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Analytical Solution of Impedance Synthesis Problem for a 2D Array of Thin Vibrators

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Abstract—A synthesis problem of a 2D array of thin linear vibrators, whose geometric centers are located at the nodes of a flat rectangular grid with double periodicity solved. The problem can be formulated as follows. A 2D antenna array radiates monochromatic electromagnetic waves into free space. Suppose that the array radiation pattern (RP) can be scanned in space by varying complex surface impedances of separate vibrators. Then, it is necessary to determine vibrator surface impedances to control the direction of the RP maximum. The analytical solution of the impedance synthesis problem, as an alternative to a numerical solution of a two-dimensional equation system, was obtained under two assumptions: the vibrators are excited by electric currents of equal amplitudes, and the RP of each radiator does not differ from that of an isolated radiator. Verification of theoretical formulas will be done by comparing them with relations known for one-dimensional equidistant arrays.

1. INTRODUCTION

As known [1, 2], the main advantage of antenna arrays consists in possibility of space surveillance by electronically scanning the RP beam. Such antennas are known as arrays with electric scanning, which can be classified into three classes: with phase, frequency and switching scanning. The most common method for controlling excitation current phases on the array vibrators is carried out by phase shifters. Amplitude and phase distributions in the array elements with frequency scanning can be obtained by varying electrical distance between the array elements due to frequency sweeping. Anyway the scanning control can be either continuous or discrete, depending on the control signal mode. Arrays that use a discreetly switching control are known as antennas with switched scanning beams.

The above classification is valid for antenna arrays with different types of radiators, including the arrays of impedance vibrators. However, in the latter case, there exists possibility to use the surface impedances as an additional control factor for the vibrator array [3, 4], which can vary vibrator electric lengths and, hence, the amplitude and phase distribution in the array. We have earlier applied this approach to study 1D antenna array [5], where the impedance synthesis method was proposed, and the problem was solved analytically and validated. This article is aimed to generalize the methodology of the analytical solution of impedance synthesis problem for 2D planar vibrator antenna arrays.

2. FORMULATION OF THE IMPEDANCE SYNTHESIS PROBLEM

Let us consider an antenna array consisting of several rows of linear symmetrical impedance vibrators located in a free space with material parameters $(\varepsilon_1; \mu_1)$. The array is placed in plane $(x0z)$ of the Cartesian coordinate system (x, y, z), whose origin coincides with the array center. The geometry of the array is shown in Fig. 1. Let d_z be a distance between neighboring rows of vibrators, d_x be a distance

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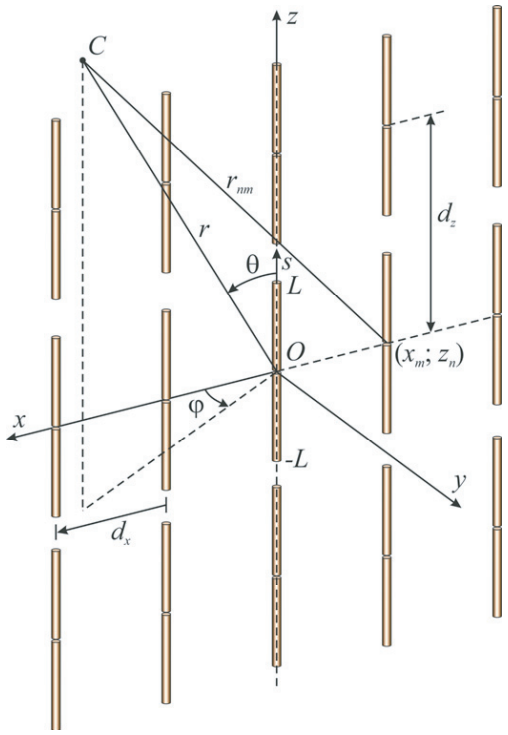

Figure 1. The geometry of the antenna array.

between the centers of the vibrators in one row, N_z be the number of rows, and N_x be the number of vibrators in one row. We will also introduce a spherical coordinate system, whose polar axis coincides with axis $\{0z\}$, and the angle φ is measured from axis $\{0x\}$.

We assume that all vibrators $N = N_z \times N_x$ are of equal length $2L$ and are characterized by different variable impedances $\bar{Z}_{nm}(s)$, normalized to the wave impedance of the medium, $Z_0 = \sqrt{\mu_1/\varepsilon_1}$ [Ohm], and integer indices $n \in [1, N_z]$ and $m \in [1, N_x]$. We also assume that the vibrator radii satisfy thin-wire approximations $\frac{\rho_{nm}}{2L} \ll 1$ and $\frac{\rho_{nm}\sqrt{\varepsilon_1\mu_1}}{\lambda} \ll 1$, and λ is the wavelength in free space. Let the vibrators be excited by monochromatic voltage-delta generators with predetermined amplitudes V_{nm} at the angular frequency ω. If time dependency is specified in the form $e^{i\omega t}$, an expression for longitudinal electric currents on vibrators can be written as [5]

$$J_{nm}(s) = -\frac{i\omega\varepsilon_1\alpha V_{nm}}{2\tilde{k}_{nm}\cos\tilde{k}_{nm}L}\sin\tilde{k}_{nm}\left(L - |s|\right) = J_{nm}\sin\tilde{k}_{nm}\left(L - |s|\right), \tag{1}$$

where $\tilde{k}_{nm}$ are the wave vectors averaged over the vibrator length, defined by the relation $\tilde{k}^2_{nm}(s) = k_1^2(1 + \frac{i2\alpha_{nm}}{k_1\rho_{nm}} \cdot \frac{1}{2L}\int\limits_{-L}^{L}\bar{Z}_{nm}(s)ds)$; J_{nm} are the complex amplitudes of the vibrator currents; $\alpha_{nm} = \frac{1}{2\ln[\rho_{nm}/(2L)]}$ are natural small parameters; $k_1 = k\sqrt{\varepsilon_1\mu_1}$, $k = \omega/c$ are the wave numbers; $c \approx 2.998 \cdot 10^{10}$ cm/s is the light velocity in vacuum. Taking into account Equation (1), we can write the main electric field component of the vibrator radiation with indices (n, m) in the wave zone as

$$\begin{aligned} E^{\theta}_{nm} &= \frac{60\pi i}{\lambda}\cdot\frac{e^{-ik_1 r_{nm}}}{r_{nm}}\sin\theta\int\limits_{-L}^{L}J_{nm}(s')e^{-ik_1 s'\cos\theta}ds' \\ &= \frac{120\pi i}{\lambda}\cdot\frac{e^{-ik_1 r_{nm}}}{r_{nm}}J_{nm}\frac{\cos\left(k_1 L\cos\theta\right) - \cos\left(\tilde{k}_{nm}L\right)}{\tilde{k}^2_{nm} - \left(k_1\cos\theta\right)^2}\tilde{k}_{nm}\sin\theta. \end{aligned} \tag{2}$$

Since the far-field components E_φ and E_r of the array are equal to zeros, the total radiation field of the array under condition that all vibrators are resonantly tuned by selecting intrinsic resistance of the δ-generators to compensate mutual influence of the vibrators can be presented as the sum of the radiation fields induced by each vibrator in the observation point $C(r,\theta,\varphi)$ as follows

$$E_\theta(r,\theta,\varphi) = \frac{120\pi i}{\lambda}\sum_{n=1}^{N_z}\sum_{m=1}^{N_x}\frac{e^{-ik_1 r_{nm}}}{r_{nm}}J_{nm}\frac{\cos(k_1 L\cos\theta)-\cos\left(\tilde{k}_{nm}L\right)}{\tilde{k}_{nm}^2-(k_1\cos\theta)^2}\tilde{k}_{nm}\sin\theta. \tag{3}$$

Thus, the impedance synthesis problem is reduced to finding impedances $\bar{Z}_{nm}(s)$ of each vibrator using Equation (3) that can steer the radiation maximum $(\theta_{\max},\varphi_{\max})$ to the predefined direction in space.

3. PROBLEM SOLUTION

First, let us represent expression (2) for the radiation field of an isolated vibrator as

$$E_{nm}^{\theta} = \frac{e^{-ik_1 r_{nm}}}{r_{nm}}\cdot\frac{60J_{nm}}{\sqrt{\varepsilon_1\mu_1}\sin\theta}\left[iFc(\theta)+\alpha_{nm}\bar{\beta}_{nm}\left(Fc(\theta)\frac{1+\cos^2\theta}{\sin^2\theta}-k_1L\sin(k_1L)\right)\right], \tag{4}$$

where $Fc(\theta)=\cos(k_1L\cos\theta)-\cos(k_1L)$ and $\bar{\beta}_{nm}=\frac{1}{2Lk_1\rho_{nm}}\int\limits_{-L}^{L}\bar{Z}_{nm}(s)ds$. Expression (4) is more convenient for further analysis, since the first term in square brackets determines the RP of an isolated perfectly conducting vibrator, while the second term defines variation of the vibrator field due to its impedance coating.

The expression for the field $E_\theta(r,\theta,\varphi)$ in the wave zone can be presented in the following form

$$\begin{aligned}E_\theta(\theta,\varphi) &= e^{-i(N_z+1)u/2}e^{-i(N_x+1)v/2}\frac{60iFc(\theta)}{\sqrt{\varepsilon_1\mu_1}\sin\theta}\\ &\times\sum_{n=1}^{N_z}\sum_{m=1}^{N_x}J_{nm}\left[1-i\alpha_{nm}\bar{\beta}_{nm}\left(\frac{1+\cos^2\theta}{\sin^2\theta}-\frac{k_1L\sin(k_1L)}{Fc(\theta)}\right)\right]e^{i(nu+mv)},\end{aligned} \tag{5}$$

where $u=k_1d_z\cos\theta$ and $\nu=k_1d_x\sin\theta\cos\varphi$. This formula is derived taking into account the expression (4) and a difference of geometric paths from neighboring vibrators to the observation point $C(r,\theta,\varphi)$. If the vibrators are perfectly conducting, and all vibrator currents are equal, $J_{nm}=J_0$, expression (5) is simplified to

$$E_\theta(\theta,\varphi) = e^{-i(N_z+1)u/2}e^{-i(N_x+1)v/2}\frac{60iFc(\theta)J_0}{\sqrt{\varepsilon_1\mu_1}\sin\theta}\sum_{n=1}^{N_z}e^{inu}\sum_{m=1}^{N_x}e^{imv}. \tag{6}$$

This expression includes two multiplayers in the form of independent power series, which represent the RPs with maxima directed along the axes $\{0z\}$ and $\{0x\}$, respectively. One can see that the maximum of the RP is achieved if $u=v=0$, and it is directed along axis $\{0y\}$, i.e., when $\theta=\pi/2$ and $\varphi=\pi/2$.

As known from the general theory of antenna arrays (see, e.g., [6]), a linear phase shift between currents of the array radiators results in radiation pattern steering. If the phase shifts between neighboring vibrators in the row and between adjacent rows are $-\Delta u$ and $-\Delta v$, the direction of maximum array radiation $(\theta_{\max};\varphi_{\max})$ can be determined by the following relations

$$\cos\theta_{\max}=\Delta u/k_1d_z \quad\text{and}\quad \sin\theta_{\max}\cos\varphi_{\max}=\Delta v/k_1d_x. \tag{7}$$

Then, expression (6) can be represented as

$$E_\theta(\theta,\varphi) = e^{-i(N_z+1)u/2}e^{-i(N_x+1)v/2}\frac{60iFc(\theta)J_0}{\sqrt{\varepsilon_1\mu_1}\sin\theta}\sum_{n=1}^{N_z}\sum_{m=1}^{N_x}e^{i(nu+mv)}e^{-i[(n-1)\Delta u+(m-1)\Delta v]}. \tag{8}$$

Expressions (5) and (8) become identical if the relations

$$1-e^{-ik_1[(n-1)d_z\cos\theta+(m-1)d_x\sin\theta\cos\varphi]}=i\alpha_{nm}\bar{\beta}_{nm}\left(\frac{1+\cos^2\theta}{\sin^2\theta}-\frac{k_1L\sin(k_1L)}{Fc(\theta)}\right)\Bigg|_{\theta=\theta_{\max};\,\varphi=\varphi_{\max}} \tag{9}$$

are valid for any n and m.

In essence, the relations in Eq. (9) for given angles $(\theta_{\max};\varphi_{\max})$ can be interpreted as equations, whose solution defines the matrix of unknowns $\{\bar{\beta}_{nm}\}$. It is easy to see that formula (9) for the 1D array (at $k_1=k$, $d_z=0$, $d_x=d$, $m=n$, $\alpha_{nm}=\alpha$, $\bar{\beta}_{nm}=\bar{\beta}_n$) can be reduced to the expression obtained in [5]: $e^{-i(n-1)kd\sin\theta\cos\varphi}=[1-i\alpha\bar{\beta}_n(\frac{1+\cos^2\theta}{\sin^2\theta}-\frac{kL\sin(kL)}{Fc(\theta)})]\Big|_{\theta=\theta_{\max};\,\varphi=\varphi_{\max}}$.

Without loss of generality, let us assume that all vibrators have different complex impedances $\bar{Z}_{nm}=\bar{R}_{nm}+i\bar{X}_{nm}$, constant along the longitudinal axes of the vibrators. Then Equation (9) can be represented in the following form

$$1-e^{-ik_1\gamma_{nm}}=i\alpha_{nm}\frac{\bar{R}_{nm}+i\bar{X}_{nm}}{k_1\rho_{nm}}\left(\frac{1+\cos^2\theta_{\max}}{\sin^2\theta_{\max}}-\frac{k_1L\sin(k_1L)}{Fc(\theta_{\max})}\right), \tag{10}$$

where $\gamma_{nm}=[(n-1)d_z\cos\theta_{\max}+(m-1)d_x\sin\theta_{\max}\cos\varphi_{\max}]$.

Since parameters α_{nm} for lossless media are real, the final formula for the real and imaginary parts of vibrator surface impedances can be written using Eq. (10) as

$$\bar{R}_{nm}=\frac{k_1\rho_{nm}\sin(k_1\gamma_{nm})}{\alpha_{nm}\left(\frac{1+\cos^2\theta_{\max}}{\sin^2\theta_{\max}}-\frac{k_1L\sin(k_1L)}{Fc(\theta_{\max})}\right)};$$

$$\bar{X}_{nm}=\frac{-k_1\rho_{nm}[1-\cos(k_1\gamma_{nm})]}{\alpha_{nm}\left(\frac{1+\cos^2\theta_{\max}}{\sin^2\theta_{\max}}-\frac{k_1L\sin(k_1L)}{Fc(\theta_{\max})}\right)}. \qquad n=1,2,\ldots,N_x;\ \ m=1,2,\ldots,N_y; \tag{11}$$

Formulas (9) and (11) remain valid for arbitrary number of vibrators in the array and arbitrary distances between the vibrators.

4. NUMERICAL RESULTS

As an example, consider a 2D array of symmetrical half-wave impedance vibrators radiating in the front half-space $0\le\theta\le\pi$, $0\le\varphi\le\pi$ (Fig. 1). The number of array elements is $25(N_x=5, N_z=5)$, and

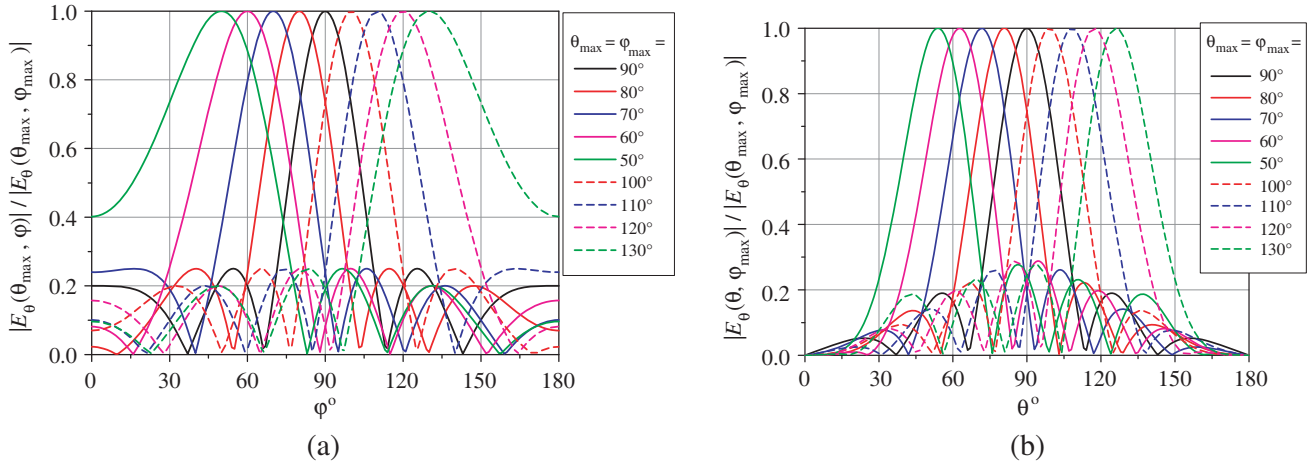

Figure 2. Normalized radiation patterns of the array with array periods $d_x=0.5\lambda$, $d_z=0.5\lambda$: (a) the angle $\theta_{\max}$ is fixed, (b) the angle $\varphi_{\max}$ is fixed.

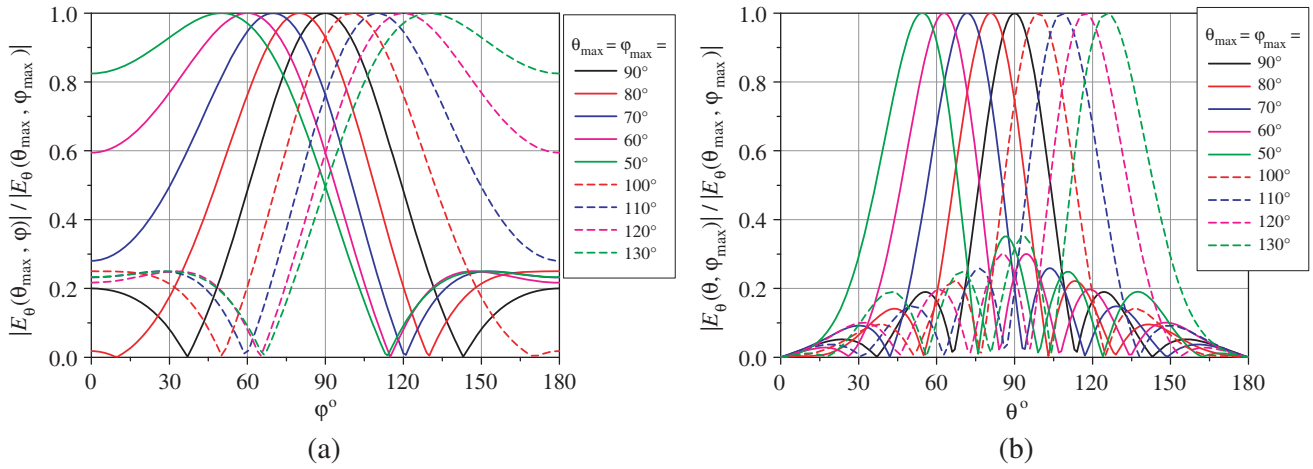

Figure 3. Normalized radiation patterns of the array with periods $d_x = 0.25\lambda$, $d_z = 0.5\lambda$: (a) the angle $\theta_{\max}$ is fixed, (b) the angle $\varphi_{\max}$ is fixed.

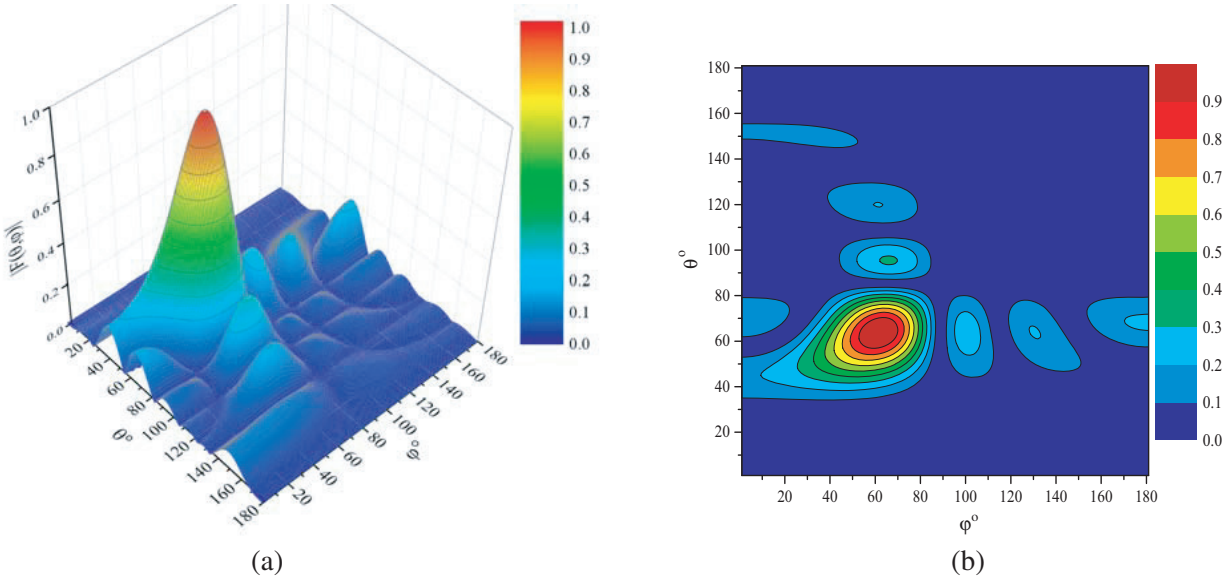

Figure 4. The 3D radiation pattern of the antenna array with parameters: $d_z = d_x = 0.5\lambda$, $\theta_{\max} = \varphi_{\max} = 60°$: (a) surface plot, (b) contour plot.

the vibrator length and radius are $L = \lambda/4$ and $\rho_{nm} = L/75$. The problem consists in finding the impedances in Eq. (11) of each vibrator such that the main lobe of the RP is oriented in a predefined direction $(\theta_{\max}; \varphi_{\max})$.

The results of numerical simulations are presented for two arrays with different geometric structures ($d_x = 0.5\lambda$, $d_z = 0.5\lambda$ and $d_x = 0.25\lambda$, $d_z = 0.5\lambda$) are shown in Fig. 2 and Fig. 3, where variations of angle $(\theta_{\max}; \varphi_{\max})$ are shown symmetrically with respect to direction $(\theta = 90°; \varphi = 90°)$ corresponding to main lobe orientation for the array with perfectly conducting vibrators.

As can be seen from the figures, the simulation results confirm the possibility to vary the direction of the array RP maximum within the entire half-space, at least theoretically, by varying the intrinsic impedances of the vibrators. As expected, the main lobe width depends on array aperture dimensions as seen from the observation angles, and therefore slightly varies during spatial scanning. As an example, Fig. 4 shows the surface and contour plots of RP of the antenna array.

It should be noted that Equations (11) and relations in Equation (9) are valid for any number

Table 1. Calculated impedances for the RP shown in Fig. 4.

$\bar{R}_{nm}$					$\bar{X}_{nm}$				
0	0.37	0.155	−0.305	−0.282	0	−0.299	−0.724	−0.602	−0.126
0.378	0.079	−0.345	−0.223	0.252	−0.378	−0.748	−0.533	−0.073	−0.096
0	−0.37	−0.155	0.305	0.282	−0.757	−0.457	−0.033	−0.155	−0.63
−0.378	−0.079	0.345	0.223	−0.252	−0.378	−0.008	−0.224	−0.684	−0.66
0	0.37	0.155	−0.305	−0.282	0	−0.299	−0.724	−0.602	−0.126

Table 2. Calculated values of the impedances.

$\bar{R}_{nm}$					$\bar{X}_{nm}$					$\arg(J_{nm})$, [°]				
0	0.26	0.368	0.26	0	0	−0.108	−0.368	−0.628	−0.735	0	−45	−90	−135	−180
0	0.26	0.368	0.26	0	0	−0.108	−0.368	−0.628	−0.735	0	−45	−90	−135	−180
0	0.26	0.368	0.26	0	0	−0.108	−0.368	−0.628	−0.735	0	−45	−90	−135	−180
0	0.26	0.368	0.26	0	0	−0.108	−0.368	−0.628	−0.735	0	−45	−90	−135	−180
0	0.26	0.368	0.26	0	0	−0.108	−0.368	−0.628	−0.735	0	−45	−90	−135	−180

of vibrators in the array and arbitrary distance between vibrators. However, in a general case, the impedances $\bar{Z}_{nm} = \bar{R}_{nm} + i\bar{X}_{nm}$ (11), thus calculated, cannot guarantee that the inequality $\bar{R}_{nm} \geq 0$, arising from energy considerations, holds. Such a situation can be observed in Table 1, where the real and imaginary parts of the vibrator impedances calculated for the RP in Fig. 4 are shown.

The normalized RPs of the array ($d_x = 0.25\lambda$, $d_z = 0.5\lambda$) are represented in main planes in Fig. 5. The simulation results show that the real parts of the impedances remain positive if angle $\theta_{\max} = 90°$ and angle $\varphi_{\max}$ varies at the interval $\varphi_{\max} \in [60°, 120°]$. As an example, calculated impedances are given in Table 2 for angles $\theta_{\max} = 90°$ and $\varphi_{\max} = 60°$. As can be seen from the table, all positive values are observed in the case when the resulting difference in the phase currents of extreme radiators in the lattice does not exceed 180°. If $\bar{R}_{nm} < 0$, the real part of impedances $\bar{R}_{nm}$ should be interpreted as effective physical quantities [7], which can be realized only as a result of electrodynamic interactions between the vibrators and additional scatterers. Impedances $\bar{Z}_{nm}$ with the real parts $\bar{R}_{nm} \geq 0$ which can be interpreted as traditional intrinsic impedances under conditions that the individual vibrators can be manufactured in the form of cylindrical structures with complex intrinsic structures [4]. Therefore, for practical applications, limiting scanning angles of the array RP should be determined for which the condition $\bar{R}_{nm} \geq 0$ is satisfied. From the point of view of physical interpretation, such studies can most obviously be carried out if the RP is scanned only in one plane.

The normalized RPs of array ($d_x = 0.25\lambda$, $d_z = 0.5\lambda$) are represented in main planes in Fig. 5. The simulation results show that the real parts of the impedances remain positive if angle $\theta_{\max} = 90°$ and angle $\varphi_{\max}$ varies at the interval $\varphi_{\max} \in [60°, 120°]$. As an example, calculated impedances are given in Table 2 for angles $\theta_{\max} = 90°$ and $\varphi_{\max} = 60°$. As can be seen from the table, all positive values are observed in the case when the resulting difference in the phase currents of the extreme radiators in the lattice does not exceed 180°. As can be seen from Table 2, positive $\bar{R}_{nm}$ are observed if in the phase shifts of the currents on the edge vibrators $\arg(J_{nm})$ do not exceed 180°. The simulation results have also shown that this condition is valid for other array geometries, particularly for two-dimensional scanning.

The maximum angle of the RP deviation relative to the normal $\Delta\varphi_{\max} = 90° - \varphi_{\max}$ for $\bar{R}_{nm}$ to remain positive can be estimated by $\Delta\varphi_{\max} = \pm\arcsin(\frac{\pi}{kd_x(N_x-1)}) = \pm\arcsin(\frac{\lambda}{2d_x(N_x-1)})$ or by $\Delta\varphi_{\max} \approx \pm\frac{\lambda}{2d_x(N_x-1)} \cdot \frac{180°}{\pi}$ valid for small deviations. This estimate relative to the main lobe width at half-power level $\Delta\varphi_{0.5} \approx \frac{\lambda}{d_x(N_x-1)} \cdot 51°$ can be express as $\Delta\varphi_{\max} \approx \pm 0.56 \cdot \Delta\varphi_{0.5}$.

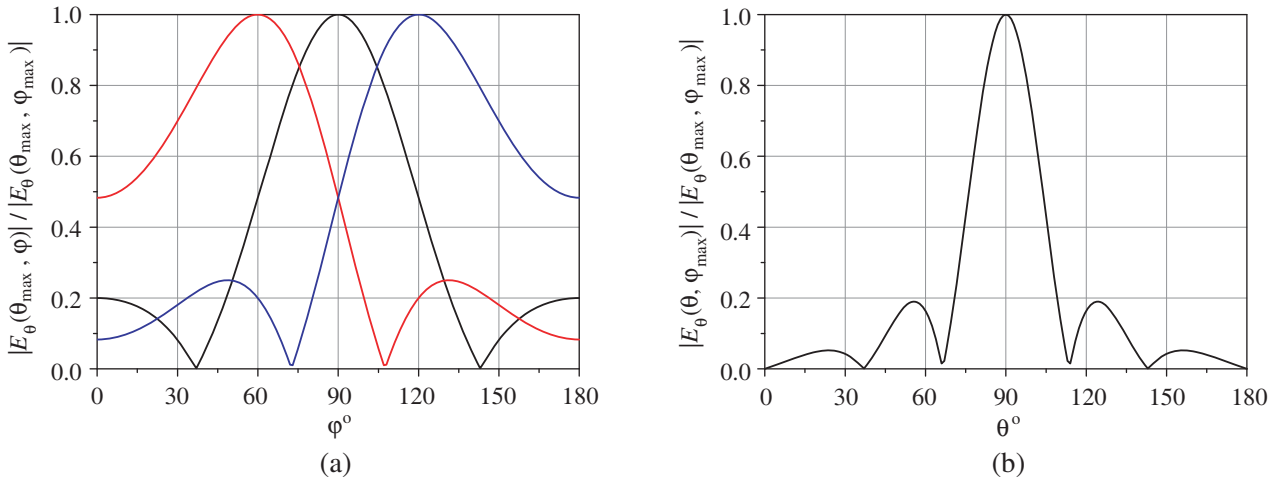

Figure 5. RPs in the main planes.

5. CONCLUSION

The method of impedance synthesis of the radiation pattern, proposed in [5] for linear vibrator structures, is generalized for the 2D vibrator arrays. The problem of impedance synthesis for the double periodic arrays is solved analytically. The derived formulas allow defining the impedance of each vibrators that permit to steer the RP maximum in a predefined direction. The formulas can be used to develop algorithms for controlling antenna arrays, for example, for spatial scanning of the RP. The approach is characterized by using the variable vibrator impedance as the averaged integral coefficient. This methodology turns out to be common for the radiators with impedance coatings of various types. The method validity is confirmed by reduction of the obtained formulas for particular cases of 1D equidistant arrays. The numerical simulation of the 2D antenna array consisting of $25 (N = 25, N_x = 5, N_z = 5)$ symmetrical half-wave vibrators $(L = \lambda/4)$ has been carried out, which confirms a possibility to scan the antenna RP in the front half-space by varying complex internal impedances of the vibrators. The physical realization of this possibility is analyzed. The results obtained in the article can be used for modeling of adaptive antenna arrays with adjustable dynamic variation of parameters and characteristics depending upon influence of external or intrinsic factors.

REFERENCES

1. Bakhrakh, L. D. and S. D. Kremenetsky, *Synthesis of Radiation Systems (Theory and Methods of Calculation)*, Sov. Radio, Moscow, 1974 (in Russian).
2. Skobelev, S. P., *Phased Array Antennas with Optimized Element Patterns*, Artech House, Boston, 2011.
3. Berdnik, S. L., V. A. Katrich, M. V. Nesterenko, and Y. M. Penkin, "Electromagnetic waves radiation by a vibrators system with variable surface impedance," *Progress In Electromagnetics Research M*, Vol. 51, 157–163, 2016.
4. Nesterenko, M. V., V. A. Katrich, Yu. M. Penkin, V. M. Dakhov, and S. L. Berdnik, *Thin Impedance Vibrators. Theory and Applications*, Springer Science+Business Media, New York, 2011.
5. Penkin, Yu. M., V. A. Katrich, and M. V. Nesterenko, "Formation of radiation fields of linear vibrator arrays by using impedance synthesis," *Progress In Electromagnetics Research M*, Vol. 57, 1–10, 2017.
6. Amitay, N. V. Galindo, and C. P. Wu, *Theory and Analysis of Phased Array Antennas*, John Wiley & Sons Inc, New York, 1972.
7. Penkin, Yu. M., V. A. Katrich, and M. V. Nesterenko, "Development of fundamental theory of thin impedance vibrators," *Progress In Electromagnetics Research M*, Vol. 45, 185–193, 2016.