2)} & \mathbf{U}_2^{(2)}\mathbf{V}_2^{(2)} \\ \mathbf{U}_3^{(2)}\mathbf{V}_3^{(2)} & \mathbf{X}_3^{(2)} \end{bmatrix} \end{bmatrix}. \tag{17}$$

First, $\mathbf{Z}_2$ is the block diagonal matrix of $\mathbf{Z}$ at the 2nd level and is expressed as

$$\mathbf{Z}_2 = \begin{bmatrix} \mathbf{X}_0^{(2)} & & & \\ & \mathbf{X}_1^{(2)} & & \\ & & \mathbf{X}_2^{(2)} & \\ & & & \mathbf{X}_3^{(2)} \end{bmatrix}. \tag{18}$$

Then, we compute the product of $\mathbf{Z}_2^{-1}\mathbf{Z}$ as

$$\mathbf{Z}_2^{-1}\mathbf{Z} = \begin{bmatrix} \begin{bmatrix} \mathbf{1} & \dot{\mathbf{U}}_0^{(2)}\mathbf{V}_0^{(2)} \\ \dot{\mathbf{U}}_1^{(2)}\mathbf{V}_1^{(2)} & \mathbf{1} \end{bmatrix} & \dot{\mathbf{U}}_0^{(1)}\mathbf{V}_0^{(1)} \\ \dot{\mathbf{U}}_1^{(1)}\mathbf{V}_1^{(1)} & \begin{bmatrix} \mathbf{1} & \dot{\mathbf{U}}_2^{(2)}\mathbf{V}_2^{(2)} \\ \dot{\mathbf{U}}_3^{(2)}\mathbf{V}_3^{(2)} & \mathbf{1} \end{bmatrix} \end{bmatrix}, \tag{19}$$

where $\mathbf{1}$ denotes the identity matrix,

$$\dot{\mathbf{U}}_i^{(2)} = \left(\mathbf{X}_i^{(2)}\right)^{-1}\mathbf{U}_i^{(2)}, \tag{20}$$

for $i = 0, 1, 2, 3$.

$$\dot{\mathbf{U}}_i^{(1)} = \begin{bmatrix} \left(\mathbf{X}_{2i}^{(2)}\right)^{-1} & \\ & \left(\mathbf{X}_{2i+1}^{(2)}\right)^{-1} \end{bmatrix} \mathbf{U}_i^{(1)}, \tag{21}$$

for $i = 0, 1$. Thus, $\mathbf{Z}_1$ is the block diagonal matrix of $\mathbf{Z}_2^{-1}\mathbf{Z}$ at the 1st level and is written as

$$\mathbf{Z}_1 = \begin{bmatrix} \begin{bmatrix} \mathbf{1} & \dot{\mathbf{U}}_0^{(2)}\mathbf{V}_0^{(2)} \\ \dot{\mathbf{U}}_1^{(2)}\mathbf{V}_1^{(2)} & \mathbf{1} \end{bmatrix} & \\ & \begin{bmatrix} \mathbf{1} & \dot{\mathbf{U}}_2^{(2)}\mathbf{V}_2^{(2)} \\ \dot{\mathbf{U}}_3^{(2)}\mathbf{V}_3^{(2)} & \mathbf{1} \end{bmatrix} \end{bmatrix}. \tag{22}$$

Finally, $\mathbf{Z}_0$ is the block diagonal matrix of $\mathbf{Z}_1^{-1}\left(\mathbf{Z}_2^{-1}\mathbf{Z}\right)$ at the 0th level,

$$\mathbf{Z}_0 = \mathbf{Z}_1^{-1}\left(\mathbf{Z}_2^{-1}\mathbf{Z}\right) = \begin{bmatrix} \mathbf{1} & \ddot{\mathbf{U}}_0^{(1)}\mathbf{V}_0^{(1)} \\ \ddot{\mathbf{U}}_1^{(1)}\mathbf{V}_1^{(1)} & \mathbf{1} \end{bmatrix}, \tag{23}$$

where

$$\ddot{\mathbf{U}}_i^{(1)} = \begin{bmatrix} \mathbf{1} & \dot{\mathbf{U}}_{2i}^{(2)}\mathbf{V}_{2i}^{(2)} \\ \dot{\mathbf{U}}_{2i+1}^{(2)}\mathbf{V}_{2i+1}^{(2)} & \mathbf{1} \end{bmatrix}^{-1} \dot{\mathbf{U}}_i^{(1)}, \tag{24}$$

for i=0,1. By using Eqs. (18)–(24), we can obtain $\mathbf{Z}_2$, $\mathbf{Z}_1$ and $\mathbf{Z}_0$. At last, Eq. (15) is used to compute the unknown coefficients. In the calculation process, the inverses of $\mathbf{Z}_{L-1}, \ldots, \mathbf{Z}_1$ and $\mathbf{Z}_0$ are required. The diagonal blocks of these matrices have the special form as

$$\begin{pmatrix} \mathbf{1} & \mathbf{A}_0\mathbf{B}_0 \\ \mathbf{A}_1\mathbf{B}_1 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{A}_0 \\ \mathbf{A}_1 & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_0 \end{pmatrix}, \tag{25}$$

where $\mathbf{0}$ denotes the zero matrix. The sizes of $\mathbf{A}_0$ and $\mathbf{A}_1$ are $M \times r$, and $\mathbf{B}_1$ has the size of $r \times M$, $r \ll M$. Directly performing its inverse through the LU decomposition, the computational complexity is $O(M^3)$. However, according to the Sherman-Morrison-Woodbury formula [14, 15], the inverse of Eq. (25) can be expressed as [11, 12]

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} - \begin{pmatrix} \mathbf{0} & \mathbf{A}_0 \\ \mathbf{A}_1 & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{B}_1\mathbf{A}_0 \\ \mathbf{B}_0\mathbf{A}_1 & \mathbf{1} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_0 \end{pmatrix}. \tag{26}$$

The calculation of $\mathbf{B}_1\mathbf{A}_0$ and $\mathbf{B}_0\mathbf{A}_1$ scales as $O(r^2M)$, and the inverse of the matrix $[\mathbf{1}, \mathbf{B}_1\mathbf{A}_0; \mathbf{B}_0\mathbf{A}_1, \mathbf{1}]$ scales as $O(r^3)$. Thus, the total computational complexity of Eq. (26) is only $O(r^2M)$. Hence, the inverses of $\mathbf{Z}_{L-1}, \ldots, \mathbf{Z}_1$ and $\mathbf{Z}_0$ can be quickly computed by Eq. (26). The computational complexity of multiplying Eq. (26) by a vector is $O(rM)$. Thus, the factorization of Eq. (14), which requires multiplying Eq. (26) by vectors, can also be performed quickly. For more details about the complexity of the SMWA, we refer the reader to [12].

4. EXAMPLES AND DISCUSSION

In order to demonstrate the accuracy and efficiency of the aforementioned method for analyzing composite conducting-dielectric arrays, in this section, several numerical results are provided. All the simulations are irradiated by the 300 MHz uniform plane wave with the incident direction of (theta = 0, phi) while the polarization is at the direction $\hat{\mathbf{x}}$. In the first example, a 4×4 array is calculated. Each array element is a conical dielectric body, the subface of which is covered by the metallic material. The base radius of every cone is 0.15 m, and the height is 0.5 m. The mutual distance between the elements of the array is 0.6 m. The relative permittivity tensor of each dielectric array element is set as [1.5 0 0; 0 2.0 0; 0 0 2.2].

The conducting surfaces are discretized into 528 RWG basis functions while the dielectric conical bodies are discretized into 9629 SWG basis functions. The total number of unknowns is 10157. The target is divided into 4 levels by the binary tree, and the 4-level SMW algorithm is used. Numerical result is shown in Fig. 2, and the bistatic RCS computed by SMWA agrees well with the one computed by conventional MoM. Table 1 shows the CPU time and memory of the SWMA and the conventional LU decomposition. It is clear that the SWMA saves much CPU time and memory.

To further demonstrate the efficiency and accuracy of the method, the second example considers

Table 1. The CPU time and memory of the first example.

Method	*Time* (*s*)	*Memory* (*MB*)
conv. MoM	156	730
SMWA	57	83

Table 2. The CPU time and memory of the second example.

Method	*Time* (*s*)	*Memory* (*MB*)
conv. MoM	2661	7533
SMWA	270	443

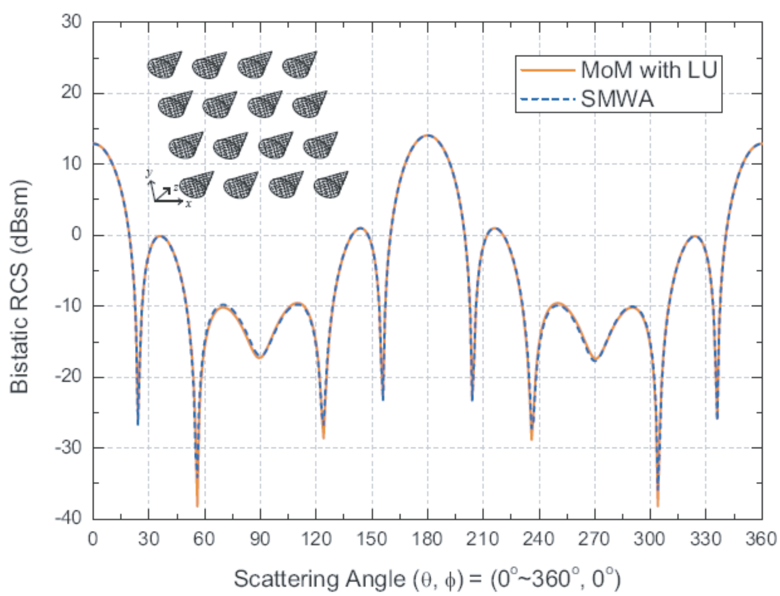

Figure 2. Bistatic RCSs of a 4×4 array of the first example.

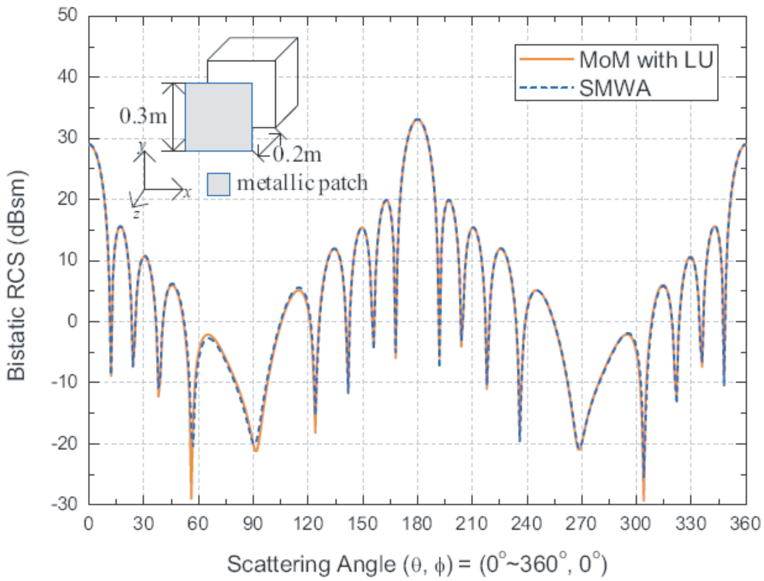

Figure 3. Bistatic RCSs of a 8×8 array of the second example.

the scattering field of a two-dimensional 8×8 array, whose elements are at a mutual distance of 0.6 m. Every array element is a $0.3\,\text{m} \times 0.3\,\text{m} \times 0.3\,\text{m}$ dielectric cube with a metallic patch hanging at the height of 0.2 m above the top surface, and the relative permittivity tensor of each dielectric array element is [2.2 0 0; 0 1.5 0; 0 0 1.3]. In this example, the total number of unknowns is 32612, including 1344 RWG and 31268 SWG basis functions. As shown in Figure 3, the results also agree well with each other. CPU time and memory are shown in Table 2. The SMWA requires 244 s CPU time and 409 MB memory while the conventional MoM requires 3276 s CPU time and 8114 MB memory. In this case, compared with the conventional LU decomposition, the total time was reduced by a factor of 13 and the memory reduced by a factor of 20 by the SMW algorithm.

5. CONCLUSION

In this article, the fast direct solver based on the Sherman-Morrison-Woodbury formula is implemented to accelerate solving the electromagnetic scattering from composite conducting-dielectric arrays using the VSIE. A remarkable reduction of CPU time and memory can be achieved by the algorithm. Hence, the algorithm presented in this paper can be applied to solve the VSIE of electrically much larger size. In addition, the SMWA is very simple for implementation and can be applied to analyse various electromagnetic problems.

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REFERENCES

1. Aksun, M. I., A. Alparslan, and E. P. Karabulut, "Determining the effective constitutive parameters of finite periodic structures: Photonic crystals and metamaterials," *IEEE Trans. Microw. Theory Tech.*, Vol. 56, No. 6, 1423–1434, Jun. 2008.

2. Wu, T. X. and D. L. Jaggard, "Scattering of chiral periodic structure," *IEEE Trans. Antennas Propag.*, Vol. 52, 1859–1870, Jul. 2004.
3. Gibson, W. C., *The Method of Moments in Electromagnetics*, *Boca Raton, FL: CRC Press*, 2007.
4. Lu, C. C. and W. C. Chew, "A coupled surface-volume integral equation approach for the calculation of electromagnetic scattering from composite metallic and material targets," *IEEE Trans. Antennas Propag.*, Vol. 48, No. 12, 1866–1868, Dec. 2000.
5. Song, J. M., C. C. Lu, and W. C. Chew, "Multilevel fast multipole algorithm for electromagnetic scattering by large complex objects," *IEEE Trans. Antennas Propag.*, Vol. 45, No. 10, 1488–1493, 1997.
6. Ewe, W. B., L. W. Li, and M. S. Leong, "Fast solution of mixed dielectric/conducting scattering problem using volume-surface adaptive integral method," *IEEE Trans. Antennas Propag.*, Vol. 52, No. 11, 3071–3077, 2004.
7. Nie, X. C., N. Yuan, L.W. Li, Y. B. Gan, and T. S. Yeo, "A fast volume-surface integral equation solver for scattering from composite conducting-dielectric objects," *IEEE Trans. Antennas Propag.*, Vol. 53 No. 2, 818–824, 2005.
8. Chen, X., C. Gu, J. Ding, Z. Li, and Z. Niu, "Multilevel fast adaptive cross-approximation algorithm with characteristic basis functions," *IEEE Trans. Antennas Propag.*, Vol. 63, No. 9, 3994–4002, Sep. 2015.
9. Kong, W. Y., J. Bremer, and V. Rokhlin, "An adaptive fast direct solver for boundary integral equations in two dimensions," *Applied and Computational Harmonic Analysis*, Vol. 31, No. 3, 346–369, 2011.
10. Ambikasaram, S. and E. Darve, "An O(NlogN) fast direct solver for partial hierarchically semi-separable matrices with application to radial basis function interpolation," *Journal of Scientific Computing*, Vol. 57, No. 3, 477–501, 2013.
11. Chen, X., C. Gu, Z. Niu, and Z. Li, "Direct solution of MoM matrix equation using Sherman-Morrison-Woodbury formula-based algorithm with ACA-SVD," *Asia-Pacific Microwave Conference*, Dec. 2015.
12. Chen, X., C. Gu, Z. Li, and Z. Niu, "Accelerated direct solution of electromagnetic scattering via characteristic basis function method with Sherman-Morrison-Woodbury formula-based algorithm," *IEEE Trans. Antennas Propag.*, to be published (Early Access), DOI 10.1109/TAP.2016.2587743.
13. Zhang, Y., X. Chen, C. Fei, Z. Li, and C. Gu, "Fast direct solution of dielectric array using adaptive cross approximation with Sherman-Morrison-Woodbury formula," *2016 IEEE 5th Asia-Pacific Conference on Antennas and Propagation (APCAP)*, 171–172, Jul. 2016.
14. Hager, W., "Updating the inverse of a matrix," *SIAM Rev.*, Vol. 31, No. 2, 221–239, 1989.
15. Golub, G. H. and C. F. Van Loan, *Matrix Computations*, The Johns Hopkins Univ., Press, Baltimore, MD, 1996.
16. Zhao, K., M. N. Vouvakis, and J.-F. Lee, "The adaptive cross approximation algorithm for accelerated method of moments computations of EMC," *IEEE Trans. Electromagn. Compat.*, Vol. 47, No. 4, 763–773, 2005.
17. Kobidze, G. and B. Shanker, "Integral equation based analysis of scattering from 3-D inhomogeneous anisotropic bodies," *IEEE Trans. Antennas and Propag.*, Vol. 52, No. 10, 2650–2658, Oct. 2004.
18. Chen, X., C. Gu, Z. Niu, and Z. Li, "Fast solution of volume-surface integral equation for scattering from composite conducting-dielectric targets using multilevel fast dipole method," *International Journal of RF and Microwave Computer-aided Engineering*, Vol. 25, No. 5, 624–631, Sep. 2012.
19. Rao, S. M., D. R. Wilton, and A. W. Glisson, "Electromagnetic scattering by surfaces of arbitrary shape," *IEEE Trans. Antennas and Propag.*, Vol. 30, No. 2, 409–418, May 1982.
20. Schaubert, D. H., D. R. Wilton, and A. W. Glisson, "A tetrahedral modeling method for electromagnetic scattering by arbitrarily shaped inhomogeneous dielectric bodies," *IEEE Trans. Antennas and Propag.*, Vol. 32, 77–85, 1984.