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Modulation of Observed Thomson Scattering Spectra in a Plasma Density Irregularity

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Abstract—Thomson scattering of an electromagnetic wave in a plasma density irregularity is considered. A new effect is found that the scattered waves generation and superposition near the electron density extremum may result in a substantial modulation of the scattered signal frequency spectrum. Due to this effect, the observable spectrum shape will be substantially different from that for the electron density fluctuations. This fact should be taken into account when interpreting Thomson scattering experiments.

1. INTRODUCTION

Thomson (incoherent) scattering technique is based on the ability of randomly distributed free electrons to exhibit weak scattering of electromagnetic waves. The scattered spectrum allows for observing plasma fluctuations of various types and getting a valuable information about the electron density, temperature, and constituents of the plasma [1–6].

The scattered spectrum consists mainly of 2 parts: the ion line produced by ion-acoustic waves and the plasma line caused by the Langmuir fluctuations with the frequency close to the electron Langmuir frequency. These parts of the spectrum are formed by different fluctuation mechanisms. In this paper, we will consider new peculiarities of the plasma line related to the inhomogeneous structure of the plasma.

In the homogeneous plasma the scattered spectrum is proportional to the electron density fluctuation spectrum multiplied by the scattering volume (see e.g., [7]). Thus, the shape of the scattered spectrum repeats that of the electron density spectrum. If the plasma is inhomogeneous, such a coincidence does not take place anymore. For example, a gradient of the electron density results in a broadening of the plasma line obsereved in most of the experiments. In order to eliminate this effect and recreate real electron density spectrum, chirp sounding pulses can be used [8, 9].

The most significant influence of the plasma irregularities on the plasma density fluctuation spectrum takes place in plasma density cavities where the Langmuir waves can be trapped, enhanced, and acquire a discrete frequency spectrum [10–12]. These effects depend essentially on the collision frequency of electrons and the spatial scale of the irregularities. For example, in small-scale ionospheric irregularities with sizes $\sim 10\,\mathrm{m}$ or less the collisional damping has little effect on the fluctuation spectra. This leads to a modulated electron density spectrum in which the modulation characteristics depend on the geometrical characteristics of the irregularities.

In this letter, we show that peculiarities of plasma density fluctuations generated in the irregularities might not be the only reason for the modulation of the spectrum which can be observed in experiments on Thomson scattering. Superposition of the scattered waves excited in different regions of the plasma irregularity may result in additional modulation of the observed frequency spectrum. In such a case correct interpretation of the scattering spectra requires to take this effect into consideration.

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2. PROBE WAVE SCATTERING IN A ONE-DIMENSIONAL LAYER

We consider a one-dimensional plasma layer where the electron concentration n_e changes in the direction x. The probe and backscattered waves are supposed to propagate along the x-direction, as well. Such a geometry corresponds to the situation when the electron density variation has the greatest impact on the scattered spectrum. Let the ambient electron density $n_e\,(x)$ depend on the coordinate x in the following way

$$n_e(x) = n_0 + \frac{1}{2} n_0 b^2 x^2 \tag{1}$$

where n_0 is the density at the extremum point. The parameter b determines the size of the irregularity. The total electron density variation $N_e\,(t,x)$ consists of the unperturbed density $n_e(x)$ and small amplitude fluctuations $\delta n_e(t,x)$

$$N_e\,(t,x) = n_e(x) + \delta n_e(t,x), \;\; |\delta n_e| \ll n_e. \tag{2}$$

Assume that the probe wave is incident on the irregularity from the right (i.e., in the negative direction). The electric field of this wave varies with the coordinate and time as follows

$$E_1\,(t,x) = a_1 \sin\left(\omega_1 t - k_1 x\right) = \frac{a_1}{2i} \exp\left\{i\left[\omega_1 t - k_1 x\right]\right\} + C.C. \tag{3}$$

where $k_1 = -\omega_1/c < 0$ is the wave number of the probe wave; $C.C.$ means complex conjugate and will be omitted in the next calculations. The wavenumber k_1 is taken not to depend on density n_e, as the frequency ω_1 in the scattering experiments is usually much more than the electron Langmuir frequency ω_p:

$$\omega_1 \gg \omega_p. \tag{4}$$

The amplitude a_1 in the approximation of Eq. (4) is a constant because only a small part of incident power is scattered by the fluctuations. The scattered wave amplitude a_2 can be found from the equation derived in [13]

$$\begin{gathered} c\frac{\partial}{\partial x} a_2(\omega_1+\omega, x) = \frac{2\pi i}{\omega_1}\frac{e_e^2}{m_e} a_1 \exp(i\Delta k x)\delta n_e(\omega, x), \\ \Delta k = k_2 - k_1 \approx 2\frac{\omega_1}{c} \end{gathered} \tag{5}$$

where Δk is the difference between the wavenumbers of the scattered and probe waves. In contrast to a_1, the scattered amplitude a_2 is a function of x and t. We consider the stationary problem, so that the average intensity of both the Langmuir fluctuations and the amplitude a_2 does not depend on time.

The functions $a_2(\omega_1+\omega, x)$ and $\delta n_e(\omega, x)$ are the Laplace transforms of the fluctuations $a_2(t,x)$, and $\delta n_e(t,x)$

$$\widehat{L}_t(\omega) = \frac{1}{2\pi}\int\limits_0^\infty dt \exp\left[-i\left(\omega - i\Delta\right)t\right], \;\; f\,(x,\omega) = \widehat{L}_t(\omega) f\,(t,x) \tag{6}$$

where $f\,(t,x)$ can be replaced with $\delta n_e(t,x)$, and $a_2(t,x)$. This Fourier-Laplace operator was proposed in [14].

In this equation f is an arbitrary function, $\Delta > 0$, and the limit $\Delta \to 0$ should be taken in the final results. Now the scattered electric field E_2 can be represented in the form similar to Eq. (3)

$$E_2(t,x) = \frac{1}{2i}\int d\omega_2 a_2\,(x,\omega_2) \exp\left\{i\left[\omega_2 t - k_2 x\right]\right\} \tag{7}$$

where $k_2 = \omega_2/c > 0$. The amplitude a_2 is nonzero when $|\Delta\omega_2|$ is of the order or less than ω_p. This means that $\omega_2 \sim \omega_1$ and $\omega_2 \gg \omega_p$. The sign of k_2 means that the wave 2 is scattered in the positive direction, opposite to the incident wave 1.

If the ambient electron density and the fluctuation amplitude vary only slightly within the plasma wavelength (which is usual for the majority of the Thomson scattering experiments in a plasma), the fluctuation $\delta n_e(t,x)$ can be represented by the geometrical optics approximation [15]

$$\delta n\,(\omega, x) = \delta\widetilde{n}\,(\omega, x) \exp\left[-i\int_0^x \psi\left(\omega, x'\right) dx'\right] \tag{8}$$

where $\delta\widetilde{n}$ is the amplitude of the fluctuation, and ψ is the wavenumber of the plasma wave. The wavenumber ψ is the solution of the dispersion relation for the Langmuir waves [15]

$$\omega^2 = \omega_0^2(x)\left(1 + 3v_{Te}^2\frac{\psi^2}{\omega_{p0}^2}\right) \tag{9}$$

where $\omega_0(x) = \left[4\pi e^2 n_e(x)/m_e\right]^{1/2}$ is the electron Langmuir frequency, and $v_{Te} = (T_e/m_e)$ is the electron thermal velocity; e, m_e, and T_e are the electron charge, mass, and temperature, respectively. ω_{p0} is the electron Langmuir frequency $\omega_0(x)$ at the point $x = 0$. The relationship in Eq. (9) assumes that the collisional and Landau damping rates of the Langmuir waves are negligible in the interval where the scattered wave is formed. As is shown below, this interval can be even less than the irregularity size.

The solution of Equation (9) takes the form

$$\begin{aligned}\psi(\omega, x) &= \psi(\omega, 0)\left[1 - x^2/d^2(\omega)\right]^{1/2},\\ \psi(\omega, 0) &= \left[2\omega_{p0}(\omega - \omega_{p0})/3v_{Te}^2\right]^{1/2},\\ d(\omega) &= \frac{2}{b}\left[(\omega - \omega_{p0})/\omega_{p0}\right]^{1/2}.\end{aligned} \tag{10}$$

where $\psi(\omega, 0)$ is the wavenumber of the plasma wave at the point $x = 0$, and $d(\omega)$ is the coordinate of the point where the Langmuir wave propagating outside the irregularity is reflected towards its center. In fact, the interval of x, where the Langmuir wave is present, has the length $2d(\omega)$, and this interval increases with an increase in the frequency shift $\omega - \omega_{p0}$.

Eq. (5) must be solved with the boundary condition $\lim\limits_{x\to-\infty} a_2 = 0$. Then, using Eq. (8) we find from Eq. (5)

$$a_2^s(\omega_1 + \omega) = \frac{2\pi i}{c\omega_1}\frac{e_e^2}{m_e}a_1\int_{-\infty}^{+\infty} dx\delta\widetilde{n}(\omega, x)\exp\left[iT(x)\right],\ T(x) = \int_0^x\left[\Delta k - \psi\left(\omega, x'\right)\right]dx' \tag{11}$$

where the left-hand side $a_2^s(\omega_1 + \omega) = a_2(\omega_1 + \omega, x = +\infty)$ represents the amplitude of the scattered wave as it leaves the plasma layer.

The integrand in the expression for a_2^s includes the exponential $\exp\left[iT(x)\right]$ which is a fast oscillating function except the interval near the point where the derivative $T'(x)$ is equal to zero. This condition is fulfilled if $\Delta k = \psi(\omega, x)$ (i.e., the scattering wavenumber is equal to that of the Langmuir wave). The point x where $\Delta k = \psi(\omega, x)$ is, in fact, the scattered wave generation point. Obviously, the parabolic irregularity has 2 generation points. The above-mentioned interval increases when x approaches zero, as $\psi(\omega, x)$ varies the slowest at its maximum point $x = 0$ (see Eq. (10)). Thus, the vicinity of the point $x = 0$ makes the main contribution to the scattered wave amplitude, and we can replace here the wavenumber difference $\Delta k - \psi(\omega, x)$ in Eq. (11) with the power series expansion

$$\Delta k - \psi(\omega, x) \approx \chi(\omega) + \beta x^2,\ \chi = \Delta k - \psi(\omega, 0),\ \beta = \frac{b^2\omega_{p0}^2}{12v_{Te}^2\Delta k}. \tag{12}$$

This relationship is valid when $|x| \ll d(\omega)$. In this interval the slowly varying fluctuation amplitude $\delta\widetilde{n}(\omega, x)$ can be replaced with its value $\delta\widetilde{n}(\omega, 0)$ at the point $x = 0$. Therefore, the integration in Eq. (11) reduces to

$$\int_0^{+\infty} dx\cos\left(\frac{\beta}{3}x^3 + \chi x\right) = \pi\beta^{-1/3}\mathsf{Ai}\left(\beta^{-1/3}\chi\right) \tag{13}$$

where Ai is the Airy function [16]. Taking this expression into account, we obtain the following relationship for the scattering amplitude a_2^s

$$a_2^s(\omega_1 + \omega) = 4\pi^2 i\frac{e_e^2 a_1}{c\omega_1 m_e}\beta^{-1/3}\mathsf{Ai}\left(\beta^{-1/3}\chi\right)\delta\widetilde{n}(\omega, 0). \tag{14}$$

3. FREQUENCY SPECTRUM OF THE SCATTERED WAVE

In view of the Fourier-Laplace transform in Eq. (6) the frequency spectrum of the scattered wave is defined as follows

$$\left\langle E_2^2 \right\rangle_\Omega = \left\langle E_2^2 \right\rangle_{\Omega,x=+\infty} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\tau \exp\left(-i\Omega\tau\right) \left\langle E_2^2 \right\rangle_{\tau,x=+\infty}, \ \left\langle E_2^2 \right\rangle_\tau = \left\langle E_2\left(t\right) E_2\left(t+\tau\right) \right\rangle \tag{15}$$

where $\Omega = \omega_1 + \omega$ and the angle brackets denote time average. This relationship corresponds to the stationary process when the statistical moments depend only on the time difference between the arguments of the random values entering into the expression for the moments. The use of the transform in Eq. (6) provides a simple way to find the frequency spectrum if the fluctuation is known [14]

$$\lim_{\Delta\to 0} 2\Delta \left\langle E_2\left(\Omega,x\right) E_2^*\left(\Omega,x\right) \right\rangle = \frac{1}{2\pi} \left\langle E_2^2 \right\rangle_{\Omega,x}. \tag{16}$$

Eqs. (3), (6), (16) enable us to relate the scattered amplitude a_2^s (14) and the spectrum $\left\langle E_2^2 \right\rangle_\Omega = \left\langle E_2^2 \right\rangle_{\Omega,x=+\infty}$

$$\lim_{\Delta\to 0} \pi\Delta \left\langle a_2^s\left(\Omega,x\right) a_2^{s,*}\left(\Omega,x\right) \right\rangle = \left\langle E_2^2 \right\rangle_{\Omega,x}. \tag{17}$$

The relationship similar to Eq. (16) is valid for the electron density fluctuation amplitude $\delta\widetilde{n}\left(\omega,x\right)$. Using this relationship and taking into account Eq. (17) we find the expression for the scattering spectrum

$$\left\langle E_2^2 \right\rangle_{\omega_1+\omega} = 4\pi^4 e_e^2 \frac{V_1^2}{c^2} \beta^{-2/3} \mathsf{Ai}^2\left(\beta^{-1/3}\chi\right) \left\langle \delta\widetilde{n}^2 \right\rangle_{\omega,x=0} \tag{18}$$

where $V_1 = \left|e_e\right| \left|a_1\right| / \left(m_e\omega_1\right)$ is the module of the oscillatory velocity of free electrons in the probe wave electric field E_1. The factor $\left\langle \delta\widetilde{n}^2 \right\rangle_{\omega,x=0}$ in Eq. (18) is the frequency spectrum of the electron dencity fluctuation amplitude $\delta\widetilde{n}$ which is defined similarly to Eq. (15).

4. DISCUSSION OF THE RESULTS

The scattering spectrum in Eq. (18) is proportional to the spectral function $\left\langle \delta\widetilde{n}^2 \right\rangle_{\omega,x=0}$ for the amplitude of the electron density fluctuations. In this letter, we will not discuss the explicit form of $\left\langle \delta\widetilde{n}^2 \right\rangle_{\omega,x=0}$ which is a separate task. An example of calculating the electron density fluctuation amplitude $\delta\widetilde{n}$ in an inhomogeneous plasma is given in [13]. It is shown in this paper that the amplitude $\delta\widetilde{n}$ is determined by the so-called fluctuation source discussed in [17, 18]. This source depends on ω only slightly in the frequency region where the plasma line is present. Thus, the spectral function of the plasma wave amplitude $\left\langle \delta\widetilde{n}^2 \right\rangle_{\omega,x=0}$ will be a slowly varying function of ω provided it is not modulated due to the effect of the plasma wave trapping inside the irregularity.

In order to illustrate most clearly the effect of the plasma irregularity on the Thomson scattering we consider the following realistic situation. The plasma wave damping is taken to be negligible in a small interval $|x| \ll d$ where the scattering spectrum (18) is formed. This corresponds to the absence of a damping term in the argument of the exponential in Eq. (8). At the same time, we suppose that the plasma wave damping is heavy in a larger interval $(0, d)$. This condition means that the plasma wave trapping effect [10–12] will be negligible. Therefore, under these conditions the dependence of $\left\langle \delta\widetilde{n}^2 \right\rangle$ on ω in Eq. (18) can be neglected, and the frequency dependence of the scattering spectrum $\left\langle E_2^2 \right\rangle_{\omega_1+\omega}$ will be determined only by the influence of the irregularity on the scattering process.

Under these assumptions, the spectrum of the plasma line in Eq. (18) is determined by the factor $\mathsf{Ai}^2\left(\beta^{-1/3}\chi\right)$ and takes the form shown in Fig. 1. This curve is calculated with the following ionospheric plasma parameters: $\omega_{p0}/\left(2\pi\right) = 5\,\mathrm{MHz}$, $\nu_{ef} = 5\times10^2\ \mathrm{s}^{-1}$, $\omega_1/\left(2\pi\right) = 233\,\mathrm{MHz}$, $v_{Te}/\omega_{p,0} = 1\,\mathrm{cm}$, and $b = 4.47\times10^{-5}\,\mathrm{cm}^{-1}$ (the electron density changes 10 percent at a distance of 100 m).

It is seen from Fig. 1 that the spectrum is a modulated function. The reason for such a modulation can be rather simple. In fact, in a parabolic irregularity we have 2 scattered waves generated in the

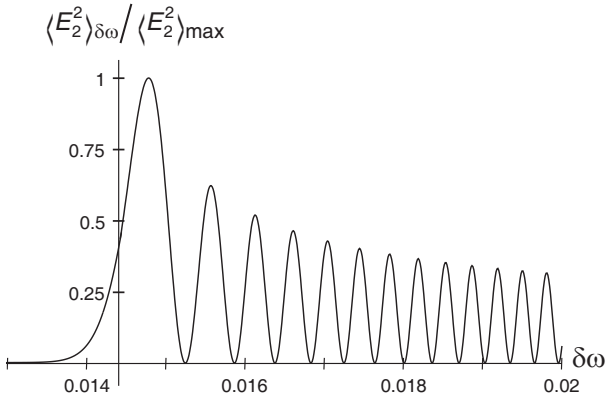

Figure 1. Plasma line spectrum versus relative frequency shift $\delta\omega = (\omega - \omega_{p0})/\omega_{p0}$. The plasma line intensity is normalized to its maximum value.

points on opposite sides of the irregularity where $\Delta k = \psi(\omega, x)$ (see above). When the waves leave the irregularity, they interfere with each other. The distance between generation points depends on the frequency ω. Thus, for different frequencies ω the scattered waves will either amplify or suppress each other.

The modulation period depends on the relative frequency shift $\delta\omega$. It is the largest near the plasma line maximum and decreases with an increase in $\delta\omega$. Knowing behavior of the Airy function, we can obtain the maximum modulation period

$$[\delta\omega]_{\max} \approx 3 \times \left(\frac{v_{Te}}{\omega_{p0}}\right)^{4/3} (b\Delta k)^{2/3}. \tag{19}$$

If the electron temperature T_e and the ambient electron density n_0 are found from other observations, all the values in Eq. (19) except b can be calculated. In such a case, measuring the modulation period $[\delta\omega]_{\max}$, one can find experimentally the inhomogeneity parameter b.

Nonstationary processes in the plasma may result in blurring of the plasma line spectrum (overlapping of the local maxima). In particular, such effect may arise from diffusion spreading of plasma irregularities. Thus, if the observation period is less than the spreading time, the modulation will be detected. Otherwise the unmodulated spectrum will be observed. Hence, by varying the observation period and detecting the period when the modulation appears, it is possible to estimate the lifetime of the irregularity.

5. CONCLUSIONS

We have shown that the process of the Thomson scattering of electromagnetic wave in a plasma irregularity may result in a frequency modulation of the observed scattering spectrum. The modulation period depends on the irregularity size. On the basis of our results, the irregularity size and lifetime can be found experimentally from the Thomson scattering spectrum. The modulation effect described in this paper should be taken into account when interpreting the Thomson scattering spectra for inhomogeneous plasmas.

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