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# Fast Converging CFIE-MoM Analysis of Electromagnetic Scattering from PEC Polygonal Cross-Section Closed Cylinders

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**Abstract**—The analysis of the electromagnetic scattering from perfectly electrically conducting (PEC) objects with edges and corners performed by means of surface integral equation formulations has drawbacks due to the interior resonances and divergence of the fields on geometrical singularities. The aim of this paper is to show a fast converging method for the analysis of the scattering from PEC polygonal cross-section closed cylinders immune from the interior resonance problems. The problem, formulated as combined field integral equation (CFIE) in the spectral domain, is discretized by means of Galerkin method with expansion functions reconstructing the behaviour of the fields on the wedges with a closed-form spectral domain counterpart. Hence, the elements of the coefficients' matrix are reduced to single improper integrals of oscillating functions efficiently evaluated by means of an analytical asymptotic acceleration technique.

## 1. INTRODUCTION

Surface integral equation formulations are well suited for analyzing scattering from PEC objects [1], and among others, electric field integral equations (EFIE) and magnetic field integral equation (MFIE) formulations are usually preferred. However, for PEC closed surfaces, both EFIE and MFIE do not admit a unique solution at the resonant frequencies of suitable interior problems. As a consequence, matrices obtained upon discretizing them become increasingly ill-conditioned, and the convergence tends to be slower and slower as the operational frequency approaches an interior resonant frequency [2]. In the literature devoted to the analysis of the scattering by PEC closed surfaces, many are the remedies proposed in order to guarantee the uniqueness of the solution even at the interior resonant frequencies: the extended boundary condition method [3, 4], combining interior and exterior field expression method [5], combined-source method [6], and dual-surface formulation method [7] deserve to be mentioned.

Among others, combined field integral equation (CFIE) formulation, i.e., a judiciously constructed linear combinations of EFIE and MFIE [8, 9], is largely preferred [2, 10–18]. The desired uniqueness property of such a kind of formulation resides in the orthogonality of the null spaces of EFIE and MFIE formulations. Unfortunately, the discretization and truncation of CFIE when its EFIE component contains a hypersingular term leads to "approximate solutions" which do not necessarily converge to the exact solution of the problem, and in any case, the sequence of condition numbers of truncated systems is divergent due to the unboundedness of the involved operator [19]. On the other hand, it has been widely noted that classical discretization schemes (such as, Rao-Wilton-Glisson discretization) applied to MFIE produced worse results than EFIE, and more sophisticated approaches have to be employed in order to achieve more accurate solutions [20–26]. The problem is even worse for scatterers with edges or corners due to the divergence of the fields on geometrical singularities. As a matter of

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fact, the second kind MFIE related to the longitudinal current is ill-posed since the functional spaces to which the unknown and the free term belong are different [2, 11].

Well-posed integral equations in the functional space to which the solution belongs can be obtained by means of analytical regularization method, i.e., by recasting the equation at hand as a second-kind Fredholm integral equation [19]. Indeed, Fredholm's theory [27] allows to state that the solution of the discretized and truncated counterpart of a second-kind Fredholm integral equation converges to the exact solution of the problem if unique. Moreover, the condition numbers of truncated systems are uniformly bounded and have a finite limit.

Particularly attractive is the possibility of combining discretization and analytical regularization in a single step by means of Galerkin method with a complete set of expansion functions making the most singular part of the integral operator invertible with a continuous two-side inverse [19]. This approach has been successfully used for studying propagation, radiation and scattering problems involving PEC and dielectric objects in homogeneous and layered media [28–54]. In these works, the selection of expansion bases reconstructing the physical behaviour of the unknowns at edges [55] has demonstrated to guarantee fast convergence.

In this paper, the analysis of the scattering from a PEC polygonal cross-section closed cylinder, when a TM/TE polarized plane wave orthogonally impinges onto the scatterer surface, is performed by means of a fast converging method immune from the interior resonance problems. The problem is formulated in terms of CFIE in the spectral domain. Galerkin method is employed in order to recast the obtained integral equation as a linear system of algebraic equations. Expansion bases reconstructing the physical behaviour of the surface current density on each side of the polygonal cross-section and even on the adjacent wedges are considered. The Fourier transform of the expansion functions used is expressed in closed form in terms of confluent hypergeometric functions of first kind. Such a result and the reciprocity theorem allow to reduce the convolution integrals to algebraic products. Finally, the elements of the obtained coefficients' matrix, which are single improper integrals of oscillating functions, are efficiently evaluated by means of an analytical asymptotic acceleration technique.

This paper is organized as follows. Section 2 is devoted to the formulation of the problem. The solution is proposed in Section 3. Numerical results are shown in Section 4 and the conclusions summarized in Section 5.

## 2. FORMULATION OF THE PROBLEM

The geometry of the problem is sketched in Figure 1: a plane wave impinges onto a polygonal cross-section PEC closed cylinder with an angle $\phi$ with respect to the $x$ axis and orthogonally with respect to the cylinder axis $z$, hence the electromagnetic field is invariant with respect to the $z$ axis. The $L$ sides of the polygonal cross-section are numbered clockwise. A local coordinate system $(x_i, y_i, z)$ is introduced

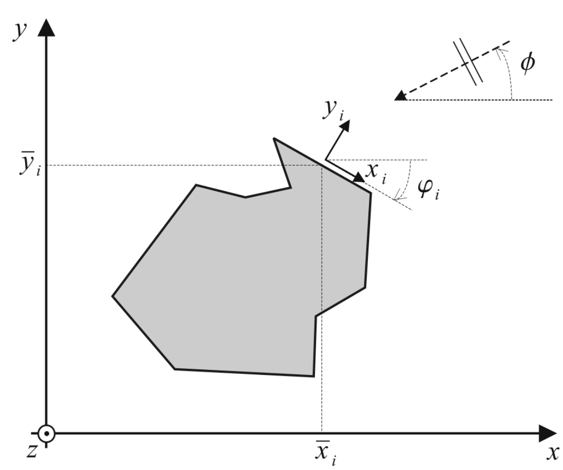



**Figure 1.** Geometry of the problem.

with the origin at the centre of the $i$-th side in position $(\bar{x}_i, \bar{y}_i)$ and the $y_i$ axis oriented in the outward direction. $\varphi_i \in ]-\pi, \pi]$ denotes the orientation of the $x_i$ axis with respect to the $x$ axis, and $2a_i$ denotes the length of the $i$-th side.

The incident field induces a surface current on the cylinder surface. Due to Meixner's theory [55], the following edge behaviour can be immediately established for the longitudinal component and the derivative of the transverse component of the current density on the $i$-th surface

$$J_{iz}(x_i), \frac{\partial}{\partial x_i} J_{ix_i}(x_i) \underset{x_i \to \pm a_i}{\sim} (1 \mp x_i/a_1)^{t_i^{\pm}} \tag{1}$$

with $i = 1, 2, \ldots, L$, where

$$t_i^{\pm} = \begin{cases} (\psi_i^{\pm} - \pi)/(2\pi - \psi_i^{\pm}) & \psi_i^{\pm} \leq 3\pi/2 \\ 1 & \psi_i^{\pm} \geq 3\pi/2 \end{cases}, \tag{2}$$

$\psi_i^{\pm}$ being the angle of the wedge at abscissas $x_i = \pm a_i$. Moreover, the transverse component of the surface current density is continuous even on the wedges, i.e., $C_1 = J_{Lx_L}(a_L) = J_{1x_1}(-a_1)$ and $C_i = J_{i-1x_{i-1}}(a_{i-1}) = J_{ix_i}(-a_i)$ for $i = 2, 3, \ldots, L$.

Starting from the edge behaviour of the longitudinal and transverse components of the surface current density on the $i$-th side, it is possible to evaluate the asymptotic behaviour of their Fourier transforms $\tilde{J}_{iz}(u)$ and $\tilde{J}_{ix_i}(u)$ with respect to $x_i$. As a matter of fact, by means of Watson's lemma [56], it can be shown that

$$\tilde{J}_{iz}(u) \overset{|u| \to +\infty}{\sim} \tilde{J}_{iz}^{\infty}(u) = \eta_i^{-} \frac{e^{-jua_i}}{u^{t_i^{-}+1}} + \eta_i^{+} \frac{e^{jua_i}}{u^{t_i^{+}+1}}, \tag{3a}$$

$$\tilde{J}_{ix_i}(u) \overset{|u| \to +\infty}{\sim} \tilde{J}_{ix_i}^{\infty}(u) = \frac{-C_i e^{-jua_i} + C_{i+1} e^{jua_i}}{2\pi j u}, \tag{3b}$$

where $\eta_i^{\pm}$ are suitable parameters depending on the problem at hand.

Stating the asymptotic behaviour in Eq. (3), the following spectral domain representation for the vector potential can be obtained by invoking the superposition principle [36]

$$\underline{A}(x, y) = -j\frac{\mu}{2} \sum_{i=1}^{L} \int_{-\infty}^{+\infty} \underline{\tilde{J}}_i(u) \frac{e^{-j|y_i|R(u)}}{R(u)} e^{-jux_i} du, \tag{4}$$

where

$$R(u) = \begin{cases} \sqrt{k^2 - u^2} & |u| \leq k \\ -j\sqrt{u^2 - k^2} & |u| \geq k \end{cases}, \tag{5}$$

$k = \omega\sqrt{\varepsilon\mu}$ is the wavenumber, $\varepsilon$ the dielectric permittivity, $\mu$ the magnetic permeability of the medium, and $\omega$ the angular frequency.

In order to obtain the integral equation governing the problem at hand, it is convenient to particularize the polarization of the impinging plane wave.

### 2.1. TM Incidence

Let us consider an orthogonally incident plane wave with transverse magnetic polarization with respect to the $z$ axis, namely

$$\underline{E}_{inc}(x, y) = E_0 \hat{z} e^{-j(k_x x + k_y y)}, \tag{6a}$$

$$\underline{H}_{inc}(x, y) = \frac{E_0}{\omega\mu} (k_y \hat{x} - k_x \hat{y}) e^{-j(k_x x + k_y y)}, \tag{6b}$$

where $k_x = -k\cos\phi$ and $k_y = -k\sin\phi$. With such a choice, only TM solutions can be searched for, i.e., the induced current is longitudinal.

Hence, the only non-vanishing component of the scattered electric field is

$$E_z(x,y) = -j\omega A_z(x,y) = -\frac{\omega\mu}{2}\sum_{i=1}^{L}\int_{-\infty}^{+\infty}\tilde{J}_{iz}(u)\frac{e^{-j|y_i|R(u)}}{R(u)}e^{-jux_i}du, \tag{7}$$

while the component of the transverse scattered magnetic field along the $x_j$ direction can be written as

$$H_{x_j}(x,y) = j\frac{1}{\omega\mu}\frac{\partial}{\partial y_j}E_z(x,y). \tag{8}$$

As a consequence of the asymptotic behaviour in Eq. (3a), it is possible to invert the derivative and the integration in Eq. (8) obtaining

$$H_{x_j}(x,y) = -\frac{1}{2}\sum_{i=1}^{L}\int_{-\infty}^{+\infty}\tilde{J}_{iz}(u)\left[\operatorname{sgn}(y_i)\,c_{i,j} + \frac{u}{R(u)}s_{i,j}\right]e^{-j|y_i|R(u)}e^{-jux_i}du, \tag{9}$$

where $c_{i,j} = \cos(\varphi_i - \varphi_j)$, $s_{i,j} = \sin(\varphi_i - \varphi_j)$, and $\operatorname{sgn}(\cdot)$ denotes the signum function.

By imposing the total electric field to be vanishing on the cylinder surface, an EFIE is obtained

$$\left.E_z(x,y)\right|_{y_j=0} = -\left.E_0 e^{-j(k_x x + k_y y)}\right|_{y_j=0} \tag{10}$$

with $|x_j| \le a_j$ and $j = 1, 2, \ldots, L$. On the other hand, by imposing the discontinuity of the tangential component of the total magnetic field on the cylinder surface, a MFIE is obtained

$$\left.H_{x_j}(x,y)\right|_{y_j=0} + J_{jz}(x_j) = \left.E_0\sqrt{\frac{\varepsilon}{\mu}}\sin(\phi - \varphi_j)\,e^{-j(k_x x + k_y y)}\right|_{y_j=0}, \tag{11}$$

with $|x_j| \le a_j$ and $j = 1, 2, \ldots, L$.

To conclude, the following CFIE can be readily stated [8]

$$\begin{aligned}&\alpha\left.E_z(x,y)\right|_{y_j=0} + (1-\alpha)\sqrt{\frac{\mu}{\varepsilon}}\left(\left.H_{x_j}(x,y)\right|_{y_j=0} + J_{jz}(x_j)\right)\\ &= \left.\left[-\alpha + (1-\alpha)\sin(\phi - \varphi_j)\right]E_0 e^{-j(k_x x + k_y y)}\right|_{y_j=0},\end{aligned} \tag{12}$$

with $|x_j| \le a_j$ and $j = 1, 2, \ldots, L$, where the choice of $0 < \alpha < 1$ will be discussed later.

### 2.2. TE Incidence

For an orthogonally incident plane wave with transverse electric polarization, namely

$$\underline{E}_{inc}(x,y) = -\frac{H_0}{\omega\varepsilon}(k_y\hat{x} - k_x\hat{y})\,e^{-j(k_x x + k_y y)}, \tag{13a}$$

$$\underline{H}_{inc}(x,y) = H_0 e^{-j(k_x x + k_y y)}\hat{z}, \tag{13b}$$

only TE solutions can be obtained, i.e., only a transverse current is induced.

The longitudinal component of the scattered magnetic field can be evaluated as

$$H_z(x,y) = \frac{1}{\mu}\left.\nabla\times\underline{A}(x,y)\right|_z, \tag{14}$$

while the component of the transverse scattered electric field along the $x_j$ axis can be expressed as

$$E_{x_j}(x,y) = -j\frac{1}{\omega\varepsilon}\frac{\partial}{\partial y_j}H_z(x,y). \tag{15}$$

The asymptotic behaviour in Eq. (3b) allows us to invert derivative and integration in Eq. (14) obtaining

$$H_z(x,y) = \frac{1}{2}\sum_{i=1}^{L}\operatorname{sgn}(y_i)\int_{-\infty}^{+\infty}\tilde{J}_{ix_i}(u)e^{-j|y_i|R(u)}e^{-jux_i}du. \tag{16}$$

However, the direct inversion of derivative and integration in Eq. (15) is not possible when $y_i = 0$. A procedure to overcome this problem, based on the extraction of the asymptotic behaviour of the integrand in Eq. (16), has been developed in [39]. In such a way, Eq. (15) can be rewritten as follows

$$E_{x_j}(x,y) = -\frac{1}{2\omega\varepsilon}\sum_{i=1}^{L}\int_{-\infty}^{+\infty}\left[\tilde{J}_{ix_i}(u)T_{i,j}(u,\mathrm{sgn}(y_i))\,e^{-j|y_i|R(u)} - \tilde{J}^{\infty}_{ix_i}(u)T^{\infty}_{i,j}(u,\mathrm{sgn}(y_i))\,e^{-|y_iu|}\right]e^{-jux_i}du \tag{17}$$

where

$$T_{i,j}(u,\mathrm{sgn}(y_i)) = R(u)c_{i,j} + u\mathrm{sgn}(y_i)\,s_{i,j}, \tag{18a}$$
$$T^{\infty}_{i,j}(u,sgn(y_i)) = -j|u|c_{i,j} + u\mathrm{sgn}(y_i)\,s_{i,j}. \tag{18b}$$

An EFIE and a MFIE are obtained by imposing the boundary conditions on the scatterer surface for the electric and magnetic fields, respectively,

$$E_{x_j}(x,y)\big|_{y_j=0} = -H_0\sqrt{\frac{\mu}{\varepsilon}}\sin(\phi-\varphi_j)\,e^{-j(k_xx+k_yy)}\Big|_{y_j=0}, \tag{19a}$$

$$H_z(x,y)\big|_{y_j=0} - J_{jx_j}(x_j) = -H_0e^{-j(k_xx+k_yy)}\Big|_{y_j=0} \tag{19b}$$

with $|x_j| \leq a_j$ and $j = 1, 2, \ldots, L$.

To conclude, the following CFIE can be readily written [8]

$$(1-\beta)\sqrt{\frac{\varepsilon}{\mu}}\,E_{x_j}(x,y)\big|_{y_j=0} + \beta\left(H_z(x,y)\big|_{y_j=0} - J_{jx_j}(x_j)\right) = [-\beta-(1-\beta)\sin(\phi-\varphi_j)]\,H_0\,e^{-j(k_xx+k_yy)}\Big|_{y_j=0}, \tag{20}$$

with $|x_j| \leq a_j$ and $j = 1, 2, \ldots, L$, where the choice of $0 < \beta < 1$ will be discussed later.

## 3. PROPOSED SOLUTION

In general, the integral equations (12) and (20) do not admit closed from solutions, hence, it is necessary to resort to numerical methods. Galerkin method will be applied in the following. In order to achieve fast convergence, suitable expansion bases reconstructing the behaviour of the unknowns even on the wedges with a closed-form spectral domain counterpart will be used.

### 3.1. TM Incidence

For TM incidence, the following expansion for the longitudinal current on the $i$-th side of the cylinder is considered

$$J_{iz}(x_i) = J^i_{-1}\chi^i\left(\frac{x_h}{a_h},\frac{x_i}{a_i}\right) + J^j_{-1}\chi^j\left(\frac{x_i}{a_i},\frac{x_j}{a_j}\right) + \sum_{n=0}^{+\infty}J^i_n\phi_n^{(\bar{t}^+_i,\bar{t}^-_i)}\left(\frac{x_i}{a_i}\right), \tag{21}$$

where $h$, $i$ and $j$ are three consecutive sides,

$$\chi^i\left(\frac{x_h}{a_h},\frac{x_i}{a_i}\right) = \begin{cases} \dfrac{\varphi_{-1}^{(t^+_h,\bar{t}^-_h)}(x_h/a_h)}{\sqrt{\left[\xi_{-1}^{(t^+_h,\bar{t}^-_h)}\right]^2+\left[\xi_{-1}^{(t^-_i,\bar{t}^+_i)}\right]^2}} & \text{for} \quad y_h = 0 \\ \dfrac{\varphi_{-1}^{(t^-_i,\bar{t}^+_i)}(-x_i/a_i)}{\sqrt{\left[\xi_{-1}^{(t^+_h,\bar{t}^-_h)}\right]^2+\left[\xi_{-1}^{(t^-_i,\bar{t}^+_i)}\right]^2}} & \text{for} \quad y_i = 0 \end{cases}, \tag{22a}$$

$$\varphi_{-1}^{(\alpha,\beta)}\left(\frac{x}{a}\right) = \frac{a^{\alpha}\xi_0^{(\alpha,\beta)}}{2^{\beta}}\varphi_0^{(\alpha,\beta)}\left(\frac{x}{a}\right), \tag{22b}$$

$$\varphi_n^{(\alpha,\beta)}\left(\frac{x}{a}\right) = \left(1-\frac{x}{a}\right)^{\alpha}\left(1+\frac{x}{a}\right)^{\beta}\frac{P_n^{(\alpha,\beta)}(x/a)}{\xi_n^{(\alpha,\beta)}}\prod\left(\frac{x}{a}\right) \tag{22c}$$

with $n = 0, 1, \ldots$,

$$\xi_{-1}^{(\alpha,\beta)} = \sqrt{\int_{-a}^{a}\left(1-\frac{x}{a}\right)^{-\alpha}\left(1+\frac{x}{a}\right)^{-\beta}\left[\phi_{-1}^{(\alpha,\beta)}\left(\frac{x}{a}\right)\right]^2 dx} = \frac{a^{\alpha}\xi_0^{(\alpha,\beta)}}{2^{\beta}}, \tag{23a}$$

$$\xi_n^{(\alpha,\beta)} = \sqrt{\int_{-a}^{a}\left(1-\frac{x}{a}\right)^{\alpha}\left(1+\frac{x}{a}\right)^{\beta}\left[P_n^{(\alpha,\beta)}\left(\frac{x}{a}\right)\right]^2 dx} = \sqrt{\frac{a2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}} \tag{23b}$$

where $n = 0, 1, \ldots$ are suitable normalization quantities; $\Pi(\cdot)$ is the unitary rectangular window; $P_n^{(\alpha,\beta)}(\cdot)$ is the Jacobi polynomial of order $n$ and parameters $\alpha$, $\beta$; $\Gamma(\cdot)$ denotes the Gamma function [57].

The coefficients $\bar{t}_i^{\pm}$ are chosen in order to reconstruct the second order behaviour of the longitudinal surface current on the wedges adjacent to the $i$-th side. It is worth noting that the first two functions in Eq. (21) are only responsible for the reconstruction of the first-order behaviour of the current on the wedges, while the residual expansion series factorizes the second-order edge behaviour of the current itself. Moreover, the property

$$\lim_{x_i\to a_i}\frac{J_{iz}(x_i)}{(a_i-x_i)^{\alpha_i}} = \lim_{x_{i+1}\to -a_{i+1}}\frac{J_{i+1z}(x_{i+1})}{(a_{i+1}+x_{i+1})^{\beta_{i+1}}}, \tag{24}$$

which can be deduced from Meixner's theory, has been imposed.

### 3.2. TE Incidence

For TE incidence, a suitable expansion series for the transverse current on the $i$-th side is

$$J_{ix_i}(x_i) = J_{-1}^i\bar{\chi}^i\left(\frac{x_h}{a_h},\frac{x_i}{a_i}\right) + J_{-1}^j\bar{\chi}^j\left(\frac{x_i}{a_i},\frac{x_j}{a_j}\right) + \sum_{n=0}^{+\infty}J_n^i\varphi_n^{\left(t_i^+ +1,t_i^- +1\right)}\left(\frac{x_i}{a_i}\right), \tag{25}$$

where $h$, $i$ and $j$ are three consecutive sides,

$$\bar{\chi}^i\left(\frac{x_h}{a_h},\frac{x_i}{a_i}\right) = \begin{cases} \dfrac{\bar{\varphi}_{-1}^{\left(t_h^+ +1,t_h^- +1\right)}(x_h/a_h)}{\sqrt{\left[\bar{\xi}_{-1}^{\left(t_h^+ +1,t_h^- +1\right)}\right]^2+\left[\bar{\xi}_{-1}^{\left(t_i^- +1,t_i^+ +1\right)}\right]^2}} & y_h = 0 \\ \dfrac{\bar{\varphi}_{-1}^{\left(t_i^- +1,t_i^+ +1\right)}(-x_i/a_i)}{\sqrt{\left[\bar{\xi}_{-1}^{\left(t_h^+ +1,t_h^- +1\right)}\right]^2+\left[\bar{\xi}_{-1}^{\left(t_i^- +1,t_i^+ +1\right)}\right]^2}} & y_i = 0 \end{cases}, \tag{26a}$$

$$\bar{\varphi}_{-1}^{(\alpha,\beta)}\left(\frac{x}{a}\right) = \frac{B_{(1+x/a)/2}(\beta,\alpha)}{B(\beta,\alpha)}\Pi\left(\frac{x}{a}\right), \tag{26b}$$

and

$$\bar{\xi}_{-1}^{(\alpha,\beta)} = \sqrt{\int_{-a}^{a}\left(1+\frac{x}{a}\right)^{-\beta}\left[\varphi_{-1}^{(\alpha,\beta)}\left(\frac{x}{a}\right)\right]^2 dx} = \sqrt{\frac{a2^{1-\beta}}{1-\beta}\left[1-\frac{2}{\alpha}\frac{B(\beta,2\alpha)}{B(\beta,\alpha)^2}\right]}, \tag{27}$$

is a suitable normalization quantity, while $B_z(\cdot,\cdot)$ and $B(\cdot,\cdot)$ are the incomplete and complete Beta functions, respectively [57].

It is worth noting that the continuity of the transverse current across the wedges is imposed.

### 3.3. Galerkin Method

The Fourier transform of the expansion functions in Eqs. (22c) and (26b) can be expressed in closed-form in terms of confluent hypergeometric functions of first kind ${}_1F_1(\cdot\,;\,\cdot\,;\,\cdot)$ [57], i.e.,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{\varphi}_{-1}^{(\alpha,\beta)}\left(\frac{x}{a}\right)e^{jux}dx = \frac{e^{jua}-e^{-jua}{}_1F_1\left(\beta;\alpha+\beta;j2ua\right)}{2j\pi u}, \tag{28a}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi_{n}^{(\alpha,\beta)}\left(\frac{x}{a}\right)e^{jux}dx = \frac{a2^{\alpha+\beta}B\left(n+\alpha+1,n+\beta+1\right)}{\pi n!\xi_n^{(\alpha,\beta)}}\left(2jua\right)^n$$
$$e^{-jua}{}_1F_1\left(n+\beta+1;2n+\alpha+\beta+2;2jua\right). \tag{28b}$$

Galerkin method leads to a matrix equation whose coefficients are double integrals which can be reduced to single integrals. Indeed, by means of reciprocity, it is simply to individuate a representation of the matrix coefficients such that the convolution integrals can be always interpreted as the Fourier transform in the complex plane of the expansion functions, i.e., they can be reduced to algebraic products [38]. Hence, all the elements of the scattering matrix can be rewritten as single improper integrals involving products of confluent hypergeometric functions of first kind. The integrands of such kind of integrals are oscillating functions with a slow asymptotic decay in the worst cases. The simple technique used to speed up the convergence of such kind of integrals consists in the extraction of the $q$th order asymptotic contribution from the integrands with $q = 0, 1, \ldots, Q$, where the choice of $Q$ depends on the case at hand, so that the integrals of the extracted contributions can be expressed in closed form [38].

## 4. NUMERICAL RESULTS

An approximate solution for the problem at hand is obtained by truncating and inverting the coefficients' matrix. To do this, for a given number of expansion functions used, the parameters $0 < \alpha,\ \beta < 1$ in Equations (12) and (20) are suitably chosen in order to minimize the condition number of the truncated coefficients' matrix.

The aim of this section is to show the fast convergence of the presented method, i.e., few expansion functions are enough to achieve highly accurate solutions, even at the resonant frequencies of suitable interior problems. All the simulations are performed on a laptop equipped with an Intel Core 2 Duo CPU T9600 2.8 GHz, 3 GB RAM, running Windows XP and the integrals evaluated by means of an adaptive Gaussian quadrature routine. The obtained numerical results are validated by means of comparisons with the commercial software CST Microwave Studio (CST-MWS). Moreover, comparisons in terms of convergence rate with EFIE and MFIE formulations discretized by means of the same technique are preformed in order to further appreciate the effectiveness of the presented method. To this purpose, the following normalized truncation error is introduced

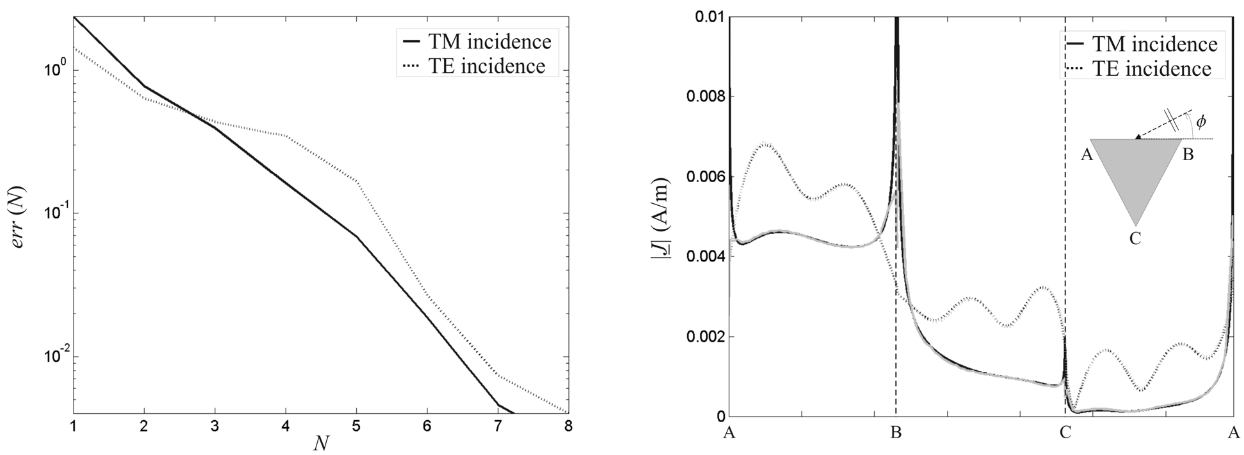
$$\mathrm{err}(N) = \frac{\|\mathbf{J}_{N+1}-\mathbf{J}_N\|}{\|\mathbf{J}_N\|}, \tag{29}$$

where $\|\cdot\|$ is the usual euclidean norm and $\mathbf{J}_M$ the vector of all the expansion coefficients of the currents on all the sides evaluated with $M$ terms on each side.

In the first example, the scattering from an equilateral triangular cross-section cylinder of side $a$ when a TM/TE polarized plane wave impinges with $\phi = \pi/3$ and $|E_0| = \sqrt{\mu/\varepsilon}|H_0| = 1\,\mathrm{V/m}$ is analyzed. In Table 1, the number of expansion functions considered on each side of the cylinder in order to achieve a normalized truncation error less than $10^{-2}$ is reported for EFIE, MFIE and CFIE formulations, when $ka$ approaches the internal resonant value $4\pi/\sqrt{3}$ associated to the $\mathrm{TM}_{10}$ and $\mathrm{TE}_{10}$ modes [58]. As can be seen, for values of $ka$ far enough from the internal resonant one, all the formulations have the same (fast) convergence. However, CFIE formulation preserves the same convergence rate for all the examined cases and even for $ka = 4\pi/\sqrt{3}$, while the number of expansion functions to be used for EFIE and MFIE formulations increases more and more when $ka$ tends to $4\pi/\sqrt{3}$. In Figure 2, for

**Table 1.** Number of expansion functions on each side of an equilateral triangular cross-section cylinder of side $a$ needed to achieve a normalized truncation error less than $10^{-2}$.

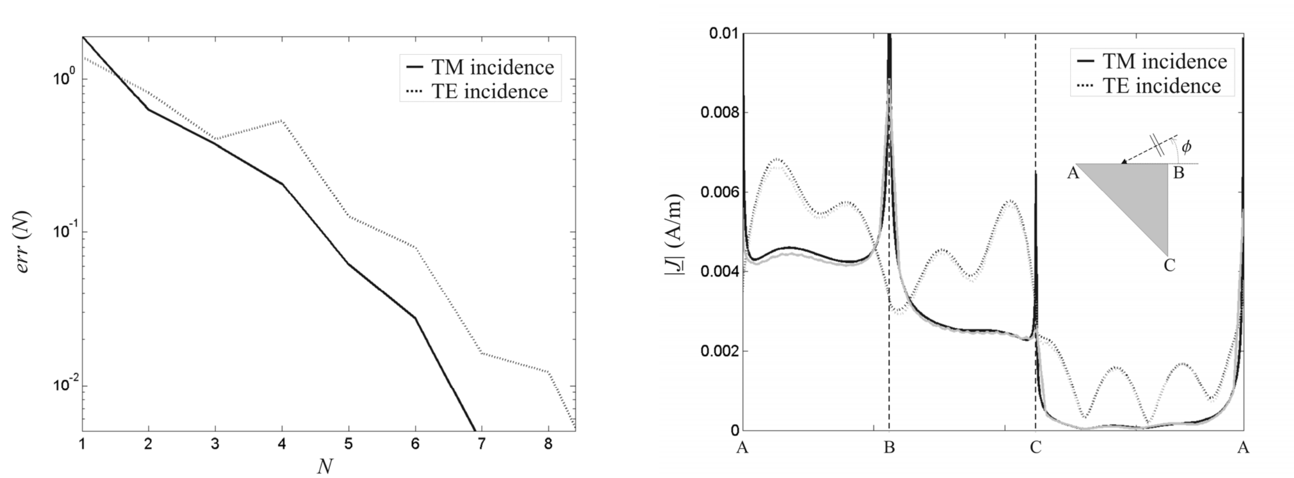
| | TM incidence | | | TE incidence | | |
|---|---|---|---|---|---|---|
| $ka$ | EFIE | MFIE | CFIE | EFIE | MFIE | CFIE |
| 7.3 | 7 | 7 | 7 | 7 | 7 | 7 |
| 7.26 | 7 | 10 | 7 | 8 | 7 | 7 |
| 7.255 | 8 | 18 | 7 | 10 | 7 | 7 |
| 7.2552 | 10 | 36 | 7 | 18 | 10 | 7 |
| 7.25520 | 10 | $---$ | 7 | 18 | 10 | 7 |
| 7.255197 | 12 | $---$ | 7 | 26 | 12 | 7 |
| $4\pi/\sqrt{3}$ | $---$ | $---$ | 7 | $---$ | $---$ | 7 |




(a) (b)

**Figure 2.** Scattering from an equilateral triangular cross-section cylinder of side $a$: (a) normalized truncation error and (b) surface current density (black lines: this method, gray lines: CST-MWS). $\overline{AB} = \overline{BC} = \overline{CA} = a$, $ka = 4\pi/\sqrt{3}$, $\phi = \pi/3$ and $|E_0| = \sqrt{\mu/\varepsilon}|H_0| = 1\,\mathrm{V/m}$.

both the polarizations of the impinging plane wave and for $ka = 4\pi/\sqrt{3}$, the normalized truncation error with varying the number of expansion functions used on each side, revealing a convergence of exponential type, and the surface current density obtained by using 7 expansion functions on each side for both the polarizations with a calculation time of at most 90 secs, agreeing very well with the one reconstructed by means of CST-MWS, are plotted. In the second example, the scattering from an isosceles right triangular cross-section cylinder of hypotenuse $a\sqrt{2}$ when a TM/TE polarized plane wave impinges with $\phi = \pi/3$ and $|E_0| = \sqrt{\mu/\varepsilon}|H_0| = 1\,\mathrm{V/m}$ is analyzed. In Table 2, the number of expansion functions considered on each side of the cylinder in order to achieve a normalized truncation error less than $10^{-2}$ is reported for EFIE, MFIE and CFIE formulations, when $ka$ approaches the internal resonant value $\pi\sqrt{5}$ associated with the $TM_{12}$, $TM_{21}$, $TE_{12}$ and $TE_{21}$ modes [58]. Even in such a case, all the formulations converge quickly when $ka$ is far enough from the internal resonant values. Moreover, the convergence rate of CFIE formulation is the same for all the examined cases and even for $ka = \pi\sqrt{5}$. On the other hand, the number of expansion functions used for EFIE and MFIE formulations rapidly increases when $ka$ approaches $\pi\sqrt{5}$. As for the previous example examined, in Figure 3, the normalized truncation error with varying the number of expansion functions used on each side and the surface current density (obtained by using 7 and 9 expansion functions on each side for TM polarization and TE polarization, respectively, with a calculation time of at most 130 secs) are plotted




(a) (b)

**Figure 3.** Scattering from an isosceles right triangular cross-section cylinder of hypotenuse $a\sqrt{2}$: (a) normalized truncation error and (b) surface current density (black lines: this method, gray lines: CST-MWS). $\overline{AB} = \overline{BC} = \overline{CA}/\sqrt{2} = a$, $ka = \pi\sqrt{5}$, $\phi = \pi/3$ and $|E_0| = \sqrt{\mu/\varepsilon}|H_0| = 1\,\mathrm{V/m}$.

**Table 2.** Number of expansion functions on each side of an isosceles right triangular cross-section cylinder of hypotenuse $a\sqrt{2}$ needed to achieve a normalized truncation error less than $10^{-2}$.

| | TM incidence | | | TE incidence | | |
|---|---|---|---|---|---|---|
| $ka$ | EFIE | MFIE | CFIE | EFIE | MFIE | CFIE |
| 7.0 | 7 | 7 | 7 | 9 | 9 | 9 |
| 7.02 | 7 | 9 | 7 | 9 | 9 | 9 |
| 7.025 | 8 | 17 | 7 | 11 | 9 | 9 |
| 7.0248 | 10 | 33 | 7 | 15 | 10 | 9 |
| 7.02481 | 10 | $---$ | 7 | 17 | 11 | 9 |
| 7.024815 | 13 | $---$ | 7 | 27 | 17 | 9 |
| $\pi\sqrt{5}$ | $---$ | $---$ | 7 | $---$ | $---$ | 9 |

**Table 3.** Number of expansion functions on each side of a rectangular cross-section cylinder of sides $a$ and $a/2$ needed to achieve a normalized truncation error less than $10^{-2}$.

| | TM incidence | | | TE incidence | | |
|---|---|---|---|---|---|---|
| $ka$ | EFIE | MFIE | CFIE | EFIE | MFIE | CFIE |
| 7.0 | 6 | 6 | 6 | 7 | 7 | 7 |
| 7.02 | 6 | 7 | 6 | 7 | 7 | 7 |
| 7.025 | 6 | 13 | 6 | 7 | 8 | 7 |
| 7.0248 | 10 | 21 | 6 | 9 | 10 | 7 |
| 7.02481 | 10 | 27 | 6 | 11 | 12 | 7 |
| 7.024815 | 14 | $---$ | 6 | 15 | 18 | 7 |
| $\pi\sqrt{5}$ | $---$ | $---$ | 6 | $---$ | $---$ | 7 |

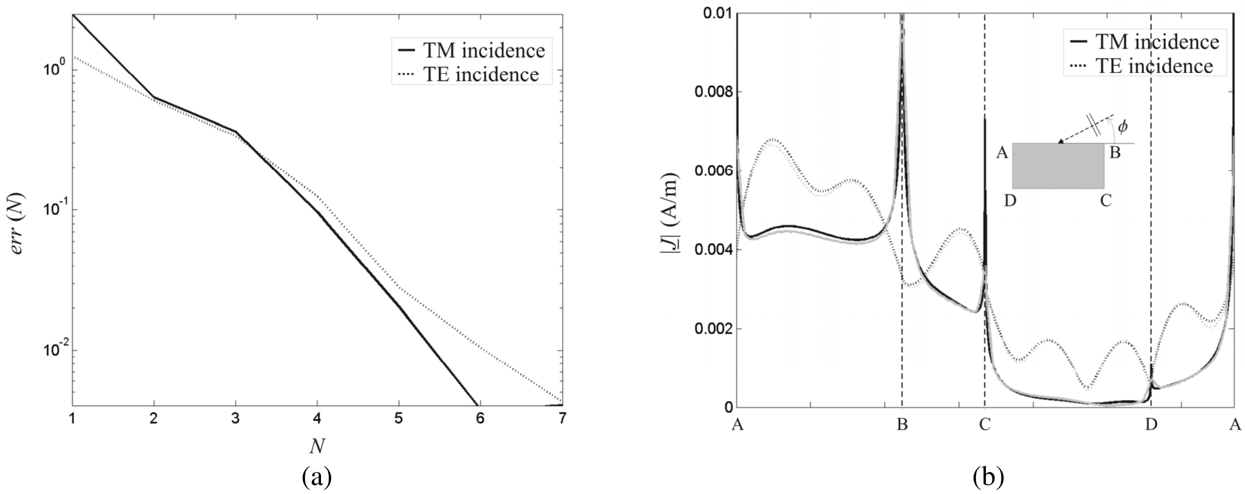



**Figure 4.** Scattering from a rectangular cross-section cylinder of sides $a$ and $a/2$: (a) normalized truncation error and (b) surface current density (black lines: this method, gray lines: CST-MWS). $\overline{AB} = 2\overline{BC} = \overline{CD} = 2\overline{DA} = a$, $ka = \pi\sqrt{5}$, $\phi = \pi/3$ and $|E_0| = \sqrt{\mu/\varepsilon}|H_0| = 1\,\text{V/m}$.

for both the polarizations of the impinging plane wave and for $ka = \pi\sqrt{5}$. Again, the convergence is of exponential type, and the agreement with the results obtained by means of CST-MWS is very good. In the last example, the scattering from a rectangular cross-section cylinder of sides $a$ and $a/2$ when a TM/TE polarized plane wave impinges with $\phi = \pi/3$ and $|E_0| = \sqrt{\mu/\varepsilon}|H_0| = 1\,\text{V/m}$ is analyzed. In Table 3, the number of expansion functions considered on each side of the cylinder in order to achieve a normalized truncation error less than $10^{-2}$ is reported for EFIE, MFIE and CFIE formulations, when $ka$ approaches the internal resonant value $\pi\sqrt{5}$ associated with the $TM_{11}$ and $TE_{11}$ modes. Once again, CFIE formulation preserves the same (fast) convergence rate in all the examined cases and even for $ka = \pi\sqrt{5}$, while the number of expansion functions to be used for EFIE and MFIE formulations increases more and more when $ka$ tends to $\pi\sqrt{5}$. To conclude, in Figure 4, for TM/TE incidence and $ka = \pi\sqrt{5}$, the exponentially convergent normalized truncation error obtained with varying the number of expansion functions used on each side and the surface current density (reconstructed by using 6 and 7 expansion functions on each side for TM polarization and TE polarization, respectively, with a calculation time of at most 125 secs) compared with good agreement with CST-MWS are plotted. It is interesting to note that in order to accurately reconstruct the surface current density for all the considered cases, the transient solver of CST-MWS requires a number of mesh-cells of about 10 millions with a calculation time of about 20 mins.

## 5. CONCLUSIONS

In this paper, a new method for the analysis of the scattering from PEC polygonal cross-section closed cylinders has been presented. As shown in the numerical results section, the presented method is very accurate and efficient even when the frequency approaches the resonance frequency of a suitable interior problem.

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