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Fraunhofer Diffraction by a Strip: Perturbation Method

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Abstract—We investigate the diffraction modeling of a plane wave by an infinitely thin and deformed perfectly conducting strip. We show that the diffraction pattern in the Fraunhofer domain can be obtained from efficiencies calculated for a periodic surface with an interpolation relationship; the elementary pattern of the periodic surface is identical to the strip. We consider the case where the deformation amplitude of the strip is small compared to its width. In this case, the propagation equation written in a curvilinear coordinate system is solved by a perturbation method inspired from quantum physics and extended to imaginary eigenvalues for evanescent waves. In the Fraunhofer approximation domain where the only waves are the propagative waves, the diffraction pattern obtained for a sinusoidal profile strip shows the phenomenon well known as apodization. Classically this phenomenon is obtained for physical optics with a slot in a screen with a variable transparency function similar to the profile function of the strip.

1. INTRODUCTION

The problem of diffraction of an electromagnetic wave by a slot or plane strip has been solved since the nineteenth century in the field of the physical optics where light is represented by a scalar function [1, 2]. In the field of the electromagnetism in which light is represented by a vector function there are many works using different theoretical and numerical approaches [3–8]. Recently Serdyuk gave exact solutions for the diffraction of an electromagnetic wave by a slot and strip following the mode-matching technique [9]. His study is done for a plane strip whose width is smaller than the wave length (electromagnetism domain). In that case (plane strip) the Rayleigh development representing the diffracted wave presents only the zero-order while in our case (deformed strip with the width greater than the wave length) there are several diffraction orders that interfere to give the Fraunhofer field.

The purpose of this study is to show that the diffraction pattern in the Fraunhofer domain obtained for a deformed strip illuminated by a plane wave can be obtained from the efficiencies of the periodic surface associated to the deformed strip with an interpolation relationship; the elementary pattern of the periodic surface is identical to the strip.

In this paper, the developments made are valid both in the case of a small deformation of the strip when the highest of the deformation is smaller than the wavelength of the incident radiation and the case in which the width of the strip is greater than the wavelength. In the former case the appropriate form of the perturbation theory can be applied, while in the latter one can use the semiclassical concept of physical optics.

We give a solution obtained in the electromagnetic optics domain with a perturbation method inspired from quantum physics [10] when the deformation of the strip is small with respect to its width. The originality of this work is the application to the electromagnetism of a perturbation method applied to a non-orthogonal curvilinear coordinates system. The interest of which is to obtain analytical solutions according to an expansion in powers of the normalized amplitude of the strip deformation.

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Section 2 presents the formulation of the problem.

Section 3 considers the simple case of an infinitely thin and perfectly conducting plane strip in vacuum and its complementary surface is obtained by interchanging the conducting surface and the vacuum space, each of them being illuminated by a plane wave. The study of the boundary conditions in these two cases and their summation [1] shows that the discretization for the problem of the diffraction of a plane wave by a plane strip can be reduced to the same discretization as for the periodic surface associated to the strip which is constituted by an infinite succession of identical strips [11].

Furthermore, in the case where the deformation of the strip is small with respect to its width, multiple reflections between patterns of grating will be considered as negligible [12]. The main physical difference with the diffraction by a periodic surface is that the spectrum of the field diffracted by the strip is continuous instead of discrete. In order to obtain the diffraction pattern of a single strip an interpolation has to be done [13].

Section 4 relates to the curvilinear coordinate method that maps the profile of the periodic surface associated to the deformed strip as a plane. This allows to express the propagation equation of fields as an eigenproblem [14–16]. The eigenvalues correspond to the diffracted directions and the eigenvectors give the diffracted fields in these directions.

Section 5 presents the essential steps of the perturbation method [17, 18]. They lead to eigenvalues and eigenvectors according to expansions up to the second order in series of powers of the amplitude of the perturbation.

In Section 6 the amplitudes of propagating and evanescent diffracted fields from boundary conditions written on the perfectly conducting periodic surface associated to the sinusoidal strip are calculated. From these results the diffracted efficiencies are expressed according to the expansions up to the fourth order in series of powers of the amplitude of the perturbation. These efficiencies obtained for peculiar directions are then utilized to plot the Fraunhofer diffraction pattern thanks to the interpolation.

2. FORMULATION OF THE PROBLEM

In Cartesian coordinate system $(Oxyz)$, we consider a cylindrical surface whose generatrix, parallel to the z-axis, is based on the line $y = h_c a(x)$ localised in the (xOy) plane with $y \geq 0$ (Figure 1). The profile function $a(x)$ is assumed to be continuously differentiable. It is is normalized so that the amplitude h_c is the maximal deviation from the y-axis so that $\max a(x) - \min a(x) = 1$. This surface represents the interface between the vacuum and an infinitely thin, perfectly conducting screen. It is illuminated by a monochromatic plane wave of angular frequency ω, wavelength λ and wave number $k = 2\pi/\lambda$. The time-dependent harmonic factor $f(t) = \exp(-i\omega t)$ is omitted everywhere.

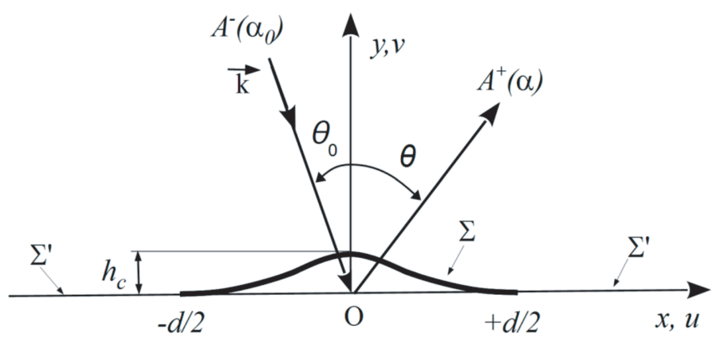

Figure 1. Sinusoidal strip Σ with incident and diffracted waves. The $A^{-}(\alpha_0)$ and $A^{+}(\alpha)$ respectively are the amplitudes of the z-component of incident and diffracted fields.

Our study deals with a finite surface Σ with width d which is a strip described by the profile function:

$$a(x) = [1 + \cos 2\pi x/d]/2 = \cos^2 \pi x/d \quad \text{with } |x| \leq d/2 \tag{1}$$

In the free space (vacuum) the remaining surface denoted Σ' with $|x| > d/2$ is such as $S = \Sigma \cup \Sigma'$ describes an infinite complete surface of equation $y = 0$ when $h_c = 0$ [2].

The incidence angle θ_0, varying from $-\pi/2$ to $+\pi/2$, is positively counted as shown in Figure 1. The diffraction angles θ are oriented in the opposite trigonometric sense. We wish to calculate the distribution of the diffracted energy as a function of the angle θ when h_c is smaller than the wide d of the Σ strip.

For 2D problems the propagation equation of the electromagnetic fields leads to taking two types of solution depending on whether the electric or magnetic field is parallel to the z-axis. The respective z-component can be represented in terms of one scalar function $F(x, y)$. For vacuum (specific conductivity equal to zero), Maxwell's equations [2] give:

- *TE* polarization

$$E_z = F(x, y) \quad H_x = -(i/kZ_0)(\partial F(x, y)/\partial y) \quad H_y = (i/kZ_0)(\partial F(x, y)/\partial x) \tag{2}$$

- *TM* polarization

$$H_z = F(x, y) \quad E_x = (iZ_0/k)(\partial F(x, y)/\partial y) \quad E_y = -(iZ_0/k)(\partial F(x, y)/\partial x) \tag{3}$$

The remaining components are equal to zero and Z_0 is the vacuum impedance.

In the vacuum, $F(x, y)$ is a solution of the following Helmholtz's equation deduced from Maxwell's equations:

$$\left[(\partial/\partial x)^2 + (\partial/\partial y)^2 + k^2\right] F(x, y) = 0 \tag{4}$$

The propagation Equation (4) is an eigenvalue equation of the form:

$$\left[(-i\partial/\partial x)^2 + (-i\partial/\partial y)^2\right] F(x, y) = k^2 F(x, y) \tag{5}$$

where $F(x, y) = f(x)g(y)$ according to the variables separation method.

According to the wave-particle duality and corresponding relations introduced in quantum physics, we associate to the wave vector $\mathbf{k}(k_x, k_y)$ of the (xOy) plane an operator

$$\hat{\mathbf{k}}(\hat{k_x} = -i\partial/\partial x, \hat{k_y} = -i\partial/\partial y) \tag{6}$$

Then $F(x, y)$ is the state function of the photon associated to the electromagnetic wave.

Let $k\alpha$ and $k\beta$ be the respective values of components k_x and k_y and the respective eigenvalues of $\hat{k_x}$ and $\hat{k_y}$ with real values (fundamental rule in quantum physics for propagating waves) $f(x)$ and $g(y)$ being the corresponding eigenvectors.

Equation (5) implies:

$$\alpha^2 + \beta^2 = 1 \tag{7}$$

The general solution $F(x, y)$ of the differential Equation (4) of the second order is a linear combination of an incoming wave $F^-(x, y) = A^- \exp(ik\alpha x)\exp(-ik\beta y)$ and an outgoing wave $F^+(x, y) = A^+ \exp(ik\alpha x)\exp(ik\beta y)$ propagating in the direction θ so that: $\sin\theta = \alpha$. The constantes A^- and A^+ are the amplitudes of associated fields. Then:

$$F(x, y) = F^-(x, y) + F^+(x, y). \tag{8}$$

Taking into account propagation Equation (5) leads to an eigenvalue problem similar to these introduced in quantum physics. We will therefore use the quantum formalism in this paper.

3. BOUNDARY CONDITIONS: DIFFRACTION OF A PLANE WAVE BY A PLANE STRIP

The purpose of this section is to show that the study of the diffraction of a plane wave by a strip can be deduced from the study of the diffraction of a plane wave by the periodic surface associated to this strip.

Let us consider the surface $S = \Sigma \cup \Sigma'$ of equation $y = 0$ in the (xOz)-plane where Σ is the infinitely thin, perfectly conducting plane strip corresponding to $|x| \leq d/2$ and Σ' the vacuum space corresponding to $|x| > d/2$ (Figure 2). The set is illuminated by a plane wave with the incidence θ_0 angle.

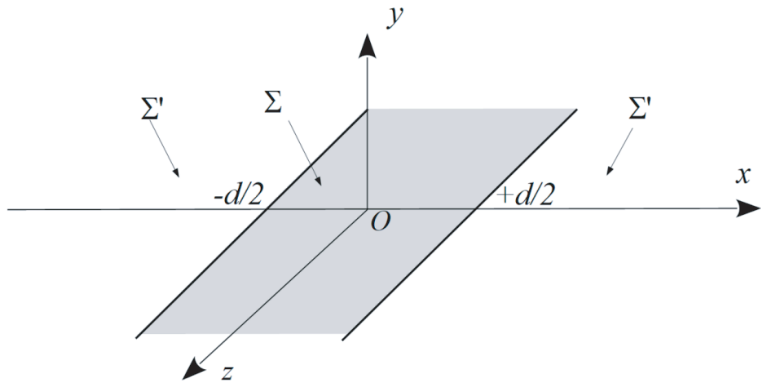

Figure 2. Plane surface $S = \Sigma \cup \Sigma'$.

3.1. Boundary Conditions: TE Polarization

The incident plane wave is represented by the E_z-component of electric field $F^-(x, y)) = A_0^- \exp(ik\alpha_0 x)\exp(-ik\beta_0 y)$. We will consider as diffracted field an additional one $F^+(x, y) = A^+ \exp(ik\alpha x)\exp(ik\beta y)$ caused by the presence of $S = \Sigma \cup \Sigma'$ surface.

The boundary conditions for the function $F(x, y)$ are derived from the known boundary conditions for the electrical and magnetic field vectors **E** and **H**: on a conducting surface, the tangential component of the vector **E** should vanish, and outside the conductor points, the continuity condition should be enforced on the tangential components of **E** and **H**. From here and from Eq. (2) one can obtain the following conditions for the function $F(x, y)$ and for its normal derivative on both sides of the $y = 0$ plane:

$$[F(x,y)]_{y=+0} = [F(x,y)]_{y=-0} \qquad \text{for any } x \tag{9}$$

$$[F(x,y)]_{y=0} = 0 \qquad \text{for } |x| \le d/2 \tag{10}$$

$$[\partial F(x,y)/\partial y]_{y=+0} = [\partial F(x,y)/\partial y]_{y=-0} \qquad \text{for } |x| > d/2 \tag{11}$$

Here, ± 0 denotes an infinitesimal positive (negative) value. The electric field is parallel to the strip boundaries, while the magnetic field is orthogonal to them, and it can have singularities on these boundaries [1, 2, 9]. That is why the points $x = \pm d/2$ are included in the boundary condition in Eq. (10) and are ignored by the condition in Eq. (11).

The condition (10) written for $y = 0$ and $x = \pm d/2$ where $F(x, y) = F^+(x, y) + F^-(x, y)$ gives $\exp[ik(\alpha - \alpha_0)d] = 1$, which leads to the quantization of diffracted directions α so that $\alpha = \sin\theta$:

$$\alpha = \alpha_m = \alpha_0 + m\lambda/d \quad \text{with} \quad m \in U \quad \text{let} \quad m = 0, \pm 1, \pm 2, \ldots, \pm N \tag{12}$$

N being the truncation order adapted to the desired accuracy for the numerical results. In this case the corresponding space of base functions U is $2N + 1$ dimensional.

Let us compare these boundary conditions to those that would be obtained for the complementary plane surface $S_c = \Sigma_c \cup \Sigma'_c$ obtained by interchanging the conducting surface and the vacuum space (Figure 3); the difference lies in the fact that the field incident on the complementary surface is no longer the same as that incident on the original surface, but is derived from it by the transformation $\mathbf{E} \to \mathbf{H}$ according to the electromagnetic form of Babinet's principle [2].

In this case and from Eq. (3) one can obtain the following boundary conditions for the function $F(x, y)$ which represents the H_z-component of the magnetic field and for its normal derivative on both sides of the $y = 0$ plane of surface S_c:

$$[\partial F(x,y)/\partial y]_{y=+0} = [\partial F(x,y)/\partial y]_{y=-0} \qquad \text{for any } x \tag{13}$$

$$[\partial F(x,y)/\partial y]_{y=0} = 0 \qquad \text{for } |x| > d/2 \tag{14}$$

$$[F(x,y)]_{y=+0} = [F(x,y)]_{y=-0} \qquad \text{for } |x| \le d/2 \tag{15}$$

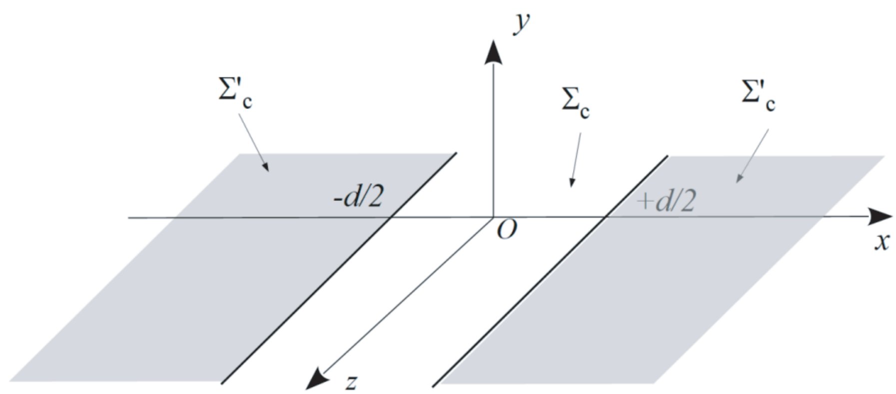

Figure 3. Complementary plane surface $S_c = \Sigma_c \cup \Sigma_c'$.

The electric field having two components E_x and E_y in Eq. (3) is not parallel to the slot boundaries. Hence, at these boundaries it cannot be equal to zero.

Therefore, the boundary points $x = \pm d/2$ are excluded from the domain where the boundary condition in Eq. (14) should be valid. On the contrary, one should take into consideration these points for the condition in Eq. (15). The difference of the values $F(x, y) = H_z$ on the both sides of the plane $y = 0$, actually determines the x-component of the current flow. Normally at the boundary of the conducting surface and directly at the boundaries $x = \pm d/2$, this current component should vanish [7–9]. Therefore, it follows from the law of the conservation of energy that the transmission of H_z-component must be total [7].

Then, in the case of complementary surface $S_c = \Sigma_c \cup \Sigma_c'$ the condition in Eq. (15) written for $y = \pm 0$ and $x = \pm d/2$ in the next form $F^-(x, y = +0) = F^+(x, y = -0)$ (transmission of H_z-component) also leads to quantization Equation (12).

3.2. Boundary Conditions: TM Polarization

In this case $F(x, y)$ represents the H_z-component of the incident wave on the $S = \Sigma \cup \Sigma'$ surface (Figure 2). From Eq. (3) one can obtain the following boundary conditions for the function $F(x, y)$ and for its normal derivative on both sides of the $y = 0$ plane.

$$[\partial F(x, y)/\partial y]_{y=+0} = [\partial F(x, y)/\partial y]_{y=-0} \qquad \text{for any } x \tag{16}$$

$$[\partial F(x, y)/\partial y]_{y=0} = 0 \qquad \text{for } |x| < d/2 \tag{17}$$

$$[F(x, y)]_{y=+0} = [F(x, y)]_{y=-0} \qquad \text{for } |x| \geq d/2 \tag{18}$$

As for Eq. (15) the condition in Eq. (18) written for $y = \pm 0$ and $x = \pm d/2$ in the form $F^-(x, y = +0) = F^+(x, y = -0)$ (transmission of H_z-component) also leads to the quantization Equation (12).

Let us compare these boundary conditions to those that would be obtained for the complementary surface $S_c = \Sigma_c \cup \Sigma_c'$ obtained by interchanging the conducting surface and the free space (Figure 3); the difference lies in the fact that the field incident on the complementary surface is no longer the same as that incident on the original surface, but is derived from it by the change $\mathbf{H} \to -\mathbf{E}$ according to the electromagnetic form of Babinet's principle [2]:

$$[F(x, y)]_{y=+0} = [F(x, y)]_{y=-0} \qquad \text{for any } x \tag{19}$$

$$[F(x, y)]_{y=0} = 0 \qquad \text{for } |x| \geq d/2 \tag{20}$$

$$[\partial F(x, y)/\partial y]_{y=+0} = [\partial F(x, y)/\partial y]_{y=-0} \qquad \text{for } |x| < d/2 \tag{21}$$

In the case of the complementary surface $S_c = \Sigma_c \cup \Sigma_c'$ the condition in Eq.(20) written for $y = 0$ and $x = \pm d/2$ where $F(x, y) = F^+(x, y) + F^-(x, y)$ also leads to quantization Equation (12).

3.3. Discretization of the Problem

Taking into account the summation method [1] in agreement with Maxwell theory, let us consider a surface S_0 that is the superposition of the surface $S = \Sigma \cup \Sigma'$ and the complementary surface $S_c = \Sigma_c \cup \Sigma'_c$. Thus surface S_0 represents an infinitely fine perfectly conducting, infinity wide plane surface illuminated by the plane wave satisfying the boundary conditions:

- the continuity of the tangential components of $\mathbf{E}$ for $y = \pm 0$ and any x (from Eqs. (9) and (13) on the one hand and from Eqs. (16) and (19) on the other hand),
- the tangential component of $\mathbf{E}$ equal to zero for $y = 0$ and any x (from Eqs. (10) and (14) on the one hand and from Eqs. (17) and (20) on the other hand),
- the continuity of the tangential components of $\mathbf{H}$ for $y = \pm 0$ and any x (from Eqs. (11) and (15) on the one hand and from Eqs. (18) and (21) on the other hand).

Here, the infinite perfectly conducting plane surface S_0 ($y = 0$ and any x) appears as a continuous reflecting perfectly plane. The field $F^+(x, y)$ is a reflected wave corresponding to $\alpha = \alpha_0$ with $A^+ = -A_0^-$ in TE polarisation and $A^+ = A_0^-$ in TM polarisation.

The continuity of tangential components of fields on the infinite plane surface S_0 defined by $y = 0$ and any x enables us to write $F^{\pm}(x, y = 0) = F^{\pm}(x + d, y = 0)$ where d arbitrary translation, can to represent the wide of the plane strip or the period of the infinite plane surface. In these conditions:

$$F^+(x+d, y=0) + F^-(x+d, y=0) = F^+(x, y=0) + F^-(x, y=0) \quad \text{for any } x \tag{22}$$

$$\text{with: } F^+(x, y=0) + F^-(x, y=0) = 0 \quad \text{for any } x \tag{23}$$

then one obtains the same quantization equation as Equation (12) for each of the polarisations TE and TM. This surface S_0 can also be considered as constituted by an infinite succession of conducting plane surface Σ identical to the plane strip.

The quantization relation (12) corresponds to the fact that a function which can be integrated $F^+(x, y)$ defined on a bounded interval $x \in [a, b]$ and $y = 0$ can be represented by a Fourier series similar to that of a periodic function of period $d = b - a$ defined over an infinite space [11]. Then:

$$F^+(x, y=0) = \sum\nolimits_m A_m^+ \exp(ik\alpha_m x) \tag{24}$$

This expansion used for the first time by Rayleigh will be called the Rayleigh expansion [12].

It follows from this result that for a given incident plane wave the amplitudes A_m^+ of the fields diffracted by the plane strip are the same as the amplitudes of the fields diffracted by the associated periodic plane surface. Furthermore, in the case where the deformation of the strip is small with respect to its width, the multiple reflections between the patterns of the associated periodic surface are negligible.

In the case of the plane strip ($h_c = 0$) and of the associated periodic plane surface S_0 the amplitudes A_m^+ are equal to zero for every m except for $m = 0$. But, if $h_c \neq 0$ then the amplitudes A_m^+ are different of zero for every m and the new corresponding surface shall called periodic surface in the next of the paper.

Then for $y \geq 0$ the field diffracted by the associated periodic surface has the form of a Rayleigh expansion that consists of a sum of propagative and evanescent waves:

$$F^+(x, y) = \sum\nolimits_m A_m^+ \exp(ik\alpha_m x) \exp(ik\beta_m y) \tag{25}$$

If $|\alpha_m| \leq 1$ ($m \in U^p$), then $\beta_m = (1 - \alpha_m^2)^{1/2} = \cos\theta_m$ is real, and the coefficients $A_m^+ = A^+(\alpha_m)$ correspond to the amplitudes of the diffracted propagative waves with angle θ_m so that $\alpha_m = \sin\theta_m$.

If $|\alpha_m| > 1$ ($m \in U^{ev}$), then $\beta_m = i(\alpha_m^2 - 1)^{1/2}$ is imaginary, and A_m^+ correspond to the amplitudes of the evanescent waves [12]. We note $U^p \cup U^{ev} = U$ given in Eq. (12).

The diffracted field by the strip ($h_c = 0$, width d) characterized by a reflectance having the transmittance distribution $\mathrm{rect}(x/d)$ in the plane $y = 0$ will be deduced from the diffracted field by the associated periodic surface $F^+(x, y = 0)$ by the next relation:

$$F_d^+(x, y=0) = \mathrm{rect}(x/d) F^+(x, y=0) \tag{26}$$

with:

$$\text{rect}(x/d) = \begin{cases} 1 & \text{if } |x| \leq d/2 \\ 0 & \text{elsewhere} \end{cases} \tag{27}$$

In the Fraunhofer approximation [13] the determination of the diffraction pattern in the far field is reduced to a Fourier transform of the diffracted field distribution immediately behind the diffracting structure. It is a decomposition of the light field $F_d^+(x, y)$ into plane waves propagating at angles α according to the spectral function $A_d^+(\alpha)$.

Then the field diffracted by the strip $F_d^+(x, y = 0)$ of which the spectral amplitude is $A_d^+(\alpha)$ in directions $\alpha = \sin\theta$ is written according to the next Fourier integral:

$$F_d^+(x, y = 0) = \int_{-\infty}^{+\infty} A_d^+(\alpha) \exp(ik\alpha x)\, d\alpha \tag{28}$$

The spectral amplitude $A_d^+(\alpha)$ is given by the inverse Fourier transform:

$$A_d^+(\alpha) = (1/d) \int_{-\infty}^{+\infty} F_d^+(x, y = 0) \exp(-ik\alpha x)\, dx \tag{29}$$

$$A_d^+(\alpha) = (1/d) \int_{-d/2}^{+d/2} \sum_m A_m^+ \exp(ik\alpha_m x) \exp(-ik\alpha x) dx \tag{30}$$

$$A_d^+(\alpha) = \sum\nolimits_m A_m^+ \sin[\pi(\alpha - \alpha_m)d/\lambda]/\pi(\alpha - \alpha_m)d/\lambda \tag{31}$$

$$A_d^+(\alpha) = \sum\nolimits_m A_m^+ \text{sinc}[(\alpha - \alpha_m)d/\lambda] \tag{32}$$

with the definition of the normalised sine cardinal:

$$\text{sinc}X = \sin\pi X/\pi X \tag{33}$$

This relation (32) represents the interpolation relation that gives the amplitudes $A_d^+(\alpha)$ of the waves diffracted by the deformed strip calculated from the diffracted waves amplitudes $A_m^+ = A^+(\alpha_m)$ obtained in directions α_m for the infinite periodic surface (period d) repeating the structure of the deformed strip in each of its period. In the Fraunhofer approximation domain corresponding to $|\alpha| \leq 1$ the amplitude $A_d^+(\alpha)$ of the field diffracted by the strip is obtained only by the superposition of the plane waves [1, 13] for each $m \in U^p$ order:

$$A_d^+(\alpha) = \sum\nolimits_{m \in U^p} A_m^+ \text{sinc}[(\alpha - \alpha_m)d/\lambda] \tag{34}$$

We underline that for $-N \leq m \leq +N$, the set of the functions $\exp(ik\alpha_m x)$ represents the orthonormal Fourier's basis of $2N + 1$ dimension denoted $\{|\exp(ik\alpha_m x) >\}$ of the subspace $€_m$ of the Hilbert functional space €. In $€_m$ the scalar product is defined as the "bracket":

$$\langle \exp(ik\alpha_m x) \,|\, \exp(ik\alpha_n x) \rangle = (1/d) \int_{-d/2}^{+d/2} (\exp(ik\alpha_m x))^\dagger \exp(ik\alpha_n x)\, dx = \delta_{m,n} \tag{35}$$

with:

$$\delta_{m,n} = \left| \begin{array}{ll} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{array} \right. \quad \text{the Kronecker symbol} \tag{36}$$

Then the calculation of the amplitude $A_d^+(\alpha)$ given in relation (34) requires the calculation of the amplitudes $A_m^+ = A^+(\alpha_m)$ of the diffracted fields by the periodic surface associated to the strip. It is the object of next sections where we give the essential steps of the calculation largely developed in [17, 18] in the case of a strip weakly deformed with respect to its width.

4. CURVILINEAR COORDINATE METHOD APPLIED TO THE DIFFRACTION BY THE PERIODIC SURFACE

4.1. The Propagation Equation in the Curvilinear Coordinate System

In order to simply express the boundary conditions along the interface of the periodic surface, we use the so-called "curvilinear coordinate system" (u, v, w) [14] defined from Cartesian coordinate system as follows:

$$u = x, \quad v = y - h_c a(x), \quad w = z \tag{37}$$

In this case the diffracting surface corresponds to $v = 0$. Note that the z-component of the electric or magnetic field (E_z or H_z) is unchanged in the new coordinate system. We call it $F(u, v)$ and it is considered as an unknown function.

In order to use reduced quantities (taking an interest in numerical calculations) we define:
$h = 2\pi h_c/d$, the normalised amplitude,
$\dot{a} = da'(x)/2\pi$, the normalised derivative of the profile which is a periodic function of d, the period.
Then, propagation Equation (4) becomes:

$$\left\{(\partial/\partial u)^2 - h\left[(\partial/\partial u)\,\dot{a} + \dot{a}(\partial/\partial u)\right](\partial/\partial v) + \left(h^2\dot{a}^2 + 1\right)(\partial/\partial v)^2 + k^2\right\} F(u, v) = 0 \tag{38}$$

It is an equation with constant coefficients according to v and periodic coefficients according to u. Consequently:

$$F(u, v) = \sum_m F_m(v) \exp(ik\alpha_m u) \tag{39}$$

with α_m satisfying Eq. (12) by analogy to the previous case, and:

$$F_m(v) = F_m \exp(ik\rho v) \tag{40}$$

where F_m and ρ are the unknowns of the problem.

4.2. Matrix Form of the Propagation Equation

In the Hilbert subspace $€_m$ to each differential operator $(-i\partial/\partial u)$, we associate a square matrix of $2N+1$ dimension noted $[(-i\partial/\partial u)] = k[\alpha]$ where $[\alpha]$ is a diagonal matrix so that $\alpha_{m,m} = \alpha_m$ as defined in Eq. (12) and $\alpha_{m,n} = 0$ if $m \neq n$.

Furthermore, we write $(-i\partial/\partial v) = k\rho$ where ρ is a numerical factor and we group the components $F_m(v)$ of the function $F(u, v)$ in the $|f\rangle$ vector.

Thus, Equation (38) is written as:

$$\left[-\left([I] - [\alpha]^2\right) - h\left([\alpha][\dot{a}] + [\dot{a}][\alpha]\right)\rho + \left(h^2[\dot{a}]^2 + [I]\right)\rho^2\right]\Big|f\Big\rangle = 0 \tag{41}$$

where $[I]$ is an identity matrix, $[\dot{a}]$ is a Toeplitz matrix formed by the Fourier coefficients $\dot{a}_{m,n}$ of the $\dot{a}(u)$ so that:

$$(\dot{a}_{m,n}) = (\dot{a}_{m-n}) = \frac{1}{d}\int_0^d \dot{a}(u) \exp\left[i(m-n)\frac{2\pi}{d}u\right] du \tag{42}$$

From diagonal matrix $[\alpha]$, we define another diagonal matrix $[\beta]$:

$$[\beta]^2 = [I] - [\alpha]^2 \tag{43}$$

4.3. Conversion of the Propagation Equation into Eigenvalues Equation

In order to solve an eigenvalues equation we introduce an auxiliary function: $F'(v) = \rho F(v)$ [16].

Thus, in the doubled subspace $€_m \oplus €_m$ with the basis B: $\{|\exp(ik\alpha_m u)\rangle, |\exp(ik\alpha_m u)\rangle\}$ we obtain the following system:

$$\begin{bmatrix} -h[\beta]^{-2}\left([\alpha][\dot{a}] + [\dot{a}][\alpha]\right) & [\beta]^{-2}\left(h^2[\dot{a}]^2 + [I]\right) \\ [I] & 0 \end{bmatrix} \begin{bmatrix} |f\rangle \\ |f'\rangle \end{bmatrix} = \rho^{-1} \begin{bmatrix} |f\rangle \\ |f'\rangle \end{bmatrix} \tag{44}$$

which appears as a $2(2N+1)$ dimensional system where ρ^{-1} is the eigenvalue and $[|\varphi\rangle] = \begin{bmatrix} |f\rangle \\ |f'\rangle \end{bmatrix}$ is the eigenvector.

Equation (44) is written in the following form $M|\varphi\rangle = \rho^{-1}|\varphi\rangle$ so that:

$$\left(M^{(0)} + hM^{(1)} + h^2 M^{(2)}\right) |\varphi\rangle = \rho^{-1} |\varphi\rangle \tag{45}$$

with:

$$M^{(0)} = \begin{bmatrix} 0 & [\beta]^{-2} \\ [I] & 0 \end{bmatrix} \tag{46}$$

$$M^{(1)} = \begin{bmatrix} -[\beta]^{-2}[A] & 0 \\ 0 & 0 \end{bmatrix} \tag{47}$$

$$M^{(2)} = \begin{bmatrix} 0 & [\beta]^{-2}[B] \\ 0 & 0 \end{bmatrix} \tag{48}$$

setting:

$$[A] = [\alpha][\dot{a}] + [\dot{a}][\alpha] \tag{49}$$

$$[B] = [\dot{a}]^2 \tag{50}$$

5. THE PERTURBATION METHOD IN THE CURVILINEAR COORDINATE SYSTEM

5.1. Principle of the Perturbation Method

Equation (45) is split into three terms: an unperturbed matrix $M^{(0)}$ and two terms of perturbation $hM^{(1)}$ (first order perturbation) and $h^2M^{(2)}$ (second order perturbation). This small dimensionless parameter h shows the intensity of the perturbation with regard to the plane surface.

The solutions of the perturbed problem are obtained from the known eigensolutions of the unperturbed problem (that is for $h = 0$). In this case the curvilinear coordinate system is the same as the Cartesian coordinate system.

The principle of the method is as follows:

- to calculate exactly the eigenvalues and eigenvectors for the unperturbed problem,
- to verify that the eigenvectors form an orthonormal basis,
- to write the matrix M in the eigenbasis of the unperturbed problem, it will be denoted **M**,
- to determine the corrective terms that must be added to the known eigensolutions of the unperturbed problem in order to obtain the approximate solutions of **M** according to an expansion in powers of h to the second-order.

5.2. Eigensolutions of the Unperturbed Problem

For zero-order perturbation the eigenvalues $(\rho_m)^{-1} = r_m$ given in a matrix form are:

$$\left[r^{(0)\pm}\right] = \pm\left[\beta^{-1}\right] \tag{51}$$

The corresponding basis eigenvectors denoted $|e_m^{\pm}\rangle$ have their components grouped together in the following matrix T:

$$T = \begin{bmatrix} [I] & [I] \\ [\beta] & -[\beta] \end{bmatrix} \tag{52}$$

This is the matrix of B to $B^{\pm}$ basis change. The new normalised $B^{\pm}$ basis: $\{|e_m^+\rangle, |e_m^-\rangle\}$ in the subspace $€_m^+ \oplus €_m^-$ is identical to the initial basis B.

In this $B^{\pm}$ basis: $\{|\exp(ik\alpha_m u)\rangle, |\exp(ik\alpha_m u)\rangle\}$ the matrix of eigenvalues is:

$$\left[r^{(0)}\right] = \begin{bmatrix} [\beta^{-1}] & 0 \\ 0 & -[\beta^{-1}] \end{bmatrix} \tag{53}$$

and the associated eigenvectors matrix is:

$$\left[\left|\psi^{(0)}\right\rangle\right] = \begin{bmatrix} |\psi^{(0)+}\rangle \\ |\psi^{(0)-}\rangle \end{bmatrix} \tag{54}$$

According to $m \in U^p$ or $m \in U^{ev}$ when m vary within the range $(-N, +N)$ the components of $|\psi^{(0)\pm}\rangle$ with their amplitude equal to the unity, representing the new basis functions required to solve the perturbed problem are respectively:

- for $€_m^{\pm}$:

$$\left|\psi_m^{(0)\pm}\right\rangle = \begin{bmatrix} \left|\psi_m^{ev(0)\pm}\right\rangle \\ \left|\psi_m^{p(0)\pm}\right\rangle \\ \left|\psi_m^{ev(0)\pm}\right\rangle \end{bmatrix} = \begin{bmatrix} \exp(\mp k\beta_m v)\,|\exp(ik\alpha_m u)\rangle \\ \exp(\pm ik\beta_m v)\,|\exp(ik\alpha_m u)\rangle \\ \exp(\mp k\beta_m v)\,|\exp(ik\alpha_m u)\rangle \end{bmatrix} \tag{55}$$

- for $€_m^{+} \oplus €_m^{-}$ we set:

$$\left|\psi_m^{(0)}\right\rangle = \begin{bmatrix} \left|\psi_m^{(0)+}\right\rangle \\ \left|\psi_m^{(0)-}\right\rangle \end{bmatrix} \tag{56}$$

These new vectors $\{|\psi_m^{(0)+}\rangle, |\psi_m^{(0)-}\rangle\}$, components of $|\psi_m^{(0)}\rangle$, are estimated with a phase term $\exp(\pm ik\beta_m v)$ for propagative waves and with an amplitude term $\exp(\mp k\beta_m v)$ for evanescent waves which decrease quickly when v increases respectively by positive or negative values. We have shown in [18] that they form an orthogonal basis of $2(2N+1)$ dimension. But for $v = 0$ on the perfectly conducting surface where the boundary conditions are written, they form an orthonormal basis identical to the $B^{\pm}$ basis: $\{|\exp(ik\alpha_m u)\rangle, |\exp(ik\alpha_m u)\rangle\}$.

Therefore for $m, n \in U = U^p \cup U^{ev}$

$$\forall v: \qquad \left\langle \psi_m^{(0)\pm} \,\middle|\, \psi_n^{(0)\pm} \right\rangle = \delta_{m,n} \tag{57}$$

$$\text{and for } v = 0: \qquad \left\langle \psi_m^{(0)\pm} \,\middle|\, \psi_n^{(0)\mp} \right\rangle = \delta_{m,n} \tag{58}$$

5.3. Expression of the Matrix M of the Perturbed Problem in the Eigenbasis $B^{\pm}$ of the Unperturbed Problem

In basis $B^{\pm}$ the matrix M is changed to $\mathbf{M}$ according to the following relation:

$$\mathbf{M} = T^{-1} M T \tag{59}$$

Thus, $\mathbf{M}$ appears in the form:

$$\mathbf{M} = \mathbf{M}^{(0)} + h\mathbf{M}^{(1)} + h^2\mathbf{M}^{(2)} \tag{60}$$

with:

$$\mathbf{M}^{(\mathbf{0})} = \begin{bmatrix} [\beta^{-1}] & 0 \\ 0 & -[\beta^{-1}] \end{bmatrix} \tag{61}$$

$$\mathbf{M}^{(\mathbf{1})} = \begin{bmatrix} [C] & [C] \\ [C] & [C] \end{bmatrix} \tag{62}$$

$$\mathbf{M}^{(\mathbf{2})} = \begin{bmatrix} [D] & -[D] \\ [D] & -[D] \end{bmatrix} \tag{63}$$

setting:

$$[C] = -[\beta^{-2}][A]/2, \tag{64}$$

$$[D] = [\beta^{-2}]\,[B]\,[\beta]\,/2. \tag{65}$$

5.4. Solutions of the Perturbed Problem

The eigenstates of the perturbed problem must satisfy the equation:

$$\mathbf{M}\,|\,\psi_m\rangle = r_m\,|\,\psi_m\rangle \tag{66}$$

with $\mathbf{M}$ given by Eq. (60) and:

$$|\,\psi_m\rangle = \begin{bmatrix} |\,\psi_m^+\rangle \\ |\,\psi_m^-\rangle \end{bmatrix} \quad \text{in subspace} \quad \text{€}_m^+ \oplus \text{€}_m^- \tag{67}$$

The principle of the perturbation theory is to expand eigenvalues r_m and eigenvectors $|\psi_m\rangle$ in series of powers of h keeping only a finite number of terms. Here, we take the expansions up to the second-order in agreement with the writing of the operator $\mathbf{M}$ in Equation (60):

$$r_m = r_m^{(0)} + h r_m^{(1)} + h^2 r_m^{(2)} \tag{68}$$

$$|\psi_m\rangle = \left|\psi_m^{(0)}\right\rangle + h\left|\psi_m^{(1)}\right\rangle + h^2\left|\psi_m^{(2)}\right\rangle \tag{69}$$

with:

$$\left|\,\psi_m^{(1)}\right\rangle = \sum_i a_{mi}^{(1)}\left|\,\psi_i^{(0)}\right\rangle \quad \text{and} \quad \left|\,\psi_m^{(2)}\right\rangle = \sum_i a_{mi}^{(2)}\left|\,\psi_i^{(0)}\right\rangle \tag{70}$$

We consider the case $v = 0$ where the set of vectors $|\psi_m^{(0)}\rangle$ forms an orthonormal basis.

The set of algebraic calculations leads to the results below for the mth perturbed eigenvalue r_m and for the mth perturbed eigenvector $|\psi_m\rangle$, correct to the second-order:

$$r_m = r_m^{(0)} + h\left\langle\psi_m^{(0)}\left|\mathbf{M}^{(1)}\right|\psi_m^{(0)}\right\rangle + h^2\left[\left\langle\psi_m^{(0)}\left|\mathbf{M}^{(2)}\right|\psi_m^{(0)}\right\rangle + \sum_{i\neq m}\left|\left\langle\psi_i^{(0)}\left|\mathbf{M}^{(1)}\right|\psi_m^{(0)}\right\rangle\right|^2 \Big/ \left(r_m^{(0)} - r_i^{(0)}\right)\right] \tag{71}$$

$$\begin{aligned}|\psi_m\,\rangle = &\left|\psi_m^{(0)}\,\right\rangle + h\sum_{i\neq m}\left[\left(\left\langle\psi_i^{(0)}|\mathbf{M}^{(1)}|\psi_m^{(0)}\right\rangle\right) \Big/ \left(r_m^{(0)} - r_i^{(0)}\right)\right]\left|\psi_i^{(0)}\right\rangle \\ &+h^2\Bigg\{\sum_{j\neq m}\sum_{i\neq m}\left[\left(\left\langle\psi_i^{(0)}\left|\mathbf{M}^{(1)}\right|\psi_m^{(0)}\right\rangle\right)\Big/\left(r_m^{(0)} - r_i^{(0)}\right)\right]\left[\left(\left\langle\psi_j^{(0)}\left|\mathbf{M}^{(1)}\right|\psi_i^{(0)}\right\rangle\right)\Big/\left(r_m^{(0)} - r_j^{(0)}\right)\right]\left|\psi_j^{(0)}\right\rangle \\ &-\sum_{j\neq m}\left[\left(\left\langle\psi_j^{(0)}|\mathbf{M}^{(1)}|\psi_m^{(0)}\right\rangle\right)\left(\left\langle\psi_m^{(0)}|\mathbf{M}^{(1)}|\psi_m^{(0)}\right\rangle\right)\Big/\left(r_m^{(0)} - r_j^{(0)}\right)^2\right]\left|\psi_j^{(0)}\right\rangle \\ &-1/2\sum_{i\neq m}\left|\left(\left\langle\psi_i^{(0)}|\mathbf{M}^{(1)}|\psi_m^{(0)}\right\rangle\right)\Big/\left(r_m^{(0)} - r_i^{(0)}\right)\right|^2\left|\psi_i^{(0)}\right\rangle \\ &+\sum_{j\neq m}\left[\left(\left\langle\psi_j^{(0)}|\mathbf{M}^{(2)}|\psi_m^{(0)}\right\rangle\right)\Big/\left(r_m^{(0)} - r_j^{(0)}\right)\right]\left|\psi_j^{(0)}\,\right\rangle\Bigg\}\end{aligned} \tag{72}$$

In matrix notation, Equation (72) is written in the form:

$$[|\,\psi\rangle\,] = \left\{[I] + h\left[a^{(1)}\right] + h^2\left[a^{(2)}\right]\right\}\left[\left|\,\psi^{(0)}\right\rangle\right] \tag{73}$$

5.5. Return to the Initial Eigenvector in Basis B

The initially sought eigenvector $|\varphi\rangle$ in Eq. (45) is obtained from $|\psi\rangle$ in Eq. (73) and from the matrix T in Eq. (52) of B to $B^{\pm}$ basis change: $[|\varphi\rangle] = [T][|\psi\rangle]$.

Let for example

$$\begin{bmatrix} |\,f\rangle \\ \left|\,f'\right\rangle \end{bmatrix} = [T]\begin{bmatrix} |\,\psi^+\rangle \\ |\,\psi^-\rangle \end{bmatrix} \tag{74}$$

The sought eigenvectors, solutions of Equation (45) are written in the matrix form according to an expanding in powers of perturbation parameter h. They are:

$$\begin{bmatrix} |f\rangle \\ |f'\rangle \end{bmatrix} = \left\{ \begin{bmatrix} [F^{(0)+}] & [F^{(0)-}] \\ [F'^{(0)+}] & [F'^{(0)-}] \end{bmatrix} + h \begin{bmatrix} [F^{(1)+}] & [F^{(1)-}] \\ [F'^{(1)+}] & [F'^{(1)-}] \end{bmatrix} + h^2 \begin{bmatrix} [F^{(2)+}] & [F^{(2)-}] \\ [F'^{(2)+}] & [F'^{(2)-}] \end{bmatrix} \right\} \begin{bmatrix} |\psi^{(0)+}\rangle & 0 \\ 0 & |\psi^{(0)-}\rangle \end{bmatrix} \begin{bmatrix} |A^+\rangle \\ |A^-\rangle \end{bmatrix} \tag{75}$$

with: $[F^{(0)\pm}] = [I]$ since $|\psi_m^{(0)+}\rangle$ form an orthonormal basis as well as $|\psi_m^{(0)-}\rangle$ and $[F'^{(0)\pm}] = \pm[\beta][F^{(0)\pm}]$. The matrix $[F^{(1)\pm}]$ and $[F^{(2)\pm}]$ are deduced from Eqs. (73) and (74).

Each of these functions is defined within an amplitude term $A_m^{\pm}$ which are the components of $|A^{\pm}\rangle$. They are calculated in the next section in order to satisfy the boundary conditions.

6. DIFFRACTION PATTERN IN THE FRAUNHOFER DOMAIN FOR THE STRIP

6.1. Calculation of Amplitudes of Diffracted Fields by the Periodic Surface

The boundary conditions on the perfectly conducting surface deal with the $E_t(u, v)$ tangential component of the electric field. It is equal to zero for $v = 0$ and all u.

6.1.1. TE Polarization

In this case

$$E_t(u, v) = \sum_m A_m^+ F_m^+(u, v) + A_m^- F_m^-(u, v) \tag{76}$$

The constants $A_m^{\pm}$ will be adjusted to satisfy the next boundary condition written with matrix notation:

$$[F^+][A^+] + [F^-][A^-] = 0 \tag{77}$$

To simplify, we set the incident wave electric field magnitude equal to unity for each incident and evanescent order:

$$[A^-] = \left[A^{(0)-}\right] = [I] \tag{78}$$

$[A^+]$ is expanded in powers of h:

$$[A^+] = \left[A^{(0)+}\right] + h\left[A^{(1)+}\right] + h^2\left[A^{(2)+}\right] \tag{79}$$

as

$$[F^{\pm}] = \left[F^{(0)\pm}\right] + h\left[F^{(1)\pm}\right] + h^2\left[F^{(2)\pm}\right] \tag{80}$$

given by Eq. (75).

We substitute the three expressions (78), (79) and (80) into (77). Then comparing coefficients in powers of h one gets the coefficients of the $[A^+]$ expansion which are:

$$\left[A^{(0)+}\right] = -\left[F^{(0)+}\right]^{-1}\left(\left[F^{(0)-}\right]\left[A^{(0)-}\right]\right) \tag{81}$$

$$\left[A^{(1)+}\right] = -\left[F^{(0)+}\right]^{-1}\left(\left[F^{(1)-}\right]\left[A^{(0)-}\right] + \left[F^{(1)+}\right]\left[A^{(0)+}\right]\right) \tag{82}$$

$$\left[A^{(2)+}\right] = -\left[F^{(0)+}\right]^{-1}\left(\left[F^{(2)-}\right]\left[A^{(0)-}\right] + \left[F^{(2)+}\right]\left[A^{(0)+}\right] + \left[F^{(1)+}\right]\left[A^{(1)+}\right]\right) \tag{83}$$

6.1.2. TM Polarization

We still must write the continuity of the tangential component $E_t(u,v)$ of the electric field on conducting surface ($v = 0$ and all u)

A classical calculation of $E_t(u,v)$ shows that it is proportional to $G(u,v)$

$$E_t\left(u,v\right) = G\left(u,v\right) Z_0 cos\varphi \tag{84}$$

with:

$$G(u,v) = h\dot{a}\alpha F(u,v) - \left(1 + h^2\dot{a}^2\right) F^{'}(u,v) \tag{85}$$

where Z_0 is the vacuum impedance and φ the angle between the tangent to the profile and the x-axis.

Furthermore in the normalised basis $B^{\pm}$ the functions $F^{\pm}(u,v)$ and $F'^{\pm}(u,v) = \pm\beta F^{\pm}(u,v)$ (($+$) for outgoing waves and ($-$) for incoming waves) correspond to functions $F(u,v)$ and $F'(u,v)$ firstly defined in the basis B.

Then, on the periodic surface we write the continuity of the sum for all mth orders of $G_m^{\pm}(u,v)$ each respectively multiplied by the $A_m^{\pm}$ arbitrary constants. These constants must be adjusted to satisfy the boundary condition $G(u, v = 0)$ for all u. In matrix notation one writes:

$$\left[G^{+}\right]\left[A^{+}\right] + \left[G^{-}\right]\left[A^{-}\right] = 0 \tag{86}$$

with:

$$\left[G^{\pm}\right] = h\left[\dot{a}\right]\left[\alpha\right]\left[F^{\pm}\right] - \left(\left[I\right] + h^2\left[\dot{a}\right]^2\right)\left[F'^{\pm}\right] \tag{87}$$

and:

$$\left[F'^{\pm}\right] = \pm\left[F^{\pm}\right]\left[\beta\right] \tag{88}$$

We apply the same method for $[A^-]$ which is taken equal to unity ($[A^-] = [A^{(0)-}] = [I]$).

Since $[F^{\pm}]$ and $[F'^{\pm}]$ are known to the second order in power of h in Eqs. (75) and (88), $[G^{\pm}]$ and $[A^+]$ will be expanded to the fourth order in powers of h:

$$\left[A^{+}\right] = \left[A^{(0)+}\right] + h\left[A^{(1)+}\right] + h^2\left[A^{(2)+}\right] + h^3\left[A^{(3)+}\right] + h^4\left[A^{(4)+}\right] \tag{89}$$

The coefficients of expansion of $[A^+]$ are calculated to the fourth order in powers of h. They are given in matrix form below:

$$\left[A^{(0)+}\right] = -\left[G^{(0)+}\right]^{-1}\left(\left[G^{(0)-}\right]\left[A^{(0)-}\right]\right) \tag{90}$$

$$\left[A^{(1)+}\right] = -\left[G^{(0)+}\right]^{-1}\left(\left[G^{(1)-}\right]\left[A^{(0)-}\right] + \left[G^{(1)+}\right]\left[A^{(0)+}\right]\right) \tag{91}$$

$$\left[A^{(2)+}\right] = -\left[G^{(0)+}\right]^{-1}\left(\left[G^{(2)-}\right]\left[A^{(0)-}\right] + \left[G^{(2)+}\right]\left[A^{(0)+}\right] + \left[G^{(1)+}\right]\left[A^{(1)+}\right]\right) \tag{92}$$

$$\left[A^{(3)+}\right] = -\left[G^{(0)+}\right]^{-1}\left(\left[G^{(3)-}\right]\left[A^{(0)-}\right] + \left[G^{(3)+}\right]\left[A^{(0)+}\right] + \left[G^{(2)+}\right]\left[A^{(1)+}\right] + \left[G^{(1)+}\right]\left[A^{(2)+}\right]\right) \tag{93}$$

$$\begin{aligned}\left[A^{(4)+}\right] = &-\left[G^{(0)+}\right]^{-1}\left(\left[G^{(4)-}\right]\left[A^{(0)-}\right] + \left[G^{(4)+}\right]\left[A^{(0)+}\right] + \left[G^{(3)+}\right]\left[A^{(1)+}\right]\right.\\ &\left. + \left[G^{(2)+}\right]\left[A^{(2)+}\right] + \left[G^{(1)+}\right]\left[A^{(3)+}\right]\right)\end{aligned} \tag{94}$$

6.2. Diffraction Efficiencies in Each Direction of Diffraction by the Periodic Surface

It is interesting to define an efficiency P equal to the fraction of the incident energy which is transmitted in each eigendirection α_m of the periodic surface. This energy is obtained from the flow of Poynting vector through the unit surface perpendicular to y-axis. Then:

$$P = |E|^2 \rho^+/\beta^- \cong |E|^2 \beta^+/\beta^- \tag{95}$$

limiting the expansion of ρ^+ to the first-order of the perturbation in order to compare our results in the same direction as in the curvilinear coordinate method.

For TE polarization

$$[P_{TE}] = \left[A^{+}\right]\left[A^{+}\right]^{*}\left(\left[\beta^{+}\right]\left[\beta^{-}\right]^{-1}\right) \tag{96}$$

For TM polarization

$$[P_{TM}] = \left[A^{+}\right]\left[A^{+}\right]^{*}\left(\left[\beta^{+}\right]\left[\beta^{-}\right]^{-1}\right) \tag{97}$$

The values of P_{TE} and P_{TM} are calculated according to a fourth-order expansion in powers of h for each of them:

$$P = P^{(0)} + hP^{(1)} + h^2P^{(2)} + h^3P^{(3)} + h^{(4)}P^{(4)} \tag{98}$$

The expansion terms are analytically known.

- For P_{TE}, they are in matrix form:

$$\begin{aligned}[P_{TE}] = &\left\{\left[A^{(0)+}\right]\left[A^{(0)+}\right]^{*} + h\left(\left[A^{(1)+}\right]\left[A^{(0)+}\right]^{*} + \left[A^{(0)+}\right]\left[A^{(1)+}\right]^{*}\right)\right.\\ &+h^2\left(\left[A^{(2)+}\right]\left[A^{(0)+}\right]^{*} + \left[A^{(1)+}\right]\left[A^{(1)+}\right]^{*} + \left[A^{(0)+}\right]\left[A^{(2)+}\right]^{*}\right)\\ &+h^3\left(\left[A^{(2)+}\right]\left[A^{(1)+}\right]^{*} + \left[A^{(1)+}\right]\left[A^{(2)+}\right]^{*}\right)\\ &\left.+h^4\left(\left[A^{(2)+}\right]\left[A^{(2)+}\right]^{*}\right)\right\}\left(\left[\beta^{+}\right]\left[\beta^{-}\right]^{-1}\right)\end{aligned} \tag{99}$$

- For P_{TM}, they are in matrix form:

$$\begin{aligned}[P_{TM}] = &\left\{\left[A^{(0)+}\right]\left[A^{(0)+}\right]^{*} + h\left(\left[A^{(1)+}\right]\left[A^{(0)+}\right]^{*} + \left[A^{(0)+}\right]\left[A^{(1)+}\right]^{*}\right)\right.\\ &+h^2\left(\left[A^{(2)+}\right]\left[A^{(0)+}\right]^{*} + \left[A^{(1)+}\right]\left[A^{(1)+}\right]^{*} + \left[A^{(0)+}\right]\left[A^{(2)+}\right]^{*}\right)\\ &+h^3\left(\left[A^{(3)+}\right]\left[A^{(0)+}\right]^{*} + \left[A^{(2)+}\right]\left[A^{(1)+}\right]^{*} + \left[A^{(1)+}\right]\left[A^{(2)+}\right]^{*} + \left[A^{(0)+}\right]\left[A^{(3)+}\right]^{*}\right)\\ &+h^4\left(\left[A^{(4)+}\right]\left[A^{(0)+}\right]^{*} + \left[A^{(3)+}\right]\left[A^{(1)+}\right]^{*} + \left[A^{(2)+}\right]\left[A^{(2)+}\right]^{*} + \left[A^{(1)+}\right]\left[A^{(3)+}\right]^{*}\right.\\ &\left.\left.+\left[A^{(0)+}\right]\left[A^{(4)+}\right]^{*}\right)\right\}\left(\left[\beta^{+}\right]\left[\beta^{-}\right]^{-1}\right)\end{aligned} \tag{100}$$

These relations (99) and (100) give the efficiencies P_{TE} and P_{TM} for each of the mth directions of diffraction when the periodic surface is illuminated with the nth incident direction.

6.3. Numerical Results: Diffraction Pattern Given by the Strip

In the case of the strip, the calculation of the amplitudes $A_d^{+}(\alpha)$ given in relation (34) requires the calculation of the amplitudes $A_m^{+} = A^{+}(\alpha_m)$ of the diffracted fields by the periodic surface associated to the strip.

Then the perturbation method is applied to the periodic surface constituted by an infinite succession of perfectly conducting sinusoidal strip identical to that given in Figure 1. The values chosen for parameters are:

- incidence angle: $\theta_0 = 1°$ (close to the normal incidence that is a degenerated point given that the perturbation method is developed for undegenerated cases),
- magnitude of grooves: $h_c/\lambda = 0.08$ (chosen in the validity domain of the perturbation method compared to the rigorous numerical method [18]),
- period: $d/\lambda = 3.9$ (electromagnetic optics domain).

This value of d/λ represents also the width of the sinusoidal strip. It is 10 times larger than the width of the plane strip studied by Serdyuk (electromagnetic domain), making the comparison between our results and those of Serdyuk impossible.

- number of space-harmonic $N = 5$ (mth real diffraction orders -3, -2, -1, 0, 1, 2, 3 and mth evanescent orders -5, -4, 4, 5) and nth $= 0$ corresponding to the incident wave.

The calculated values of the efficiencies P_{TE} in Eq. (99) and P_{TM} in Eq. (100), noted $P_{TE}(\alpha_m)$ and $P_{TM}(\alpha_m)$ are given in Table 1. The corresponding values obtained through the rigorous numerical method from Equation (44) [14–16] are also given with parentheses in Table 1. Their summation $\sum_m P$ from $m = -2$ to $m = +2$ must be very near to unity that is verified in Table 1. Indeed the comparison of efficiency values obtained from the algebraic expression of the perturbation method and from the rigorous numerical method shows that the relative uncertainty of efficiencies $\Delta P_{TE}/P_{TE}$ and $\Delta P_{TM}/P_{TM}$ between the two methods for each TE and TM polarisation is less than 0.06 with $\Delta P/P = |P_{\text{perturbation}} - P_{\text{numerical}}|/P_{numerical}$. This result is in good agreement with the two methods and confirms the validity of the perturbation method.

Table 1. Values of diffracted efficiencies P_{TE} and P_{TM} calculated (perturbation method) in directions $\alpha_m = \sin\theta_m$ for diffraction orders mth $= -3, -2, -1, 0, +1, +2, +3$ with incident order nth $= 0$ ($\theta_0 = 1°$). For comparison the values obtained with the rigorous numerical method [14–16] are also given with parentheses.

mth	α_m	θ_m (deg.)	$P_{TE}(\alpha_m)$	$P_{TM}(\alpha_m)$
-3	-0.7518	-48.75	0.0000 (0.0000)	0.0000 (0.0000)
-2	-0.4954	-29.70	0.0008 (0.0008)	0.0010 (0.0009)
-1	-0.2390	-13.83	0.0613 (0.0578)	0.0659 (0.0615)
0	0.0175	1.00	0.8817 (0.8834)	0.8681 (0.8756)
+1	0.2739	15.90	0.0607 (0.0573)	0.0644 (0.0610)
+2	0.5303	32.03	0.0008 (0.0008)	0.0009 (0.0009)
+3	0.7867	51.88	0.0000 (0.0000)	0.0000 (0.0000)
$\sum_m P$			1.0053 (1.0001)	1.0003 (0.9999)

In the case of the sinusoidal strip relation (34) shows that for all $-1 \leq \alpha \leq +1$ the amplitudes $A_d^+(\alpha)$ of the diffracted field depend on the amplitudes $A_m^+ = A^+(\alpha_m)$ for the eigendirections of the diffracted field for the associated periodic surface and on normalised function $\sin c[(\alpha - \alpha_m)d/\lambda]$ which satisfy:

$$\langle \text{sinc}\left[(\alpha - \alpha_n)\, d/\lambda\right] \mid \text{sinc}\left[(\alpha - \alpha_m)\, d/\lambda\right]\rangle = \text{sinc}\left[(\alpha_n - \alpha_m)\, d/\lambda\right] = \delta_{m,n} \tag{101}$$

with δ_{mn} the Kronecker symbol in Eq. (36).

In the case of the Fraunhofer approximation domain corresponding to $|\alpha| \leq 1$ the efficiencies $P_{TE}(\alpha)$ and $P_{TM}(\alpha)$ in the directions α of diffracted fields by the strip (for $y \geq 0$) are obtained only with a superposition of the efficiencies of the plane waves [1] according to the next interpolation relations:

$$P_{TE}(\alpha) = \sum\nolimits_m P_{TE}(\alpha_m) \left\{\sin c\left[(\alpha - \alpha_m)\, d/\lambda\right]\right\}^2 \tag{102}$$

$$P_{TM}(\alpha) = \sum\nolimits_m P_{TM}(\alpha_m) \left\{\sin c\left[(\alpha - \alpha_m)\, d/\lambda\right]\right\}^2 \tag{103}$$

Table 1 shows that the values of $P_{TE}(\alpha_m)$ and $P_{TM}(\alpha_m)$ are very similar. Then one single curve representing the evolution of the efficiency or energy density $P_{TE}(\theta)$ is given (solid line) in Figure 4 for $-90° \leq \theta \leq +90°$. We can notice on the curve that the values of the efficiencies remain unchanged when $\alpha = \alpha_m$ (or $\theta = \theta_m$) since the sine cardinal function is equal to unity in these cases.

In the case of the plane strip corresponding to $h_c = 0$ with the incidence α_0 we have:

$$P_{TE}(\alpha_m) = P_{TM}(\alpha_m) = \left| \begin{array}{ll} 1 & \text{if} \quad m = 0 \\ 0 & \text{if} \quad m = \pm 1, \pm 2, \pm 3 \end{array} \right. \tag{104}$$

Consequently:

$$P_{TE}(\alpha) = P_{TM}(\alpha) = \left\{\text{sinc}\left[(\alpha - \alpha_m)\, d/\lambda\right]\right\}^2 \tag{105}$$

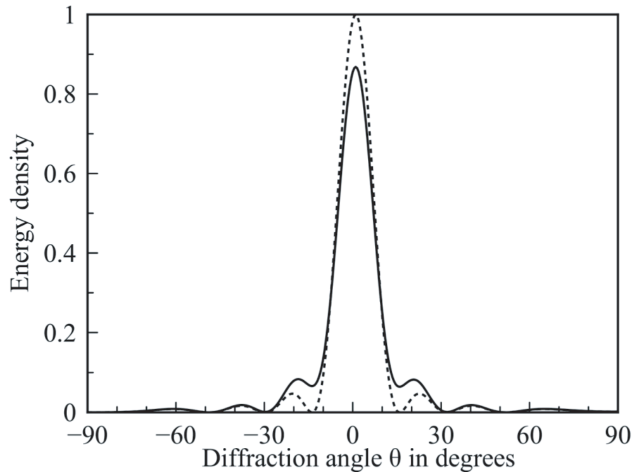

Figure 4. Diffracted energy density $P_{TE}(\theta)$ with $-90° \leq \theta \leq +90°$ and $\theta_0 = 1°$ incidence angle by the sinusoidal strip (solid line) and the plane strip (dashed line).

The corresponding curve $P_{TE}(\theta)$ is drawn (dashed line) in Figure 4. It presents a central maximum equal to 1 for $\alpha - \alpha_m = 0$ with $m = 0$ and lateral minimums equal to zero for $\alpha - \alpha_m = m\lambda/d$ and $m = \pm 1, \pm 2, \pm 3$.

The comparison of the Fraunhofer diffraction patterns the first one obtained for the sinusoidal strip and for the plane strip gives prominence to the apodization phenomenon similar to that one obtained with physical optics. There is on the one hand an increase of the principal lobe width with decrease of the maximal efficiency and on the other hand a reduction of secondary lobes with minimums not equal to zero but equal respectively to $P_{TE}(\alpha_{m)}$ and $P_{TM}(\alpha_m)$.

7. CONCLUSION

The present work has been devoted to the study of the diffraction of a plane wave by an infinitely thin and perfectly conducting deformed strip the deformation of which is small with respect to its width and which, itself, is greater than the wave length.

In the plane strip case, our study explored in detail the boundary conditions applied to two complementary surfaces has shown that the discretization of the problem is similar to that which is obtained for the associated periodic surface. Consequently in the electromagnetic optics domain the modelling of the diffraction of a plane wave by a strip can be reduced to the modelling of the diffraction of a plane wave by a periodic surface the elementary pattern of which is identical to the considered strip.

As the deformation of the strip is small with respect to its width, it has been possible to use the perturbation method, inspired by quantum physics methods. This perturbation method has been largely developed in the case of the propagative and evanescent waves diffracted by the periodic surfaces. It presents the advantage of producing analytical solutions. Thus eigenvalues and eigenvectors are obtained from known solutions of the unperturbed problem which is a plane surface. Furthermore, the efficiencies are expressed as a function of the geometrical parameters.

We have shown that it is possible to determine by interpolation the Fraunhofer diffraction pattern of a deformed strip only from the associated periodic surface efficiencies. The resulting curve highlights the phenomenon of the apodization used to remove Airy disks caused by diffraction around an intensity peak. In our case the apodization is obtained with an opaque deformed screen whereas, in optics this phenomenon is obtained for a slit in a screen from a transparency function. Here the geometry of the device plays the same role as the refractive index of Airy disks.

The problem of the apodization presents some interest in radiation diagram antennas and slits or different apertures in screens: indeed the results will also apply to the complementary distribution of obstacles.

This study of the Fraunhofer diffraction by a strip or screen may be applied in a similar manner for other kind of screens, the calculations being particularly simple when the curvilinear coordinate system can be chosen so that one of the coordinate lines coincides with the boundary of the screen.

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