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Cross Section Equivalence between Photons and Non-Relativistic Massive Particles for Targets with Complex Geometries

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Abstract—The quantum radar cross section (QRCS) is a concept that gives information on the amount of returns (or scattered energy towards the detector) one can expect from a particular target when being illuminated with a small number of photons. This cross section is highly dependent on the target's geometry, as well as the illumination angle and the scattering angle from the target. The expression for the quantum radar cross section equation has been derived in the context of photon scattering. In this paper, it will be shown that an equivalent cross section expression, including the alternate form written in terms of Fourier transforms, can be derived using quantum scattering theory applied to non-relativistic, massive particles. Both single particle and multiple particle illumination are considered. Although this approach is formulated based upon massive, non-relativistic particle scattering, its equivalence to the expression based upon photon scattering provide many valuable insights of representing and interpreting these equations in the context of quantum radar. This includes an improved algorithm to simulate the QRCS response of an object illuminated with any number of photons desired.

1. INTRODUCTION

Quantum radar has the potential to provide a significant advantage in observing stand off targets in the field of remote sensing [1–5]. In this context, the quantum radar cross section has been defined to characterize how electromagnetically "large" an object appears to a quantum radar. It must be mentioned that this kind of cross section is highly dependent on the macroscopic target geometry (e.g., the shape of an airborne vehicle). This is in contrast to the kind of cross section normally discussed in physics within the context of particle collision experiments, where target geometry is relatively simple (e.g., a flat surface).

Objects viewed with a quantum radar exhibit increased QRCS sidelobe returns, resolution, and signal-to-noise ratio (SNR) in comparison to classical radar with the same transmit power under certain regimes [1, 2]. There are many mechanisms by which a quantum radar achieves an advantage over classical radar [6–8], and it is these results that prompt further study into this promising field.

This paper uses the formalism of quantum scattering theory for non-relativistic massive particles to obtain equivalent expressions for the QRCS found in the literature [2, 9, 10]. It will be shown that a scattering cross section can be derived for massive particles that is equivalent to the QRCS in quantum radar under certain conditions. Using this theory, the alternate form of the QRCS equation written in terms of Fourier transforms can also be derived in a very natural manner. Using the equivalence between the two cross sections, a more efficient method is developed to simulate the QRCS response of an object in the case of many illuminating photons.

The driving point behind our results is that the same QRCS expression is obtained using non-relativistic, massive particles (which we will hereby just refer to as "particles"), as opposed to photons. By "massive", we simply mean particles with mass. This leads one to the conclusion that the underlying

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mechanism of the quantum radar cross section comes purely from the concept of quantum superposition, and therefore any quantum particle probe will generate an analogous cross section response. This result suggests an interesting possibility of using not only photons for standoff target detection, but other quantum particles as well.

2. ILLUMINATION WITH A SINGLE PARTICLE

The theory of particle scattering in quantum mechanics is well understood and well established [11]. We will denote the state of an incoming particle's wave function as $|\psi\rangle$, traveling in a direction defined by its wave vector $\mathbf{k}$. After the particle interacts with the target and scatters in another direction $\mathbf{k}'$, the outgoing wave function, which we will denote as $|\psi'\rangle$, is the sum of the incident plane wave of the free particle, plus an outgoing spherical wave from the target. This is given by the following:

$$\langle \mathbf{x}|\psi'\rangle = \frac{1}{(2\pi\hbar)^{3/2}}\left[e^{i\mathbf{k}\cdot\mathbf{x}} + f\left(\mathbf{k},\mathbf{k}'\right)\frac{e^{ikr}}{r}\right] \tag{1}$$

where $f(\mathbf{k},\mathbf{k}')$ is the scattering amplitude and describes how the particle is scattered in a particular direction $\mathbf{k}'$, which is in general, different from $\mathbf{k}$, and r is the distance from the center of the target to the observation point (which is far removed) and is equal to $r = |\mathbf{x}|$. Note that in this development, we are ignoring tunneling and diffraction effects. The expression for the scattering amplitude under the Born approximation is given by the following [11]:

$$f\left(\mathbf{k},\mathbf{k}'\right) = -\frac{1}{2\pi}\frac{m}{\hbar^2}\int e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}'}V\left(\mathbf{x}'\right)d\mathbf{x}' \tag{2}$$

where $\mathbf{x}'$ is the distance from the center of the object (and coordinate system) to a particular point on the object, $V(\mathbf{x}')$ the potential that represents the scatterer, or target in space, m the mass of the particle, and $\hbar = h/2\pi$ where h is Planck's constant. Typically in these types of scattering problems in physics, one is interested in particle collision scattering, namely, two particle beams colliding with each other or against a 2D target of simple geometry. However, within the context of quantum radar, the scatterer of interest is actually a macroscopic target with complex geometry made up of a collection of atoms. We treat each atom as a point scatterer, having no long range interaction with the incoming particles. This can be modeled adequately by a collection of spatial impulse functions. Therefore, we make use of the Fermi pseudo potential typically used in neutron scattering theory [12] and express

$$V\left(\mathbf{x}'\right) = \frac{2\pi\hbar^2}{m}\sum_{n=1}^{N} b_n\delta\left(\mathbf{x}' - \mathbf{x}^{(n)}\right) \tag{3}$$

where $\mathbf{x}^{(n)}$ represents the position of the n-th atom in an arbitrary coordinate system, N the total number of atoms, and b_n the scattering length of each atom, which is a description of how large the atom looks to a quantum particle based on the particle's wavelength and the atom's size. One can think of b_n as being analogous to the atom's individual scattering cross section, and the potential for the entire system is proportional to the summation of all these cross sections. We now obtain the following for the scattering amplitude after substituting this expression for the potential function and assuming that the scattering lengths for each atom are the same ($b_n = b, \forall n$)

$$\begin{aligned} f\left(\mathbf{k},\mathbf{k}'\right) &= -b\int e^{-i\mathbf{k}'\cdot\mathbf{x}'}\sum_{n=1}^{N}\delta\left(\mathbf{x}' - \mathbf{x}^{(n)}\right)e^{i\mathbf{k}\cdot\mathbf{x}'}d\mathbf{x}' \\ &= -b\sum_{n=1}^{N}\int e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}'}\delta\left(\mathbf{x}' - \mathbf{x}^{(n)}\right)d\mathbf{x}' \\ &= -b\sum_{n=1}^{N} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}^{(n)}} \end{aligned} \tag{4}$$

For the monostatic case, the observation direction is opposite to the direction of incidence, thus $\mathbf{k} = -\mathbf{k}'$, and $|\mathbf{k} - \mathbf{k}'| = 2k$. So, we obtain,

$$f\left(\mathbf{k}, \mathbf{k}'\right) = -b \sum_{n=1}^{N} e^{2i\mathbf{k}\cdot\mathbf{x}^{(n)}} \tag{5}$$

Taking the magnitude squared gives us:

$$\left|f\left(\mathbf{k}, \mathbf{k}'\right)\right|^2 = |b|^2 \left|\sum_{n=1}^{N} e^{2i\mathbf{k}\cdot\mathbf{x}^{(n)}}\right|^2 \tag{6}$$

We now insert a factor of 1 by multiplying by $|e^{2i\mathbf{k}\cdot\mathbf{d}}|$, where $\mathbf{d}$ is the distance from the center of the object, to the radar. This gives us:

$$\begin{aligned}\left|f\left(\mathbf{k}, \mathbf{k}'\right)\right|^2 &= |b|^2 \left|\sum_{n=1}^{N} e^{2i\mathbf{k}\cdot\left(\mathbf{d}+\mathbf{x}^{(n)}\right)}\right|^2 \\ &= |b|^2 \left|\sum_{n=1}^{N} e^{2i|\mathbf{k}|\left|\mathbf{d}+\mathbf{x}^{(n)}\right|\cos\theta}\right|^2\end{aligned} \tag{7}$$

where θ is the angle between the vector $\mathbf{k}$ and $(\mathbf{d}+\mathbf{x}^{(n)})$, and $(\mathbf{d}+\mathbf{x}^{(n)})$ is the distance from the receiver to the n-th atom. If the object is very far away, this angle will always be very small and we can set it equal it to be zero. This gives:

$$\left|f\left(\mathbf{k}, \mathbf{k}'\right)\right|^2 = |b|^2 \left|\sum_{n=1}^{N} e^{2i|\mathbf{k}|\left|\mathbf{d}+\mathbf{x}^{(n)}\right|}\right|^2 \tag{8}$$

We recognize the term $2|\mathbf{d} + \mathbf{x}^{(n)}|$ to be the round trip distance from the radar, to a particular atom, otherwise known as the total interferometric distance. We denote this as ΔR_n. Thus, we have

$$\left|f\left(\mathbf{k}, \mathbf{k}'\right)\right|^2 = |b|^2 \left|\sum_{n=1}^{N} e^{ik\Delta R_n}\right|^2 \tag{9}$$

This term accounts for the superposition of all the wave functions emitted from each atom in the target. The origin for this superposition of wave functions is the multi-path nature of the particle. When the particle impinges on the target, it has a probability of interacting with any of the atoms in the object and being scattered in another direction. This situation is analogous to a double slit experiment. The uncertainty in the exact path of the particle manifests as a summation of all possible paths the particle can take [13].

Our next objective is to develop an expression for the quantum radar cross section of the target. A reasonable definition of the quantum radar cross section σ_Q is to define it in terms of intensity [2], analogous to the classical case [14]. Thus,

$$\sigma_Q = \lim_{R\to\infty} 4\pi R^2 \frac{\langle I_{\text{scattered}}\rangle}{\langle I_{\text{Incident}}\rangle}. \tag{10}$$

where R is the distance that the target is away from the radar, $\langle I_{\text{scattered}}\rangle$ the expectation value of the scattered intensity, and $\langle I_{\text{Incident}}\rangle$ the expectation value of the incident intensity. It can be shown that the magnitude squared of the scattering amplitude, $|f(\mathbf{k}, \mathbf{k}')|^2$ is equal to the differential scattering cross section of the target, $d\tilde{\sigma}/d\Omega$ [11]. We have added a tilde above σ to differentiate it from the σ used for the radar cross section, which has units of square meters, m^2. The definition of this particular cross section normally used in physics is *the number of incident particles crossing a plane perpendicular to the incident direction per unit area per unit time* [11]. This particular cross section, although sharing the same nomenclature, is different from the type of cross section considered for radar and standoff sensing. While both cross sections provide information on the amount of return expected at a particular angle, the radar cross section takes into account potentially complex 3D target geometries, whereas $\tilde{\sigma}$ does not because it is associated with a collision from a simple target. Based on the definition, the units of $\tilde{\sigma}$ are $(1/s)/m^2$. This suggests that the scattering amplitude is directly proportional to the incident intensity density, as all one needs to do is multiply by the energy per particle, thereby obtaining units of $\mathrm{W/m^2}$, where W is Watts. We will call this energy $E_{\mathbf{k}'}$.

$$\langle I_{\text{scattered}} \rangle = E_{\mathbf{k}'} \left| f\left(\mathbf{k}, \mathbf{k}'\right) \right|^2. \tag{11}$$

To obtain the expression for the total incident intensity on the object, we employ the high frequency approximation expounded in Ref. [2], which is valid for any wavelength smaller than the target dimensions. We recognize that the incident intensity will simply be equal to the scattered intensity density, integrated over all viewing angles (assuming there is no loss of energy during the scattering process)

$$\int_{S_T} \langle I_{\text{incident}} \rangle dS \approx \iint_{\mathcal{S} \supset T} E_{\mathbf{k}'} R^2 \left| f\left(\mathbf{k}, \mathbf{k}'\right) \right|^2 \sin\theta d\theta' d\phi' \tag{12}$$

where S_T is the surface of the target, and $\mathcal{S} \supset T$ denotes the solid angle of a large half sphere that surrounds the target. It is only a half sphere because the object does not emit fields behind itself, as the atoms that would be responsible for this behavior are in the shadow region of the illumination. Henceforth, we will orient our coordinate system such that this half sphere is associated with the angles $0 \le \phi' \le 2\pi$ and $0 \le \theta' \le \pi/2$, If the wavelength of the photon is small in comparison to the object, then we can assume that the incident intensity is uniform over the target, and we obtain the following:

$$\int_{S_T} \langle I_{\text{incident}} \rangle dS \approx \langle I_{\text{incident}} \rangle A_\perp \tag{13}$$

where $A_\perp$ is the projected cross sectional area of the target. Thus, the incident intensity of the particle is given by the following:

$$\langle I_{\text{incident}} \rangle = \frac{E_{\mathbf{k}'}}{A_\perp} \int_0^{2\pi} \int_0^{\pi/2} R^2 \left| f\left(\mathbf{k}, \mathbf{k}'\right) \right|^2 \sin\theta d\theta' d\phi'. \tag{14}$$

Substituting Equations (11) and (14) into Equation (10) gives:

$$\sigma_Q(\theta, \phi) = 4\pi A_\perp \frac{\left| f\left(\mathbf{k}, \mathbf{k}'\right) \right|^2}{\displaystyle\int_0^{2\pi} \int_0^{\pi/2} \left| f\left(\mathbf{k}, \mathbf{k}'\right) \right|^2 \sin\theta' d\theta' d\phi'} \tag{15}$$

or equivalently:

$$\sigma_Q(\theta, \phi) = 4\pi A_\perp \frac{\left| \displaystyle\sum_{n=1}^{N} e^{ik\Delta R_n} \right|^2}{\displaystyle\int_0^{2\pi} \int_0^{\pi/2} \left| \sum_{n=1}^{N} e^{ik\Delta R'_n} \right|^2 \sin\theta' d\theta' d\phi'} \tag{16}$$

Thus, using purely particle considerations, we have arrived at the exact same equation for the QRCS using single photon illumination [2]. We can go a step further and relax the monostatic geometry assumption in Equation (5), as well as not imposing the extra factor of 1 introduced by a complex exponential in terms of the target distance used in Equation (7). This provides coordinates that are only dependent on target geometry, not object distance. These changes will give us a more general result for any bistatic setup, as follows:

$$\sigma_Q(\theta, \phi) = 4\pi A_\perp \frac{\left| \displaystyle\sum_{n=1}^{N} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}^{(n)}} \right|^2}{\displaystyle\int_0^{2\pi} \int_0^{\pi/2} \left| \sum_{n=1}^{N} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}^{(n)}} \right|^2 \sin\theta' d\theta' d\phi'} \tag{17}$$

Notice that σ_Q does not depend on b, the scattering length of the atoms even though the scattering amplitude explicitly depends on it in Equation (4). The reason for this is because we assumed it to be constant for every atom in the object (namely, all of the atoms are the same in the target) and it canceled in the ratio in the QRCS expression. Also notice that the limit as $R \to \infty$ is no longer present. This is because when substituting in the expression for $\langle I_{\text{incident}} \rangle$, the R^2 term cancels the R^2 term in the numerator of the expression.

It is important to mention, by no means do we suggest that we are only sending one singular particle towards the target in total. In actuality, one would need to send many single particles towards the target, and make many single particle measurements. In the next section we will explore the scenario wherein small clusters of particles are sent towards the target instead of single particles.

3. ILLUMINATION WITH MULTIPLE PARTICLES

In a general scenario, many particles per pulse would be used to illuminate a target. The method presented in the previous section can be generalized to any number of particles. We will start by analyzing the case for two-particle illumination and then generalize from there.

We assume the two particles occupy different Hilbert spaces. Thus, the total wave function can be represented as a tensor product [11], $|\psi_1\rangle \otimes |\psi_2\rangle = |\psi_1, \psi_2\rangle$. Equation (1) represents the solution for each particle separately, and therefore the total wave function solution is the product of the two independent solutions.

$$\begin{aligned}\left\langle \mathbf{x}_1, \mathbf{x}_2 | \psi_1', \psi_2' \right\rangle = \left(\frac{1}{(2\pi\hbar)^{3/2}}\right)^2 \Bigg[& e^{i\mathbf{k}_1\cdot\mathbf{x}_1} e^{i\mathbf{k}_2\cdot\mathbf{x}_2} + e^{i\mathbf{k}_1\cdot\mathbf{x}_1} \frac{e^{ik_2 r_2}}{r_2} f_2\left(\mathbf{k}_2, \mathbf{k}_2'\right) + e^{i\mathbf{k}_2\cdot\mathbf{x}_2} \frac{e^{ik_1 r_1}}{r_1} f_1\left(\mathbf{k}_1, \mathbf{k}_1'\right) \\ & + \frac{e^{ik_1 r_1}}{r_1} \frac{e^{ik_2 r_2}}{r_2} f_1\left(\mathbf{k}_1, \mathbf{k}_1'\right) f_2\left(\mathbf{k}_2, \mathbf{k}_2'\right) \Bigg]\end{aligned} \tag{18}$$

where the first term represents the case where neither particles interact with the target, the second and third terms represent the cases where only one particle interacts with the target, and the last term represents the case where both particles interact with the target. In the interest of quantum radar, we are discussing the case where both terms interact with the target, so all of other terms can be disregarded. We use the same expression as in the single particle case for both scattering amplitudes $f_{1,2}(\mathbf{k}, \mathbf{k}')$.

$$\left\langle \mathbf{x}_1, \mathbf{x}_2 | \psi_1', \psi_2' \right\rangle = \left(\frac{1}{(2\pi\hbar)^{3/2}}\right)^2 \frac{e^{ik_1 r_1}}{r_1} \frac{e^{ik_2 r_2}}{r_2} \left(b \sum_{n=1}^{N} e^{i(\mathbf{k}_1 - \mathbf{k}_1')\cdot\mathbf{x}^{(n)}} \right) \left(b \sum_{m=1}^{N} e^{i(\mathbf{k}_2 - \mathbf{k}_2')\cdot\mathbf{x}^{(m)}} \right) \tag{19}$$

In general, for M particles, we would need to take M tensor products of Equation (1). This will produce many terms, each one representing a different scenario of particles that do, and do not, interact with the target. Since we are only interested in the case when all particles interact with the target, we again only take final term in the expansions and the procedure just shown will be identical (this situation corresponds to using a narrow beam to illuminate the target). Therefore, for any number of particles M, and any arbitrary bistatic radar geometry, we have:

$$\left\langle \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N | \psi_1', \psi_2', \ldots, \psi_N' \right\rangle = \left(\frac{1}{(2\pi\hbar)^{3/2}}\right)^M \left[b^M \prod_{q=1}^{M} \frac{e^{ik_q r_q}}{r_q} \left(\sum_{n=1}^{N} e^{i\left(\mathbf{k}_q - \mathbf{k}_q'\right)\cdot\mathbf{x}_q^{(n)}} \right) \right]. \tag{20}$$

Since all particles are incident and observed in the same respective directions, we let $\mathbf{k}_q \to \mathbf{k}$ and $\mathbf{k}_q' \to \mathbf{k}'$. We again use Equation (15) as the definition of the quantum radar cross section to obtain the following.

$$\sigma_Q = \frac{4\pi A_\perp \left| \prod_{q=1}^{M} \left(\sum_{n=1}^{N} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}_q^{(n)}} \right) \right|^2}{\int_0^{2\pi} \int_0^{\pi/2} \left| \prod_{q=1}^{M} \left(\sum_{n=1}^{N} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}_q^{(n)}} \right) \right|^2 \sin\theta' d\theta' d\phi'} \tag{21}$$

In the single particle illumination case, we saw that we exactly obtained the QRCS equation for a photon. How does this multiparticle solution compare with the multiphoton solution in the literature [2]? Since each summation will yield the same value, it is tempting to write $\prod_i^\gamma(\sum_n)_i \rightarrow (\sum_n)^\gamma$. However, when expanding out the product of summations in Equation (21), we will obtain terms that represent contributions from the same atom, but from different incident particles. In the context of photon scattering, we are assuming the scattering interactions are first order, namely, Feynman diagrams with only 2 vertices [2]. Therefore, this restricts interactions from multiple photons originating from the same atom in a single term in the expansion [15]. In other words, more than one photon cannot interact with an atom simultaneously. Therefore, Equation (21) has extra contributions that are not present in the photon case.

As an example to illustrate this idea, let us imagine we are illuminating a target consisting of three atoms with two particles. We then have (after setting $\mathbf{K} = \mathbf{k} - \mathbf{k}'$):

$$\begin{aligned}\prod_{q=1}^{2}\left(\sum_{n=1}^{3} e^{i\mathbf{K}\cdot\mathbf{x}_q^{(n)}}\right) &= \left(\sum_{n=1}^{3} e^{i\mathbf{K}\cdot\mathbf{x}_1^{(n)}}\right)\left(\sum_{m=1}^{3} e^{i\mathbf{K}\cdot\mathbf{x}_2^{(m)}}\right)\\ &= \left(e^{i\mathbf{K}\cdot\mathbf{x}_1^{(1)}} + e^{i\mathbf{K}\cdot\mathbf{x}_1^{(2)}} + e^{i\mathbf{K}\cdot\mathbf{x}_1^{(3)}}\right)\\ &\quad\times\left(e^{i\mathbf{K}\cdot\mathbf{x}_2^{(1)}} + e^{i\mathbf{K}\cdot\mathbf{x}_2^{(2)}} + e^{i\mathbf{K}\cdot\mathbf{x}_2^{(3)}}\right)\end{aligned} \tag{22}$$

Expanding the right hand side of Equation (22) yields:

$$\begin{aligned}&e^{i\mathbf{K}\cdot\left(\mathbf{x}_1^{(1)}+\mathbf{x}_2^{(1)}\right)} + e^{i\mathbf{K}\cdot\left(\mathbf{x}_1^{(1)}+\mathbf{x}_2^{(2)}\right)} + e^{i\mathbf{K}\cdot\left(\mathbf{x}_1^{(1)}+\mathbf{x}_2^{(3)}\right)}\\ &e^{i\mathbf{K}\cdot\left(\mathbf{x}_1^{(2)}+\mathbf{x}_2^{(1)}\right)} + e^{i\mathbf{K}\cdot\left(\mathbf{x}_1^{(2)}+\mathbf{x}_2^{(2)}\right)} + e^{i\mathbf{K}\cdot\left(\mathbf{x}_1^{(2)}+\mathbf{x}_2^{(3)}\right)}\\ &e^{i\mathbf{K}\cdot\left(\mathbf{x}_1^{(3)}+\mathbf{x}_2^{(1)}\right)} + e^{i\mathbf{K}\cdot\left(\mathbf{x}_1^{(3)}+\mathbf{x}_2^{(2)}\right)} + e^{i\mathbf{K}\cdot\left(\mathbf{x}_1^{(3)}+\mathbf{x}_2^{(3)}\right)}\end{aligned}$$

The terms that are excluded are the terms where the contribution is from the same atom, but from different photons, namely the terms, $e^{i\mathbf{K}\cdot(\mathbf{x}_1^{(1)}+\mathbf{x}_2^{(1)})}$, $e^{i\mathbf{K}\cdot(\mathbf{x}_1^{(2)}+\mathbf{x}_2^{(2)})}$, and $e^{i\mathbf{K}\cdot(\mathbf{x}_1^{(3)}+\mathbf{x}_2^{(3)})}$, which correspond to the squared terms in the product of sums. The expansion above can be rewritten as:

$$\sum_{n=1}^{3} e^{i\mathbf{K}\cdot\left(\mathbf{x}_1^{(n)}+\mathbf{x}_2^{(n)}\right)} + \sum_{m=l}^{3}\sum_{l=1}^{3}(1-\delta_{ml})e^{i\mathbf{K}\cdot\left(\mathbf{x}_1^{(m)}+\mathbf{x}_2^{(l)}\right)} \tag{23}$$

In general, for N atoms, this can be generalized to:

$$\sum_{n=1}^{N} e^{i\mathbf{K}\cdot\left(\mathbf{x}_1^{(n)}+\mathbf{x}_2^{(n)}\right)} + \sum_{m=l}^{N}\sum_{l=1}^{N}(1-\delta_{ml})e^{i\mathbf{K}\cdot\left(\mathbf{x}_1^{(m)}+\mathbf{x}_2^{(l)}\right)} \tag{24}$$

In this particular example, N is very small, so the number of squared terms is on the same order as the number of cross terms (namely, 3 and 6 respectively). However as N grows large, the number of squared and cross terms will be N and $N!/(N-2)!$ (the number of ways to arrange N items, taken 2 at a time). If we take the ratio between these two numbers, and take the limit as $N \to \infty$, we obtain:

$$\lim_{N\to\infty}\frac{N}{N!/(N-2)!} = \lim_{N\to\infty}\frac{(N-2)!}{(N-1)!} = 0 \tag{25}$$

Therefore, in the limit of a large number of atoms, the squared (or excluded) terms do not contribute to the overall interference pattern of the response. Thus, in this limit, the multiple particle illumination solution is equivalent to the multiple photon illumination solution.

For a larger number of illuminating photons, there will be more modes that are not allowed. For example, for three particle illumination, one can show that the product of summations becomes:

$$\begin{aligned}\prod_{q=1}^{3}\left(\sum_{n=1}^{N} e^{i\mathbf{K}\cdot\mathbf{x}_q^{(n)}}\right) &= \sum_{p=1}^{N}\sum_{m=1}^{N}\sum_{n=1}^{N}(1-\delta_{mn})(1-\delta_{pn})(1-\delta_{pm})\,e^{i\mathbf{K}\cdot\left(\mathbf{x}_1^{(n)}+\mathbf{x}_2^{(m)}+\mathbf{x}_3^{(p)}\right)}\\ &\quad+3\sum_{k=1}^{N}\sum_{j=1}^{N}(1-\delta_{jk})\,e^{i\mathbf{K}\cdot\left(\mathbf{x}_1^{(j)}+\mathbf{x}_2^{(j)}+\mathbf{x}_3^{(k)}\right)} + \sum_{l=1}^{N} e^{i\left(\mathbf{K}\cdot\mathbf{x}_1^{(l)}+\mathbf{K}\cdot\mathbf{x}_2^{(l)}+\mathbf{K}\cdot\mathbf{x}_3^{(l)}\right)}\end{aligned} \tag{26}$$

In other words, the first term represents the case when none of the atom position vectors is equal to any other at the same time (the allowed terms), the second term when two are equal at the same time, and the last term when all three are equal (the excluded terms). The number of terms in the triple summation is simply the number of ways one can arrange N items, taken 3 at a time (since there are three distinct position vectors), i.e., $N!/(N-3)!$. The number of terms in the double sum term will be the amount of ways one can arrange N items, taken 2 at a time, multiplied by 3, i.e., $3N!/(N-2)!$ (we are treating $\mathbf{x}_1$ and $\mathbf{x}_2$ as one item since they have the same counting index). Lastly, the number of terms in the single sum is the number of ways to arrange N items, taken 1 at a time, which simplifies to N. Therefore, similar to the first case, the sum of the last two terms over the first is:

$$\lim_{N\to\infty} \frac{3N!/(N-2)! + N}{N!/(N-3)!} = \lim_{N\to\infty} \frac{3}{N-2} + \frac{(N-3)!}{(N-1)!} = 0 \tag{27}$$

This pattern will continue for any number of illuminating particles. For M particles, the product of summations will be equal to a term that does not repeat any indices in the atom position vectors, plus a term that repeats indices twice, plus a term that repeats indices thrice, and so on, all the way to a sum that repeats indices M times. Following the established pattern, the ratio of terms would be (the sums may have constant multiples such as the double sum did in the previous case, and we denote these terms as C_k):

$$\lim_{N\to\infty} \frac{\displaystyle\sum_{k=1}^{M-1} \frac{C_k N!}{(N-k)!}}{\dfrac{N!}{(N-M)!}} = \lim_{N\to\infty} \sum_{k=1}^{M-1} C_k \frac{(N-M)!}{(N-k)!} \tag{28}$$

Since M will always be larger than any value k can take on, $(N-M)!$ will always be smaller than $(N-k)!$; therefore every term in the sum will go to zero when taking the limit as N goes to infinity.

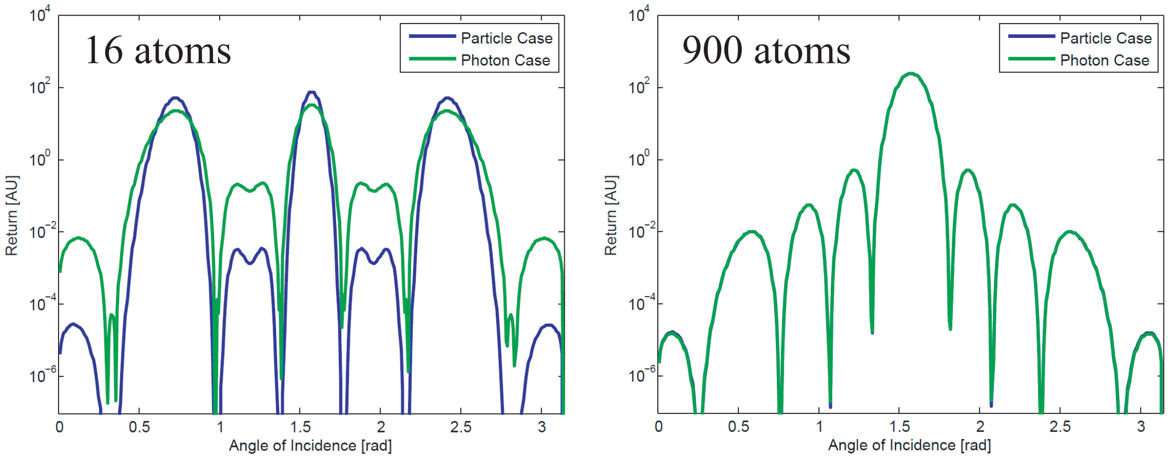
These results tell us that no matter the number of illuminating particles, in the limit of infinite atoms, the excluded scattering states do not contribute to the overall response.

Using the concepts developed here, we can write the photon and particle cross section equation for M illuminating photons or particles as the following (with the assumption of a large number of atoms):

$$\sigma_Q = 4\pi A_\perp \frac{\left|\displaystyle\sum_{n=1}^{N} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}^{(n)}}\right|^{2M}}{\displaystyle\int_0^{2\pi}\int_0^{\pi/2} \left|\sum_{n=1}^{N} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}^{(n)}}\right|^{2M} \sin\theta' d\theta' d\phi'} \tag{29}$$

Interestingly enough, this particle analysis makes evident a more useful approximation for multi-photon illumination, even though we are not dealing with photons. The realization that one can simply exponentiate the summation to a power equal to the number of photons, instead of determining all of the allowed wave function combinations, allows for much faster computations. Multiple photon illumination has always been the crutch to QRCS simulations [2]. This result enables the simulation of multiple photon illumination to be extremely simple computationally compared to previous methods.

Figures 1(a) and 1(b) illustrate this concept of large atom equivalence for a rectangular plate target illuminated by 2 particles/photons. When the number of atoms is only 16, the squared terms in the particle response have a relatively large contribution and therefore cause a large change in the interference pattern. As the number of atoms grows to 900, the two responses become equivalent. This behavior has been observed to always be true in subsequent simulations, regardless of target dimensions or incident wavelength. Although it is desirable that the wavelength be much smaller than the target dimensions, the wavelength used is just one-half of the target size in these figures. We determined via simulations that the peak returns were quite close in both cases. However, the plots of smaller wavelengths resulted in a very dense lobe structure making it impossible to observe the close correspondence between the particle and the photon case.



(a) (b)

Figure 1. QRCS response of a 1 meter × 1 meter rectangular plate illuminated with 2 particles/photons with a wavelength of 0.5 meters for both the particle and photon case. (a) When the atom number is only 16, the two responses are very different. (b) However as the atom number grows, the two responses become equivalent.

4. WRITING THE QRCS EXPRESSION IN TERMS OF FOURIER TRANSFORMS

This section serves to illustrate how one can rewrite the QRCS equation in terms of Fourier transforms using the formalism presented, for easier computation. We start with the simplest case, namely the single particle illumination case. If we return to Equation (2), and denote $\mathbf{k}-\mathbf{k}'$ as $\mathbf{K}$, we have the following.

$$f\left(\mathbf{k},\mathbf{k}'\right)=-\frac{m}{2\pi\hbar^2}\int e^{-i\mathbf{K}\cdot\mathbf{x}'}V\left(\mathbf{x}'\right)d\mathbf{x}'=-\frac{m}{2\pi\hbar^2}\mathcal{F}\left(V(\mathbf{x})\right). \tag{30}$$

Thus, we see that the scattering amplitude is simply the Fourier transform of the atom distribution [11]. Note that nothing in this theory sheds any light on what atoms in the object are most important. In this case, we are considering solid objects which are not transparent. Thus, the photon will only interact with the atoms on the surface of the object.

What was done earlier, namely, defining the interaction potential as a summation of delta functions, can be thought of as sampling a continuous distribution of atoms, since the atoms are so numerous and so close together. More explicitly, for the continuous case, we can define $V(\mathbf{x})$ to be the following:

$$V\left(\mathbf{x}'\right)=\begin{cases}1, & \text{if } \mathbf{x}'\in S\\ 0, & \text{else}\end{cases}. \tag{31}$$

where S defines the surface of the object under study. The constant multipliers required for dimensional analysis have been neglected since they will cancel upon substitution into the equation for the QRCS. With this definition in mind, we can now go through the same steps to re-write the QRCS formula in terms of Fourier transforms.

$$\sigma_Q\left(\mathbf{k}-\mathbf{k}'\right)=\frac{4\pi A_\perp\left|\mathcal{F}\left(V(\mathbf{x})\right)\right|^2}{\displaystyle\int_0^{2\pi}\int_0^{\pi/2}\left|\mathcal{F}\left(V(\mathbf{x})\right)\right|^2\sin\theta' d\theta' d\phi'} \tag{32}$$

For multi-photon illumination, the procedure is very similar. We present the case for two-photon illumination as an example. The scattering amplitude for each particle is given by Equation (2). Therefore for the product of both of scattering amplitudes, the total scattering amplitude $f_T(\mathbf{k},\mathbf{k}')$ is (after dropping the multiplication factors):

$$f_T\left(\mathbf{k},\mathbf{k}'\right)=\int e^{i\mathbf{K}_1\cdot\mathbf{x}_1'}V\left(\mathbf{x}_1'\right)d^3\mathbf{x}_1'\int e^{i\mathbf{K}_2\cdot\mathbf{x}_2'}V\left(\mathbf{x}_2'\right)d^3\mathbf{x}_2'$$

$$= \mathcal{F}\left(V(\mathbf{x}_1)\right)\mathcal{F}\left(V(\mathbf{x}_2)\right) = \mathcal{F}\left(V(\mathbf{x})\right)^2 \tag{33}$$

This result is expected because of what was found in the previous section, namely that as the number of atoms grows to infinity, the contribution from the terms not allowed in the expansion compared to the terms that are allowed goes to zero, and the summation for M photons/particles, can simply be raised to the Mth power. This allowed us to write the product of two summations as a square of one summation. When going to the continuous case, we assume the same principles apply. We can now rewrite Equation (32) as the following for any number of photons or particles M.

$$\sigma_Q\left(\mathbf{k}-\mathbf{k}'\right) = \frac{4\pi A_\perp \left|\mathcal{F}\left(V\left(\mathbf{x}'\right)\right)\right|^{2M}}{\displaystyle\int_0^{2\pi}\int_0^{\pi/2} \left|\mathcal{F}\left(V\left(\mathbf{x}'\right)\right)\right|^{2M} \sin\theta' d\theta' d\phi'} \tag{34}$$

It must be mentioned here that although we obtain the same equations for the QRCS using a particle method, the response of a particle will be vastly different than the response of a photon. This is purely due to the difference in wavelengths, which is manifested in the expression for $\mathbf{k}$ in the complex exponentials. The degree in which the atomic wave functions interfere with each other depends heavily on the wavelength, this interference dictates the QRCS response of the target; therefore a large difference in wavelength leads to a large difference in the response of the target.

To obtain some insight into this statement, let us calculate what the velocity of a particle (say a neutron for example) would need to be, to obtain a wavelength of 0.3 meters. The De Broglie wavelength is given by:

$$\lambda = \frac{h}{mv} \tag{35}$$

To obtain this wavelength, the velocity would need to be 2.08×10^{-7} meters per second, which is, for all physical purposes, essentially stationary. Therefore, the examples given earlier in this paper were simply used for comparison purposes and do not represent practical or realistic scenarios.

A more realistic situation is an electron traveling in an electron microscope [16]. In this scenario, the electron is traveling at roughly 70% of the speed of light, therefore relativistic effects must be taken into account. Doing so will give us the following expression:

$$\lambda = \frac{h}{\sqrt{2m_0 eU}} \frac{1}{\sqrt{1+\dfrac{eU}{2m_0c^2}}} \tag{36}$$

where m_0 is the rest mass of the electron, U the electric potential used to accelerate the electron and e the elementary charge. Calculating the wavelength some some typical accelerating potentials (on the order of 10^1–10^3 kV), we find that the wavelength is on the order of 10^{-12} meters, which is several orders of magnitude less than microwave frequencies, with wavelengths on the order of 10^{-3} to 10^1 meters.

5. CONCLUSIONS

The QRCS provides information on how much return one can expect in a particular scattering direction from a target, when being illuminated at a particular incident direction, from a single photon or cluster of photons. This cross section is heavily dependent on the geometry of the target as the photon-atom interaction for each atom on the objects surface contributes to the overall cross section response.

In this paper, we developed a theory which focuses on illuminating an object with non-relativistic, massive particles instead of photons. It was found that in the single photon/particle illumination case, the two cross section expressions are identical. Following this, we also showed that in the multiple photon/particle illumination case, the two cross section expressions become equivalent in the limit of large atom number. This therefore lead us to obtain a very convenient approximation of the multi-photon illumination case using the result of the multiparticle illumination case. This convenient form allows one to very quickly calculate the QRCS response with multiple photons.

These results should not be entirely surprising. Indeed, both types of objects, photons and neutral, non-relativistic particles, both interact in a superposition state with the atoms in the target. The nature of this superposition interaction is different between these two types of particles, but the form of this coherent interaction is the same.

REFERENCES

1. Lanzagorta, M., "Low brightness quantum radar," *Proceedings of the SPIE Conference on Radar Sensor Technology XIX and Active and Passive Signatures VI*, 946113, Baltimore, MD, 2015.
2. Lanzagorta, M., *Quantum Radar*, Morgan & Claypool, San Rafael, CA, 2012.
3. Lanzagorta, M., "Quantum radar cross sections," *Proceedings of the SPIE Conference on Quantum Optics*, 77270K, Brussels, Belgium, 2010.
4. Lin, Y., L. Guo, and K. Cai, "An efficient algorithm for the calculation of quantum radar cross section of flat objects," *PIERS Proceedings*, 39–43, Guangzhou, China, August 25–28, 2014.
5. Kang, L., H.-T. Xiao, and H.-Q. Fan, "Analysis and simulation of quantum radar cross section," *Chin. Phys. Lett.*, Vol. 31, 034202, 2014.
6. Barzanjeh, S., S. Guha, C. Weedbrook, D. Vitali, J. Shapiro, S. Pirandola, "Microwave quantum illumination," *Phys. Rev. Lett.*, Vol. 114, 080503, 2015.
7. Jiang, K., H. Lee, C. C. Gerry, and J. P. Dowling, "Super-resolving quantum radar: Coherent-state sources with homodyne detection suffice to beat the diffraction limit," *J. Appl. Phys.*, Vol. 114, 193102, 2013.
8. Didomenico, L. D., H. Lee, P. Kok, and J. P. Dowling, "Quantum interferometric sensors," *Proceedings of SPIE Conference on Quantum Sensing and Nanophotonic Devices*, 169–176, San Jose, CA, 2004.
9. Brandsema, M. J., R. M. Narayanan, and M. Lanzagorta, "Theoretical and computational analysis of the quantum radar cross section for simple geometrical targets," *Quantum Information Science*, Vol. 16, 32, Springer, 2017.
10. Brandsema, M. J., R. M. Narayanan, and M. Lanzagorta, "Analytical formulation of the quantum electromagnetic cross section," *Proceedings of the SPIE Conference on Radar Sensor Technology XX*, 98291H, Baltimore, MD, 2016.
11. Sakurai, J. and S. Tuan, *Modern Quantum Mechanics*, Addison-Wesley, Reading, MA, 1994.
12. Liang, L., R. Rinaldi, and H. Schober, *Neutron Applications in Earth, Energy and Environmental Sciences*, Springer, New York, NY, 2009.
13. Feynman, R. P. and A. R. Hibbs, *Quantum Mechanics and Integrals*, McGraw-Hill, New York, NY, 1965.
14. Balanis, C. A., *Advanced Engineering Electromagnetics*, 2nd Edition, Wiley, New York, NY, 2012.
15. Berestetskii, V. B., E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics*, 2nd Edition, Pergamon Press Ltd., Oxford, UK, 1982.
16. Thomas, G. and M. J. Goringe, *Transmission Electron Microscopy of Materials*, John Wiley, New York, NY, 1979.