

Inertial Properties of the TE Waveguide Fields

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Abstract—Inertial properties of the TE-waveguide modal fields are studied in time-domain making use of an analytical method, named as evolutionary approach to electrodynamics (*EAE*). To achieve inertial characteristics, electric field vector with dimension of *volt per meter* and magnetic field vector with dimension of *ampere per meter* in Maxwell's equations are factorized in SI units to obtain new electric and magnetic field vectors with their common dimensions of *inverse meter*. Having the fields with the common dimensions makes them *summable*. Using *EAE*, modal basis elements that depend on transverse coordinates and modal amplitudes that depend on time and longitudinal coordinate are obtained by solving the boundary eigenvalue problem. As a result of using the new electric and magnetic field vectors, the energetic properties are derived as real-valued functions of coordinates and time. Then, the inertial properties (that is, electromagnetic mass and momentum) of the TE-waveguide modes are obtained as the functions of time.

1. INTRODUCTION

Time-harmonic theory for electromagnetic fields in waveguides has been pioneered by Lord Rayleigh at the end of 19th century under his supposition that all the fields vary in time harmonically [1]. This postulate eliminates the time derivative from original Maxwell's equations and changes their *hyperbolic* type to *elliptic* kind as partial differential equations (PDE). The time-harmonic solutions for the elliptic type Maxwell's equations do not satisfy the causality principle. Computational electromagnetics activated progress in this area since the second half of 20th century. Numerical methods are capable to produce a big amount of numerical data. However, physical content can be extracted from those numerical results in presence mostly of additional analytical information.

During the last decade, so-called microwave resonant cavity *thruster* has been a hot topic in electrodynamics. Yet notwithstanding this device already exists off-the-shelf [2], a dispute about the physical processes within the thruster still continues [3, 4]. This fact motivated our analytical studying electromagnetic energy and concomitant mechanical inertia of the waveguide and cavity fields in the time domain.

Kaiser studied inertia of the time-domain fields in *free space* under his strict condition that the electric and magnetic fields have their *common* physical dimensions [5]. It is true for operations within the framework of CGS units. In SI units, however, electric field has the dimension of *volt per meter*, [Vm^{-1}], but magnetic field has the dimension of *ampere per meter*, [Am^{-1}]. Our time-domain study is executed within the framework of SI metric system for the system of Maxwell's equations supplemented with appropriate boundary conditions. To this aim, we apply the Evolutionary Approach to Electrodynamics (EAE, shortly) proposed in [6]. One can find the schemes of cavity and waveguide versions of the EAE for the standard Maxwell's equations in [7] and [8], respectively. Applying Kaiser's technique to the SI metric system, we rearrange standard Maxwell's equations in SI units to a novel format (in SI units, as well), in which the electric and magnetic fields have their *common*

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physical dimensions. Our results in this aspect were presented recently in [9, 10]. As for the other applications of the EAE for solving various practical time-domain problems, one can get an overview from publications [11–18].

The article is organized as follows. In Section 2, an outline of a waveguide version of the EAE, where the novel format of Maxwell's equations in SI units participates, is presented. Maxwell's equations in that format involve only one *fundamental* physical constant, namely: light velocity in vacuum, c , (see (1a)–(1c) below). Besides, the new field vectors, \mathbb{E} and \mathbb{H} , have their common dimension of *inverse meter*, $[\text{m}^{-1}]$. The standard field vectors can be recovered by the scaling formulas (2) at any step of analysis when needed. In Section 3, the real-valued time-domain modal waves in lossy waveguides and oscillations in a cavity are considered. In Section 4, the energetic characteristics of the waveguide fields are derived as the functions of time. In Section 5, inertial properties of the waveguide modes varying in time (namely, equivalent mass and mechanical momentum) are presented. In Section 6, the concluding remarks are listed.

2. OUTLINE OF A WAVEGUIDE VERSION OF THE EAE

2.1. The Novel Format of Maxwell's Equations in SI Units

The novel format of Maxwell's equations was proposed in [9] and implemented in [10]. Finally, their format has become, in SI units, much simpler than the original one, namely:

$$\nabla \times \mathbb{H}(\mathbf{r}, t) = \frac{1}{c} \frac{\partial}{\partial t} \mathbb{E}(\mathbf{r}, t) + \mathbb{J}(\mathbf{r}, t) \quad (1a)$$

$$\nabla \times \mathbb{E}(\mathbf{r}, t) = -\frac{1}{c} \frac{\partial}{\partial t} \mathbb{H}(\mathbf{r}, t) \quad (1b)$$

$$\nabla \cdot \mathbb{E}(\mathbf{r}, t) = \varrho(\mathbf{r}, t), \quad \nabla \cdot \mathbb{H}(\mathbf{r}, t) = 0 \quad (1c)$$

where c is the speed of light. It is appropriate to emphasize that only this *fundamental* physical constant participates in Eqs. (1a)–(1c). The new field vectors, \mathbb{E} and \mathbb{H} , have *common* their physical dimension of *inverse meter*, $[\text{m}^{-1}]$. The standard field vectors, $\mathcal{E}(\mathbf{r}, t)$ and $\mathcal{H}(\mathbf{r}, t)$, which have the dimensions of *volt per meter*, $[\text{Vm}^{-1}]$, and *ampere per meter*, $[\text{Am}^{-1}]$, respectively, can be recovered at any step of analysis by formulas

$$\underbrace{\mathcal{E}(\mathbf{r}, t)}_{[\text{Vm}^{-1}]} = \underbrace{\epsilon_0^{\text{V}}}_{[\text{V}]} \underbrace{\mathbb{E}(\mathbf{r}, t)}_{[\text{m}^{-1}]}, \quad \underbrace{\mathcal{H}(\mathbf{r}, t)}_{[\text{Am}^{-1}]} = \underbrace{\mu_0^{\text{A}}}_{[\text{A}]} \underbrace{\mathbb{H}(\mathbf{r}, t)}_{[\text{m}^{-1}]}, \quad (2)$$

which are presented in Eqs. (1a)–(1c). The scaling factors, ϵ_0^{V} and μ_0^{A} , were specified in [9] as

$$\epsilon_0^{\text{V}} \stackrel{\text{def.}}{=} \sqrt{N/\epsilon_0} \cong 3.3607 \times 10^5 [\text{V} \equiv \text{NmA}^{-1}\text{s}^{-1}] \quad (3a)$$

$$\mu_0^{\text{A}} \stackrel{\text{def.}}{=} \sqrt{N/\mu_0} \cong 8.9206 \times 10^2 [\text{A}] \quad (3b)$$

where $N \equiv \text{kgms}^{-2}$ is the force unit, *newton*, and ϵ_0 and μ_0 are the free-space constants, which have been heuristically defined in SI metric system (see [19]) as follows:

$$\epsilon_0 = \frac{1}{c_0^2 4\pi \times 10^{-7}} [\text{Fm}^{-1} \equiv \text{A}^2 \text{s}^2 \text{N}^{-1} \text{m}^{-2}] \quad (4a)$$

$$\mu_0 = 4\pi \times 10^{-7} [\text{Hm}^{-1} \equiv \text{NA}^{-2}]. \quad (4b)$$

Herein, $c_0 = 2.99792458 \times 10^8$ is *number*, which is installed in Eq. (4a) as the *quantity symbol* of the light speed in vacuum. One can verify by formulas (4a)–(4b) that $1/\sqrt{\epsilon_0 \mu_0} = c_0 [\text{ms}^{-1}] = c$ what is the fundamental physical constant.

In the standard Maxwell's equations, the electric current density, $\mathcal{J}(\mathbf{r}, t)$, has the physical dimension of *ampere per meter*², $[\text{Am}^{-2}]$. The vector $\mathcal{J}(\mathbf{r}, t)$ should be scaled as $\mathcal{J}(\mathbf{r}, t) = \mu_0^{\text{A}} \mathbb{J}(\mathbf{r}, t)$ which yields dimension of $\mathbb{J}(\mathbf{r}, t)$ as $[\text{m}^{-2}]$. It is also proper sometimes to present the standard electric current density via Ohm's law as $\mathcal{J} = \sigma \mathcal{E}$ where a given parameter of constant conductivity, σ , has the dimension of *siemens per meter*, $[\text{Sm}^{-1}]$. In this case, the new vector of electric current density, $\mathbb{J}(\mathbf{r}, t)$ in Eq. (1a), is presentable as $\mathbb{J}(\mathbf{r}, t) = \gamma_0 \mathbb{E}(\mathbf{r}, t)$ where $\gamma_0 = \sigma_0 376.73$. Herein, *number* σ_0 is *quantity symbol* of σ (i.e.,

σ_0 is a *number* of units [Sm^{-1}] in σ given originally in SI units). The factor γ_0 has the same dimension as the field \mathbb{E} : i.e., [m^{-1}].

The standard charge density, $\rho(\mathbf{r}, t)$, has the dimension of *coulomb per meter*³, [Cm^{-3}]. That should be scaled as $\rho = \sqrt{N\epsilon_0}\varrho$ which yields a new charge density, $\varrho(\mathbf{r}, t)$, presented now in Eq. (1c) with its dimension of [m^{-2}]. The scaling factor, $\sqrt{N\epsilon_0}$, has the numerical value 2.9756×10^{-6} and has the dimension of *coulomb per meter*, [Cm^{-1}].

2.2. Description of the Waveguide and the Boundary Conditions

The waveguide under study is homogeneous along Oz axis. Its cross section, S , is bounded by closed contour, L . A triplet of unit vectors, $\mathbf{z} \times \mathbf{l} = \mathbf{n}$, will be used where unit vector \mathbf{z} is axial, and \mathbf{l} and \mathbf{n} are tangential and normal to L unit vectors, respectively. The waveguide surface is perfectly conducting. To model possible losses, the waveguide is supposed as filled with a medium with a constant of conductivity, σ . Maxwell's Equations (1a)–(1c) should be supplemented with the boundary conditions as

$$\mathbf{l} \cdot \mathbb{E}(\mathbf{r}, t)|_{L=0}, \quad \mathbf{z} \cdot \mathbb{E}(\mathbf{r}, t)|_{L=0}, \quad \mathbf{n} \cdot \mathbb{H}(\mathbf{r}, t)|_{L=0}. \quad (5)$$

Such waveguides can support propagation of the transverse-electric (TE) and transverse-magnetic (TM) fields. In total, the set of Equations (1a)–(1c) and (5) composes the standard statement of the boundary-value problem for the study of propagation in waveguides for the time-varying waveforms.

3. TIME-DOMAIN MODES IN THE LOSSY WAVEGUIDES

3.1. Solving the Problem for the TE Time-Domain Modal Fields

Transverse-electric (TE) waveguide modes are presented as

$$\mathbb{E}^{\text{TE}} = \mathcal{A}(z, t) \mathbf{e}(\mathbf{r}_\perp), \quad \mathbb{E}_z^{\text{TE}} \equiv 0 \quad (6a)$$

$$\mathbb{H}^{\text{TE}} = \mathcal{B}(z, t) \mathbf{h}(\mathbf{r}_\perp) + h(z, t) \mathbf{h}_z(\mathbf{r}_\perp) \quad (6b)$$

where the *modal basis elements* are the vectorial factors that should be found out firstly in domain S ; \mathbf{r}_\perp is projection of the position vector, $\mathbf{r} = \mathbf{r}_\perp + \mathbf{z}z$ on S . Then, the scalar *modal amplitudes*, \mathcal{A} , \mathcal{B} , and h , have to be obtained. A complete set of eigensolutions to the Neumann eigenvalue problem for transverse part of Laplacian, $\nabla_\perp^2 = \nabla^2 - \frac{\partial^2}{\partial z^2}$, has been used as a generator of the modal basis for TE-waveform, that is,

$$(\nabla_\perp^2 + \nu_n^2) \psi_n(\mathbf{r}_\perp) = 0, \quad \mathbf{n} \cdot \nabla_\perp \psi_n(\mathbf{r}_\perp)|_{\mathbf{r}_\perp \in L} = 0 \quad (7a)$$

$$(\nu_n^2/S) \int_S \psi_n^2 ds = 1, \quad n = 1, 2, \dots \quad (7b)$$

where $\nu_n^2 > 0$ are the eigenvalues, and ψ_n are corresponding eigensolutions. The subscript (n) is responsible for the distribution of ν_n^2 on real axis in their numerical values order. The boundary conditions in Eq. (5) enforce us to generate the elements of modal basis, which participate in Eqs. (6a)–(6b), in the following format:

$$\mathbf{e}(\mathbf{r}_\perp) \equiv \mathbf{e}_n(\mathbf{r}_\perp) = \nabla_\perp \psi_n(\mathbf{r}_\perp) \times \mathbf{z} \quad (8a)$$

$$\mathbf{h}(\mathbf{r}_\perp) \equiv \mathbf{h}_n(\mathbf{r}_\perp) = \nabla_\perp \psi_n(\mathbf{r}_\perp) \quad (8b)$$

$$\mathbf{h}_z(\mathbf{r}_\perp) \equiv \mathbf{h}_{zn}(\mathbf{r}_\perp) = \mathbf{z} \nu_n \psi_n(\mathbf{r}_\perp) \quad (8c)$$

where $\psi_n(\mathbf{r}_\perp)$ is a dimensionless scalar function. Thus, Equations (6a)–(6b) look as

$$\mathbb{E}_n^{\text{TE}} = \mathcal{A}_n(z, t) \mathbf{e}_n(\mathbf{r}_\perp), \quad \mathbb{E}_{zn}^{\text{TE}} \equiv 0 \quad (9a)$$

$$\mathbb{H}_n^{\text{TE}} = \mathcal{B}_n(z, t) \mathbf{h}_n(\mathbf{r}_\perp) + h_n(z, t) \mathbf{h}_{zn}(\mathbf{r}_\perp) \quad (9b)$$

where $\mathcal{A}_n(z, t)$, $\mathcal{B}_n(z, t)$, and $h_n(z, t)$ are the unknown *modal amplitudes* for the TE waveguide modes, which are unknown as yet. The modal basis elements in Eqs. (8a)–(8c) have dimension of [m^{-1}], but the modal amplitudes should be dimensionless. Substituting the fields in Eqs. (9a)–(9b) to (1a)–(1c) yields Klein-Gordon equation (KGE) for h_n as

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} h_n(z, t) - \frac{\partial^2}{\partial z^2} h_n(z, t) + \nu_n^2 h_n(z, t) = 0, \quad (10)$$

and the direct formulas for amplitudes \mathcal{A}_n and \mathcal{B}_n , which are expressible via h_n as follows:

$$\mathcal{A}_n(z, t) = -\frac{1}{c} \frac{\partial}{\partial t} h_n(z, t), \quad \text{and} \quad \mathcal{B}_n(z, t) = \frac{\partial}{\partial z} h_n(z, t). \quad (11)$$

In the analysis that follows, we shall operate with the dimensionless variables as

$$\xi = \nu_n z \quad \text{and} \quad \tau = \nu_n c t \quad \text{for TE-modes} \quad (12)$$

which provides the requirement that the amplitudes \mathcal{A}_n and \mathcal{B}_n are also dimensionless.

Notice that KGE (10) carries all information about a shape of contour L , which is accumulated in the coefficient ν_n^2 herein. If the shape of L is composed of the coordinate lines (Cartesian, cylindrical, elliptic, or their combinations, *etc.*), the eigenvalues, ν_n^2 , can be obtained analytically via solving the problem in Eq. (7) by separation of the transverse coordinates. If the shape of contour L is rather arbitrary, the value of ν_n^2 can be obtained by solving Eq. (7) numerically. After substitution of those numerical values to Eq. (10), solving the KGE can be continued analytically.

3.2. Real-Valued Time-Harmonic Modal Waves in Lossy Waveguides

In terms of dimensionless variables in Eq. (12), KGE (10)–(11) look as

$$h_n(\xi, \tau) = \vartheta(\xi, \tau) \exp(-\alpha_n \tau) \quad (13a)$$

$$\left(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \xi^2} + 1 - \alpha_n^2 \right) \vartheta(\xi, \tau) = 0 \quad (13b)$$

$$\mathcal{A}_n = -\frac{\partial}{\partial \tau} h_n(\xi, \tau), \quad \mathcal{B}_n = \frac{\partial}{\partial \xi} h_n(\xi, \tau) \quad (13c)$$

where $\alpha_n = \gamma_0/(2\nu_n) < 1$ is the dimensionless lossy parameter, $\gamma_0 = \sigma_0/376.73$.

Equation (13b) supports propagation of the time-harmonic waves presentable as

$$\vartheta(\xi, \tau) = C \sin(\varphi + \varpi_n \tau - \beta_n \xi) \quad (14)$$

where C and $0 \leq \varphi \leq 2\pi$ are real-valued parameters; $\varpi = \omega/(\nu_n c)$ is dimension-free format of the frequency parameter ω ; β_n is the propagation constant specified as

$$\beta_n = \sqrt{\varpi_n^2 - (1 - \alpha_n^2)}. \quad (15)$$

Factually, formula (14) involves two linearly independent *sine* and *cosine* solutions to KGE (13b) due to presence in Eq. (14) of the free constants, C and φ . Indeed,

$$\vartheta(\xi, \tau) = C \sin(\varphi + \varpi_n \tau - \beta_n \xi) \equiv a \sin(\varpi_n \tau - \beta_n \xi) + b \cos(\varpi_n \tau - \beta_n \xi) \quad (16)$$

where $a = C \cos \varphi$ and $b = C \sin \varphi$. The condition $\beta_n = 0$ gives the cutoff frequencies, $\omega_{n, \text{cut-off}}$, for the TE-modes, which propagate in the lossy waveguides, as

$$\omega_{n, \text{cut-off}} = \nu_n c \sqrt{1 - (\gamma_0/2\nu_n)^2} \quad \text{where} \quad n = 1, 2, \dots \quad (17)$$

In the lossless waveguide, where $\gamma_0 = 0$, the cutoff frequencies are $\omega_{n, \text{cut-off}}^0 = \nu_n c$.

In Fig. 1, time dependences of the real-valued time-harmonic amplitudes at $\xi = 0$ are exhibited for the lossless, Fig. 1(a), and lossy, Fig. 1(b), cases, respectively.

3.3. TE-Oscillations in a Short-Circuited Piece of Waveguide

Find the TE-modal fields in a short-circuited piece of the waveguide located between two cross sections at $z = 0$ and $z = \ell$, which are perfectly conducting. So, we have to solve Equation (10) supplemented with appropriate boundary conditions as

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} h_n(z, t) + \nu_n^2 h_n(z, t) = \frac{\partial^2}{\partial z^2} h_n(z, t) = -\lambda^2 h_n(z, t), \quad \text{and} \quad h_n(z, t)|_{z=0, \ell} = 0 \quad (18)$$

where λ^2 is a constant of separation of the variables. Separating z - and t -variables yields $\lambda^2 \equiv \lambda_m^2 = (\pi m/\ell)^2$, $m=1, 2, \dots$, and the amplitudes for the field components as

$$h_{nm}(z, t) = \exp(-\alpha_n \tau) \exp(i\omega_{nm}^{\text{TE}} t) \sin(\pi m z/\ell) \quad (19a)$$

$$\mathcal{A}_{nm} = -\frac{1}{c} \frac{\partial}{\partial t} h_{nm}(z, t), \quad \mathcal{B}_{nm} = \frac{\partial}{\partial z} h_{nm}(z, t) \quad (19b)$$

where $\omega_{nm}^{\text{TE}} = c\sqrt{\nu_n^2 - \gamma_0^2 + (\pi m/\ell)^2}$ is the eigenfrequency of oscillations.

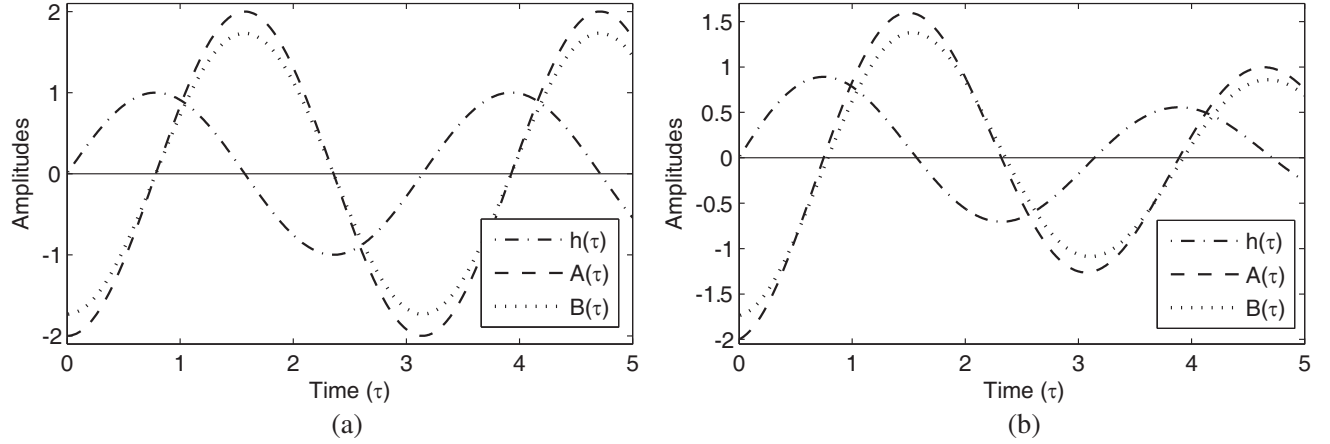


Figure 1. Time dependences of the modal amplitudes: (a) in the lossless and (b) in lossy waveguides.

4. ENERGETIC CHARACTERISTICS OF THE TE-WAVEGUIDE FIELDS

Electromagnetic energy density is specified via the field vectors, $(\mathcal{E}, \mathcal{H})$ and (\mathbb{E}, \mathbb{H}) , as

$$\mathcal{W}(\mathbf{r}, t) = (1/2) (\epsilon_0 \mathcal{E} \cdot \mathcal{E} + \mu_0 \mathcal{H} \cdot \mathcal{H}) \text{ [Jm}^{-3} \equiv \text{Nm}^{-2}] \quad (20a)$$

$$= (1/2) (\epsilon_0 \epsilon_0^V \mathbb{E} \cdot \epsilon_0^V \mathbb{E} + \mu_0 \mu_0^A \mathbb{H} \cdot \mu_0^A \mathbb{H}) = \mathbb{W}(\mathbf{r}, t) \text{ [Nm}^{-2}] \quad (20b)$$

where the quantity \mathcal{W} has the dimension of *joule per meter*³ [Jm⁻³]. Result of substituting Eqs. (2) to Eq. (20a) is shown in Eq. (20b). Notation \mathbb{W} herein corresponds to that part of \mathcal{W} , which is expressible via the vectors \mathbb{E} and \mathbb{H} and has dimension of [m⁻²], i.e.,

$$\mathbb{W}(\mathbf{r}, t) = (1/2) (\mathbb{E} \cdot \mathbb{E} + \mathbb{H} \cdot \mathbb{H}) \text{ [Nm}^{-2}]. \quad (21)$$

For the TE-modes, the value of the energy density averaged over the waveguide cross section, which we denote as W_n , is presentable via the modal amplitudes as follows:

$$W_n(\tau, \xi) = \frac{1}{2} [\mathcal{A}_n^2(\tau, \xi) + \mathcal{B}_n^2(\tau, \xi) + h_n^2(\tau, \xi)] \text{ [m}^{-2}]. \quad (22)$$

Derivation of the Poynting vector, \mathcal{S} , in terms of the field vectors \mathbb{E} and \mathbb{H} is

$$\mathcal{S}(\mathbf{r}, t) = [\mathcal{E} \times \mathcal{H}] = [\epsilon_0^V \mathbb{E} \times \mu_0^A \mathbb{H}] = c [\mathbb{E} \times \mathbb{H}] \text{ [Wm}^{-2}] \quad (23)$$

where $c = 1/\sqrt{\epsilon_0 \mu_0} = c_0 \text{ [ms}^{-1}]$ is the speed of light, and W is *watt* identical to *joule per second*, [W=Js⁻¹]. Introduce a new vector, \mathbb{S} , which is expressible via the field vectors, \mathbb{E} and \mathbb{H} , and have the same dimension as \mathbb{W} in Eq. (21), i.e.,

$$\mathbb{S}(\mathbf{r}, t) = \frac{1}{c} \mathcal{S}(\mathbf{r}, t) = [\mathbb{E} \times \mathbb{H}] \text{ [Nm}^{-2}]. \quad (24)$$

The vector \mathbb{S} averaged over the waveguide cross section is

$$\mathbb{S}_n(\tau, \xi) = \mathbf{z} \mathcal{A}_n(\tau, \xi) \mathcal{B}_n(\tau, \xi) \text{ [m}^{-2}] \quad (25)$$

where \mathbf{z} is the axial unit vector.

It had been proved by Umov that a velocity of transportation of energy in *any* wave process is a simple fraction between the power flow and the stored energy [20]. Accordingly to Poynting's theorem [21], velocity of transportation of the modal field energy is expressed below in terms of the standard field vectors and via the new field vectors, namely:

$$\mathcal{V}(\mathbf{r}, t) = \frac{\mathcal{S}(\mathbf{r}, t)}{\mathcal{W}(\mathbf{r}, t)} = \frac{2 [\mathcal{E} \times \mathcal{H}]}{\epsilon_0 \mathcal{E} \cdot \mathcal{E} + \mu_0 \mathcal{H} \cdot \mathcal{H}} \text{ [ms}^{-1}] \quad \text{and} \quad \mathbb{V}(\mathbf{r}, t) = c \frac{\mathbb{S}(\mathbf{r}, t)}{\mathbb{W}(\mathbf{r}, t)} = c \frac{2 [\mathbb{E} \times \mathbb{H}]}{\mathbb{E} \cdot \mathbb{E} + \mathbb{H} \cdot \mathbb{H}} \text{ [ms}^{-1}]. \quad (26)$$

In terms of the modal amplitudes, the velocity, \mathbf{v}_n , normalized by c is

$$\mathbf{v}_n(\tau, \xi) = \frac{1}{c} \mathbf{S}_n(\tau, \xi) / W_n(\tau, \xi) \leq 1. \quad (27)$$

In Figs. 2(a) and 2(b), the energetic properties for the TE-modes are exhibited when $\xi = 0$ for the lossless and lossy cases, respectively. Velocity, \mathbf{v}_n , is one-component vector colinear to the waveguide axis. Absolute value of the scaled by c velocity $|\mathbf{v}_n|$ varies in time provided that $|\mathbf{v}_n| \leq 1$ in accordance with Einstein [22].

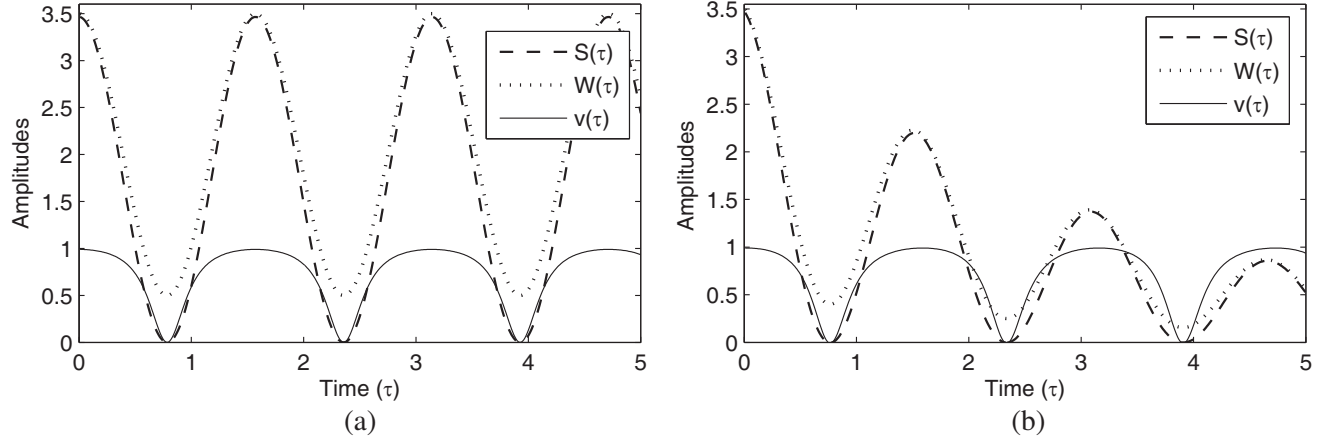


Figure 2. Time dependences of the energetic field properties and the velocity of transportation of energy for the cases: (a) in a lossless waveguide, (b) in a lossy waveguide.

Propagation of the modal waves is accompanied with an energetic wave process, in which exchange by energy occurs between the energy stored in the longitudinal field component, $w_n = \frac{1}{2} h_n^2$, and the quantity $dW_n = \frac{1}{2} (\mathcal{A}_n^2 - \mathcal{B}_n^2)$. The latter has physical sense of a *surplus* of energy stored in the transverse field components.

In Figs. 3(a) and 3(b), the energetic wave process and variation of velocity of transportation of the field energy are exhibited in the cross section $\xi=0$.

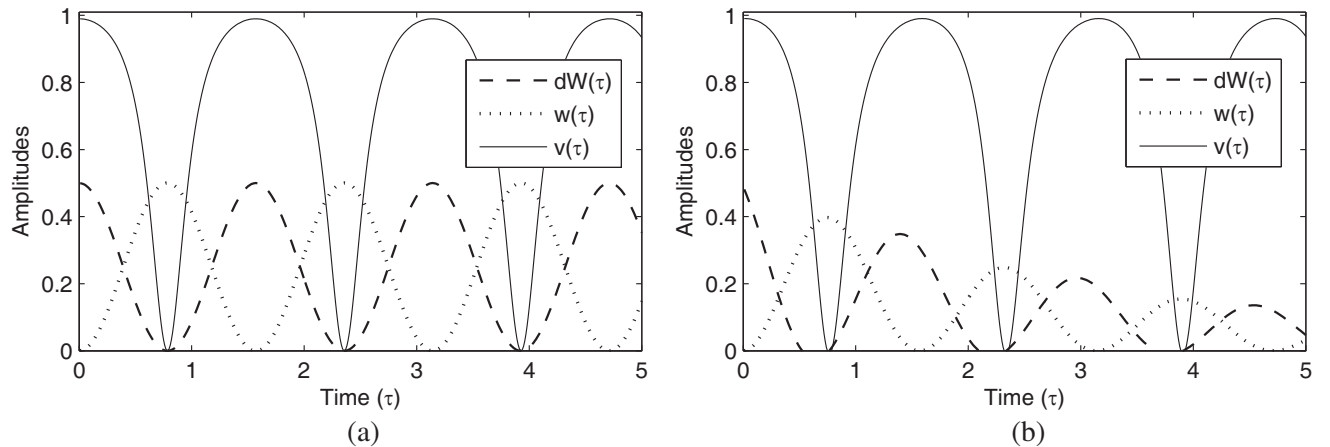


Figure 3. The energetic wave process and the velocity \mathbf{v}_n for the (a) lossless and the (b) lossy waveguides.

5. INERTIAL PROPERTIES OF THE TE WAVEGUIDE FIELDS

Notice first that *scalar* \mathcal{W} in Eq. (21) and *vector* \mathbb{S} in Eq. (24) have common dimension, i.e.,

$$\mathcal{W}(\mathbf{r}, t) = (1/2) (\mathbb{E} \cdot \mathbb{E} + \mathbb{H} \cdot \mathbb{H}) [\text{Nm}^{-2}] \quad \text{and} \quad \mathbb{S}(\mathbf{r}, t) = [\mathbb{E} \times \mathbb{H}] [\text{Nm}^{-2}] \quad (28)$$

Following Kaiser’s technique [5], compose two new *scalars* as follows:

$$\mathfrak{U}(\mathbf{r}, t) = \sqrt{\frac{1}{4} (\mathbb{E}^2 + \mathbb{H}^2)^2} = \sqrt{\frac{1}{4} (\mathbb{E}^4 + 2\mathbb{E}^2\mathbb{H}^2 + \mathbb{H}^4)} \quad (29a)$$

$$\mathfrak{J}(\mathbf{r}, t) = \sqrt{(\mathbb{E} \times \mathbb{H}) \cdot (\mathbb{E} \times \mathbb{H})} = \sqrt{\mathbb{E}^2\mathbb{H}^2 - (\mathbb{E} \cdot \mathbb{H})^2} \quad (29b)$$

where the dot product, $[\mathbb{E} \times \mathbb{H}] \cdot [\mathbb{E} \times \mathbb{H}]$, is found out by applying identity (B.8) in [23]. By [5], *reactive (rest) energy density* was defined by combination of \mathfrak{U} and \mathfrak{J} as

$$\mathbf{R}(\mathbf{r}, t) = \sqrt{\mathfrak{U}^2 - \mathfrak{J}^2} = \frac{1}{2} \sqrt{(\mathbb{E}^2 - \mathbb{H}^2)^2 + 4(\mathbb{E} \cdot \mathbb{H})^2} [\text{J/m}^3]. \quad (30)$$

In relativistic mechanics, the energy E of a particle in the state of rest is denoted as

$$E = c\sqrt{p^2 + m^2c^2}, \quad p = m\mathbf{v}/\sqrt{1 - v^2/c^2} \quad (31)$$

where m , p , and \mathbf{v} are mass, momentum, and velocity of the particle, respectively [24]. Inertial properties of the electromagnetic modal waves can be derived by making use of this statement. First, in the reference frame of rest, where $\mathbf{v} = \mathbf{0}$, Equation (31) yields $E = mc^2$, which is Einstein’s famous formula. And vice versa, as a measure of its inertia, the mass m of the particle is expressible via its energy as

$$m = E/c^2. \quad (32)$$

Electromagnetic inertia (in the reference frame of rest), m_{EM} , was specified by analogy via replacing the energy E in Eq. (32) by the electromagnetic energy R given in Eq. (30):

$$m_{\text{EM}} = \mathbf{R}(\mathbf{r}, t) / c^2 = \frac{1}{2c_0^2} \sqrt{(\mathbb{E}^2 - \mathbb{H}^2)^2 + 4(\mathbb{E} \cdot \mathbb{H})^2} [\text{kg/m}^3]. \quad (33)$$

Two important comments should be made at this point: **1)** the quantities $\mathbb{E}^2 - \mathbb{H}^2$ and $\mathbb{E} \cdot \mathbb{H}$ are invariants to the inertial reference frames, and **2)** in all the waveguide modes, \mathbb{E} and \mathbb{H} fields are orthogonal, and therefore $(\mathbb{E} \cdot \mathbb{H})=0$. So, according to Eqs. (33), (30), and the condition $(\mathbb{E} \cdot \mathbb{H})=0$, for the reference frame of rest as we have:

$$m_{\text{EM}} = \frac{1}{c^2} \mathbf{R}(\mathbf{r}, t) = \frac{1}{2c_0^2} |\mathbb{E}^2 - \mathbb{H}^2| [\text{J/m}^3]. \quad (34)$$

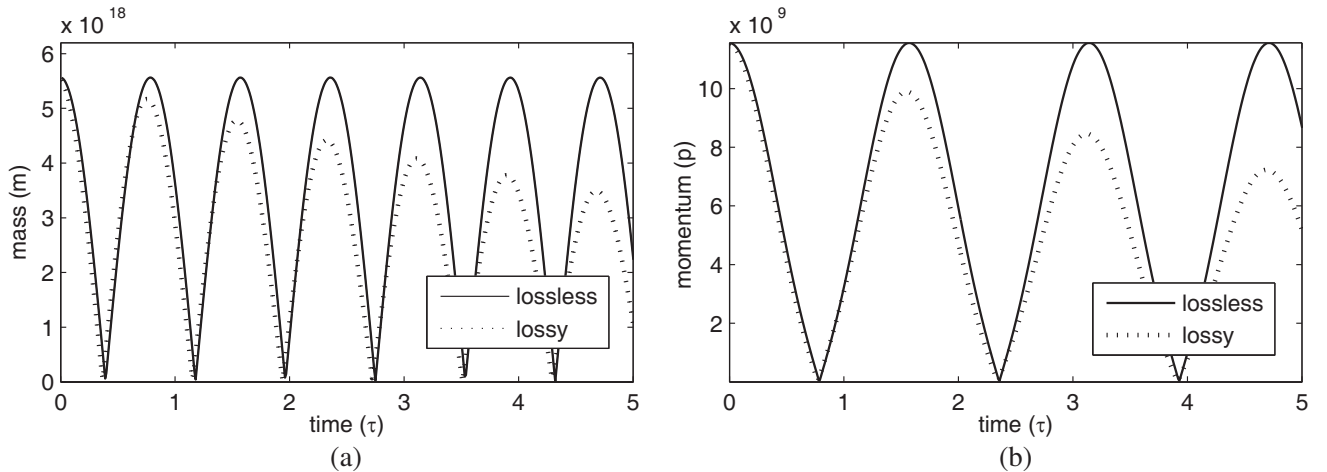


Figure 4. Time dependences of the (a) electromagnetic mass, and (b) electromagnetic momentum for the fields in a lossless and a lossy waveguide.

Landau specifies momentum as $p = E/c$ in formula (9.9) in [24], when the energy E of the particle is large compared to its rest energy, mc^2 . Replacing E by \mathfrak{J} (see Eq. (29b)) in $p = E/c$ yields the momentum of modal fields. The calculations for volumetric distribution of a *mechanical momentum of the waveguide modes* are presented in Eq. (35):

$$p = E/c = \mathfrak{J}/c \quad \text{and} \quad p_{EM} = \mathfrak{J}/c = \frac{1}{c_0} |\mathbb{E}| |\mathbb{H}| \left[\text{kg} \frac{\text{m}}{\text{s}} / \text{m}^3 \right]. \quad (35)$$

Graphical results for inertial properties, m_{EM} and p_{EM} , are exhibited herein in Figs. 4(a) and 4(b), provided that the dot products of the vectors (i.e., \mathbb{E}^2 and \mathbb{H}^2) are averaged over the waveguide cross section S .

6. CONCLUSION

1) Inertial field characteristics for the waveguide fields, that is, electromagnetic mass and momentum, are derived and presented as the real-valued functions varying in time within the framework of the *EAE* [7].

2) First, we solved the problem of propagation of the free waveguide modal waveforms for Equation (1) supplemented with the boundary conditions for the perfect electric conductor over the closed waveguide surface. This yields the vector modal fields, \mathbb{E} and \mathbb{H} , which have their common physical dimension in SI units.

3) Then, having the fields with their common dimension, we presented the energetic and inertial characteristics of the fields. Thus, it became possible to employ Kaiser's definitions [5] for the inertial characteristics of the electromagnetic fields.

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