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A Virtual Space-Frequency Matrix Method for Joint DOA-Frequency Estimation

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Abstract—The joint direction-of-arrival (DOA) and frequency estimation problem has received significant attention recently in some applications, including pulsed Doppler radar, multipath parameter estimation, etc. This paper presents a novel virtual space-frequency matrix method to estimate the DOA and frequency jointly. Via the temporal smoothing technique, a virtual space-frequency matrix is defined, which includes the information of the incident DOAs and frequencies. Making using of the proposed method, both the frequencies and DOAs can be estimated by eigenvalues and the corresponding eigenvectors of the new defined virtual space-frequency matrix, respectively. Therefore, the pairing of the estimated DOAs and frequencies is automatically determined. Compared with related works, the proposed method can provide superior performance, such as higher estimation accuracy, without the procedure of parameter search or parameter matching. Simulation results are presented to demonstrate the efficacy of the proposed approach.

1. INTRODUCTION

In wireless communications, the joint direction-of-arrival (DOA) and frequency estimation problem of multiple sources has received considerable attention in the field of array signal processing. Many high-resolution algorithms, such as MUSIC-like peak search-based algorithms [1, 2] and the algorithms [3–5] based on the shift-invariance structure of the array response matrix, have been developed. A two-dimensional (2-D) multiple signal classification (MUSIC)-based algorithm is presented in [1]. This method performs the joint DOA-frequency estimation via high dimensional eigen-decompositions of covariance matrices and 2-D search on the DOA-frequency plane, which results in enormous computations. To alleviate the computational overhead, an FSF-MUSIC method is presented [2], which describes a tree-structured frequency-space-frequency (FSF) MUSIC-based algorithm for the joint DOA and frequency estimation problem. In the presented method, the estimated DOAs and frequencies are automatically paired without extra processing. However, this method employs three 1-D MUSIC-type algorithms, i.e., two F-MUSICs and one S-MUSIC, which is computationally not very attractive since it requires multiple 1-D searches. In order to reduce effectively the computational complexity, several algorithms based on the shift-invariance structure have received much attention. For example, the MR-ESPRIT algorithm is proposed by extending the ESPRIT direction finding algorithm to antenna arrays with multiple baselines [3]. By taking advantage of the temporal smoothing, spatial smoothing, and forward-backward averaging techniques, a C-JAFE algorithm is addressed [4], which can achieve more accurate results than the previous ESPRIT-based techniques. Despite its high-resolution capability, the C-JAFE still involves the joint diagonalization processing apart from the singular value decomposition (SVD) or eigenvalue decomposition (EVD). It is well known that the joint diagonalization processing is a complex nonlinear optimization procedure. In addition, the C-JAFE algorithm needs an extra

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pairing procedure to match the separately estimated DOAs and frequencies. The parameter matching makes the joint DOA-frequency estimation more difficult to solve. In order to solve these problems, an effective Unitary-JAFE algorithm is presented [5], where the incoming DOAs and frequencies can be estimated by making use of the real and imaginary parts of the eigenvalues of the space-time factor matrix so that the pairing of the estimated frequency and DOA is automatically determined. Based on FSF-MUSIC, an FSF-ESPRIT algorithm is proposed [6], which is a hybrid of one-dimensional (1-D) ESPRIT and spatial/temporal filtering processes. In other words, two temporal and one spatial 1-D ESPRIT algorithms are employed alternatively to estimate the frequencies and the DOAs, respectively. The estimated frequencies and DOAs are automatically paired without extra computational overhead. A space-time matrix method is proposed [7]. The frequencies and incident angles can be estimated by the eigenvalues and the corresponding eigenvectors of the defined space-time matrix, respectively. Thus, the presented method can automatically determine the paring of the estimated angles and frequencies. However, it needs an extra noise power estimation procedure. By exploiting the multiple delay output, ESPRIT-based joint angle and frequency estimation algorithms are proposed to improve the estimation accuracy [8–10]. By combining the outputs of a uniform linear array (ULA) and delay network, a space-time-Euler-ESPRIT method is presented, which can provide automatically paired frequencies and their DOAs [11].

As stated before, most of the existing joint DOA and frequency estimation methods call for two-dimensional (2-D) search, or multiple 1-D searches, or multiple delay taps, and/or complex pair matching processing, etc. These facts make the joint DOA and frequency estimation problem more difficult to solve. In this paper, an effective virtual space-frequency matrix method is proposed. The 3-factor temporal smoothing technique is utilized to add a structure of the received data model for the implementation of the proposed method based on multiple antennas without delay taps. The proposed approach makes use of the covariance matrices with different virtual delay data to obtain the space-frequency matrix which can avoid the noise power estimation. The rest of the paper is organized as follows. The data model is described in Section 2. Section 3 introduces the presented method. Section 4 shows some simulation results. Finally, the conclusion is given in Section 5.

2. DATA MODEL

Consider a ULA of M elements equispaced by d. Employing the first element of the ULA as the phase reference, the array manifold can be written as

$$\mathbf{a}_M(f,\theta) = \left[1, e^{j\mu}, \ldots, e^{j(M-1)\mu}\right]^{\mathrm{T}} \tag{1}$$

where $\mu = (2\pi f/c)d\sin(\theta)$ with f and θ denoting the frequency and the DOA, respectively. c stands for the velocity of light. d is equal to the interelement spacing.

Suppose that there are q narrowband signals $s_i(t)$ $(i = 1, \ldots, q)$ simultaneously imping on the ULA with the ith source having a carrier frequency of f_i and a DOA θ_i. The signal received at the kth antenna is [4, 5]

$$x_k(t) = \sum_{i=1}^{q} a_k(f_i, \theta_i) e^{j2\pi f_i t} s_i(t) + w_k(t) \tag{2}$$

where $a_k(f_i, \theta_i)$ is the antenna response of the kth antenna to the signal from the direction θ_i, and $w_k(t)$ represents the output of the additive noise of the kth sensor.

The observed signals at the ULA can be expressed as

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{w}(t) \tag{3}$$

The matrices and the vectors in Eq. (1) have the following forms

$$\begin{aligned}
\mathbf{x}(t) &= [x_1(t), \ldots, x_M(t)]^{\mathrm{T}} \\
\mathbf{A} &= [\mathbf{a}_M(f_1, \theta_1), \ldots, \mathbf{a}_M(f_q, \theta_q)] \\
\mathbf{s}(t) &= [s_1(t), \ldots, s_q(t)]^{\mathrm{T}} \\
\mathbf{w}(t) &= [w_1(t), \ldots, w_M(t)]^{\mathrm{T}}
\end{aligned}$$

where $x_k(t)$ and $w_k(t)$ $(k = 1, \ldots, M)$ denote the output signal and the additive noise of the kth sensor, respectively. $\mathbf{a}_M(f_k, \theta_k)$ $(k = 1, \ldots, q)$ denotes the steering vector for the kth source, which is defined by Eq. (1). The superscript $(\cdot)^{\mathrm{T}}$ represents the transpose operation. Assume that $w_k(t)$ is a complex Gaussian random process with zero-mean and variance σ_n^2, and the noise $w_k(t)$ is uncorrelated with $s_i(t)$.

Under the above assumptions, it can be easily seen that $\mathbb{E}\{\mathbf{w}(t)\mathbf{w}^{\mathrm{H}}(t)\} = \sigma_n^2 \mathbf{I}_M$, where $\mathbb{E}\{\cdot\}$ represents the statistical average operation, and the superscript $(\cdot)^{\mathrm{H}}$ denotes the Hermitian operation. $\mathbf{I}_M$ is the $M \times M$ identity matrix.

Assume that P is the sample rate, which is much higher than the data rate of each source. The data samples at the receiver are

$$\mathbf{x}\left(\frac{n}{P}\right) = \sum_{i=1}^{q} \mathbf{a}\left(f_i, \theta_i\right) e^{j(2\pi/P)f_i n} s_i\left(\frac{n}{P}\right) + \mathbf{w}\left(\frac{n}{P}\right) \tag{4}$$

In matrix form, this can be written as

$$\mathbf{x}\left(\frac{n}{P}\right) = \mathbf{A}\mathbf{\Phi}^n \mathbf{s}\left(\frac{n}{P}\right) + \mathbf{w}\left(\frac{n}{P}\right) \tag{5}$$

where $\mathbf{\Phi} = \mathrm{diag}\{(\phi_1, \ldots, \phi_q)\}$ with $\phi_i = e^{j(2\pi/P)f_i}$ for $i = 1, \ldots, q$, which includes the frequency information of the incident signal sources, e.g., we refer to matrix $\mathbf{\Phi}$ as frequency factor matrix and its diagonal element as frequency factor.

Assume that we have collected N samples of the array output $\mathbf{x}(t)$ at a rate P into the $M \times N$ data matrix, i.e.,

$$\mathbf{X}\left(\frac{n}{P}\right) = \mathbf{A}\left[\mathbf{s}(0), \mathbf{\Phi}\mathbf{s}\left(\frac{1}{P}\right), \ldots, \mathbf{\Phi}^{N-1}\mathbf{s}\left(\frac{N-1}{P}\right)\right] + \mathbf{W} \tag{6}$$

where $\mathbf{W}$ is a matrix collecting N samples of the $M \times 1$ array noise vector.

3. ALGORITHM FORMULATION

To effectively estimate the DOAs and frequencies, a virtual space-time matrix method is proposed by the temporal smoothed technique. We begin the development by the 3-factor temporal smoothed technique for the original data matrix $\mathbf{X}(\frac{n}{P})$. This results in the following data matrix

$$\mathbf{X}_3 = \begin{bmatrix} \mathbf{A}\left[\mathbf{s}(0), \mathbf{\Phi}\mathbf{s}\left(\frac{1}{P}\right), \ldots, \mathbf{\Phi}^{N-3}\mathbf{s}\left(\frac{N-3}{P}\right)\right] \\ \mathbf{A}\mathbf{\Phi}\left[\mathbf{s}\left(\frac{1}{P}\right), \mathbf{\Phi}\mathbf{s}\left(\frac{2}{P}\right), \ldots, \mathbf{\Phi}^{N-2}\mathbf{s}\left(\frac{N-2}{P}\right)\right] \\ \mathbf{A}\mathbf{\Phi}^2\left[\mathbf{s}\left(\frac{2}{P}\right), \mathbf{\Phi}\mathbf{s}\left(\frac{3}{P}\right), \ldots, \mathbf{\Phi}^{N-1}\mathbf{s}\left(\frac{N-1}{P}\right)\right] \end{bmatrix} + \mathbf{W}_3 \tag{7}$$

where $\mathbf{W}_3$ represents the noise term constructed from $\mathbf{W}$ in a similar way as $\mathbf{X}_3$ obtained from $\mathbf{X}$.

Assume that the signals are narrow band, i.e., $s(t) \approx s(t+1/P) \approx s(t+2/P)$. In this case, $\mathbf{X}_3$ has the following factorization

$$\mathbf{X}_3 = \mathbf{A}_3\mathbf{F}_s + \mathbf{W}_3 \tag{8}$$

where $\mathbf{F}_s = \left[\mathbf{s}(0), \mathbf{\Phi}\mathbf{s}(\frac{1}{P}), \ldots, \mathbf{\Phi}^{N-3}\mathbf{s}(\frac{N-3}{P})\right]$. $\mathbf{A}_3$ is given by

$$\mathbf{A}_3 = \begin{bmatrix} \mathbf{A} \\ \mathbf{A}\mathbf{\Phi} \\ \mathbf{A}\mathbf{\Phi}^2 \end{bmatrix} \tag{9}$$

where $\mathbf{A} = [\mathbf{a}_M(f_1, \theta_1), \ldots, \mathbf{a}_M(f_q, \theta_q)]$. $\mathbf{\Phi} = \mathrm{diag}\{(\phi_1, \ldots, \phi_q)\}$ with $\phi_i = e^{j(2\pi/P)f_i}$ for $i = 1, \ldots, q$.

Define three selection matrices $\mathbf{J}_k$ $(k = 1, 2, 3)$ as follows

$$\begin{cases} \mathbf{J}_1 = [\mathbf{I}_M, \mathbf{0}_{M\times 2M}] \\ \mathbf{J}_2 = [\mathbf{0}_{M\times M}, \mathbf{I}_M, \mathbf{0}_{M\times M}] \\ \mathbf{J}_3 = [\mathbf{0}_{M\times 2M}, \mathbf{I}_M] \end{cases} \tag{10}$$

where $\mathbf{0}_{m\times n}$ stands for the the $m \times n$ zero matrix.

Making use of the defined selection matrices in Eq. (10), we can obtain the following equations:

$$\begin{cases} \mathbf{Y}_1 = \mathbf{J}_1\mathbf{X}_3 = \mathbf{A}\mathbf{F}_s + \mathbf{J}_1\mathbf{W}_3 \\ \mathbf{Y}_2 = \mathbf{J}_2\mathbf{X}_3 = \mathbf{A}\mathbf{\Phi}\mathbf{F}_s + \mathbf{J}_2\mathbf{W}_3 \\ \mathbf{Y}_3 = \mathbf{J}_3\mathbf{X}_3 = \mathbf{A}\mathbf{\Phi}^2\mathbf{F}_s + \mathbf{J}_3\mathbf{W}_3 \end{cases} \tag{11}$$

then, it can be easily seen that

$$\begin{cases} \mathbf{R}_{21} = \mathbb{E}\left\{\mathbf{Y}_2\mathbf{Y}_1^{\mathrm{H}}\right\} = \mathbf{A}\mathbf{\Phi}\mathbf{R}_s\mathbf{A}^{\mathrm{H}} \\ \mathbf{R}_{31} = \mathbb{E}\left\{\mathbf{Y}_3\mathbf{Y}_1^{\mathrm{H}}\right\} = \mathbf{A}\mathbf{\Phi}^2\mathbf{R}_s\mathbf{A}^{\mathrm{H}} \end{cases} \tag{12}$$

where the matrix $\mathbf{R}_s = \mathbb{E}\{\mathbf{F}_s\mathbf{F}_s^{\mathrm{H}}\} = \mathrm{diag}\{(\sigma_1^2, \ldots, \sigma_q^2)\}$, in which σ_k^2 $(k = 1, \ldots, q)$ is the average signal power of the kth signal.

Making use of $\mathbf{R}_{21}$ and $\mathbf{R}_{31}$, a virtual space-time matrix $\mathbf{R}$ is defined as follows

$$\mathbf{R} = \mathbf{R}_{31}\mathbf{R}_{21}^{\dagger} \tag{13}$$

where the superscript $(\cdot)^{\dagger}$ denotes matrix pseudo inverse.

It is clear that the space-time matrix $\mathbf{R}$ includes the information of the DOAs and frequencies for all incoming signals. Therefore, we have the following Theorem 1.

Theorem 1. Assume that there are q narrow-band sources, with the complex baseband representations $s_k(t)$ $(1 \le k \le q)$ such that the kth source has a carrier frequency of f_k and arrives a ULA from direction θ_k. If there are no same elements on the diagonal of matrix $\mathbf{\Phi}$, and $\mathbf{R}_s$ is the a full rank matrix, then, the q nonzero eigenvalues of $\mathbf{R}$ are equal to the q elements on the diagonal of matrix $\mathbf{\Phi}$, and the corresponding eigenvectors are equal to the corresponding column vectors of $\mathbf{A}\mathbf{\Phi}$, namely, $\mathbf{R}\mathbf{A}\mathbf{\Phi} = \mathbf{A}\mathbf{\Phi}^2$.

Proof: Under the above assumptions, it is easy to know that $\mathbf{A}$ is a full rank matrix. Furthermore, we can draw a conclusion that $\mathrm{rank}(\mathbf{R}_{21}) = \mathrm{rank}(\mathbf{R}_{31}) = \mathrm{rank}(\mathbf{A}) = \mathrm{rank}(\mathbf{R}_s) = q$. Thus, we have the following equation

$$\mathbf{R}_{21}^{\dagger} = \mathbf{A}\mathbf{R}_s\left(\mathbf{R}_s\mathbf{A}^{\mathrm{H}}\mathbf{A}\mathbf{R}_s\right)^{-1}\left(\mathbf{\Phi}^{\mathrm{H}}\mathbf{A}^{\mathrm{H}}\mathbf{A}\mathbf{\Phi}\right)^{-1}\mathbf{\Phi}^{\mathrm{H}}\mathbf{A}^{\mathrm{H}} \tag{14}$$

where $(\cdot)^{-1}$ denotes matrix inverse.

From Eqs. (12)–(14), the following equation can be obtained

$$\begin{aligned} \mathbf{R}\mathbf{A}\mathbf{\Phi} &= \mathbf{R}_{31}\mathbf{R}_{21}^{\dagger}\boldsymbol{A}\mathbf{\Phi} = \left(\mathbf{A}\mathbf{\Phi}^2\mathbf{R}_s\mathbf{A}^{\mathrm{H}}\right)\left(\mathbf{A}\mathbf{R}_s\left(\mathbf{R}_s\mathbf{A}^{\mathrm{H}}\mathbf{A}\mathbf{R}_s\right)^{-1}\left(\mathbf{\Phi}^{\mathrm{H}}\mathbf{A}^{\mathrm{H}}\mathbf{A}\mathbf{\Phi}\right)^{-1}\mathbf{\Phi}^{\mathrm{H}}\mathbf{A}^{\mathrm{H}}\right)\mathbf{A}\mathbf{\Phi} \\ &= \left(\mathbf{A}\mathbf{\Phi}^2\right)\left(\mathbf{R}_s\mathbf{A}^{\mathrm{H}}\mathbf{A}\mathbf{R}_s\right)\left(\mathbf{R}_s\mathbf{A}^{\mathrm{H}}\mathbf{A}\mathbf{R}_s\right)^{-1}\left(\mathbf{\Phi}^{\mathrm{H}}\mathbf{A}^{\mathrm{H}}\mathbf{A}\mathbf{\Phi}\right)^{-1}\left(\mathbf{\Phi}^{\mathrm{H}}\mathbf{A}^{\mathrm{H}}\mathbf{A}\mathbf{\Phi}\right) \\ &= \mathbf{A}\mathbf{\Phi}^2. \end{aligned} \tag{15}$$

This concludes the proof. ■

Remarks:

(1) From Theorem 1, it is easily seen that the array steering matrix $\mathbf{A}$ and diagonal matrix $\mathbf{\Phi}$ can be obtained by computing the eigendecomposition of the space-time matrix $\mathbf{R}$. Then the carrier frequency f_k and incoming DOA θ_k can be estimated by making use of the kth eigen-pair (λ_k, η_k) of the matrix $\mathbf{R}$, that is, the paring of the estimated two-dimensional parameters is automatically determined.

(2) If several sources are close in the angle of incidence θ or the range f, while there are no same elements on the diagonal of matrix $\mathbf{\Phi}$, then Theorem 1 is still true. In other words, it can resolve the incoming rays with very close DOAs or very close frequencies under the aforementioned conditional restriction.

The procedure of the proposed method is concluded as follows.

(1) Collect the data matrix $\mathbf{X}$ and construct the 3-factor temporal smoothed matrix $\mathbf{X}_3$, according to (7).

(2) Calculate $\mathbf{R}_{21}$ and $\mathbf{R}_{31}$ according to (12).

(3) Compute the eigen-pairs (λ_k, η_k) of the space-time matrix $\mathbf{R}$ given by (13), where λ_k is the kth eigenvalue of $\mathbf{R}$ and η_k is the corresponding eigenvector, for $k = 1, \ldots, q$.

(4) According to the eigen-pairs (λ_k, η_k), estimate frequency f_k and DOA θ_k as follows

$$\begin{cases} \hat{f}_k = \text{angle}(\lambda_k) \times P/(2\pi) \\ \hat{\theta}_k = \sin^{-1}\left(\text{angle}(\eta_{k2}/\lambda_k) \times c/(2\pi d \hat{f}_k)\right) \end{cases} \tag{16}$$

where η_{k2} is the second element of η_k.

4. SIMULATION RESULTS

In this section, several simulation results are provided to illustrate the performance of the proposed method. Consider a ULA with $M = 7$ antenna elements and equispaced by $d = 30\,\text{m}$. Assume that there are four far-field equal power signals $s_1(t)$, $s_2(t)$, $s_3(t)$, $s_4(t)$ impinging on the antenna array. Their DOAs are $\theta_1 = -60°$, $\theta_2 = -30°$, $\theta_3 = 30°$, $\theta_4 = 0°$, respectively. Their corresponding center frequencies are $f_1 = 5\,\text{MHz}$, $f_2 = 1\,\text{MHz}$, $f_3 = 2\,\text{MHz}$, $f_4 = 1.2\,\text{MHz}$, respectively. The source signals are narrowband (25 kHz) amplitude-modulated signals. The sampling rate is set to 15 MHz.

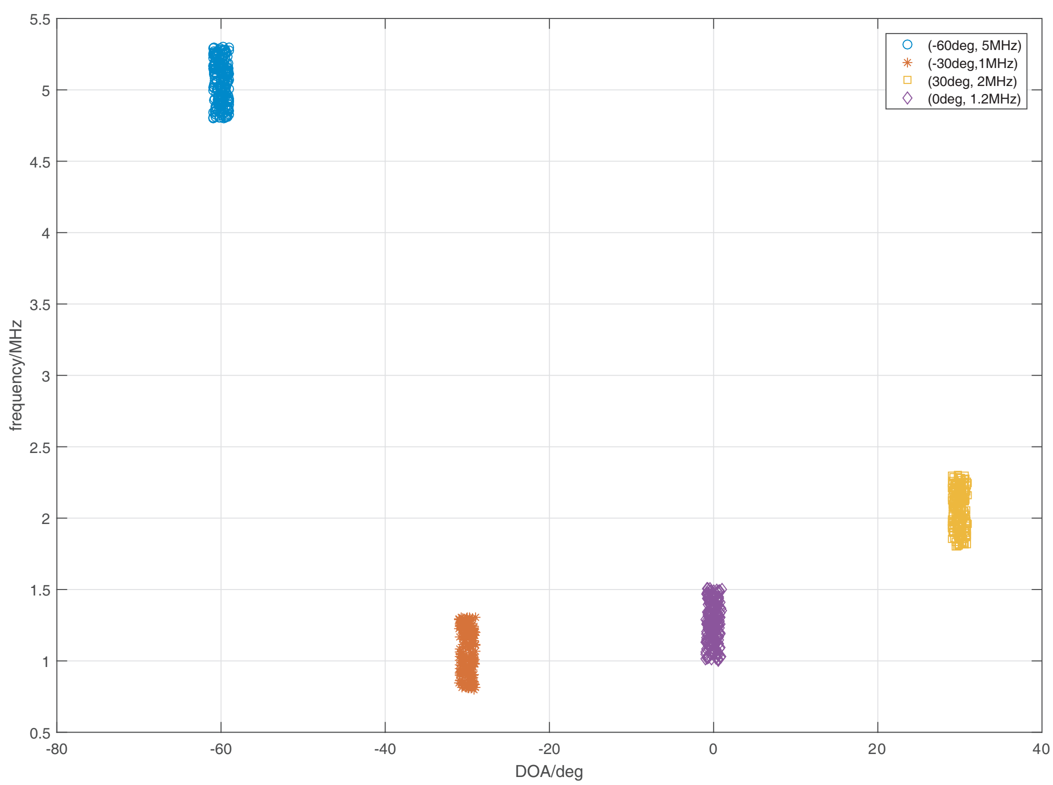

Figure 1. The scatter of the estimates based on STM.

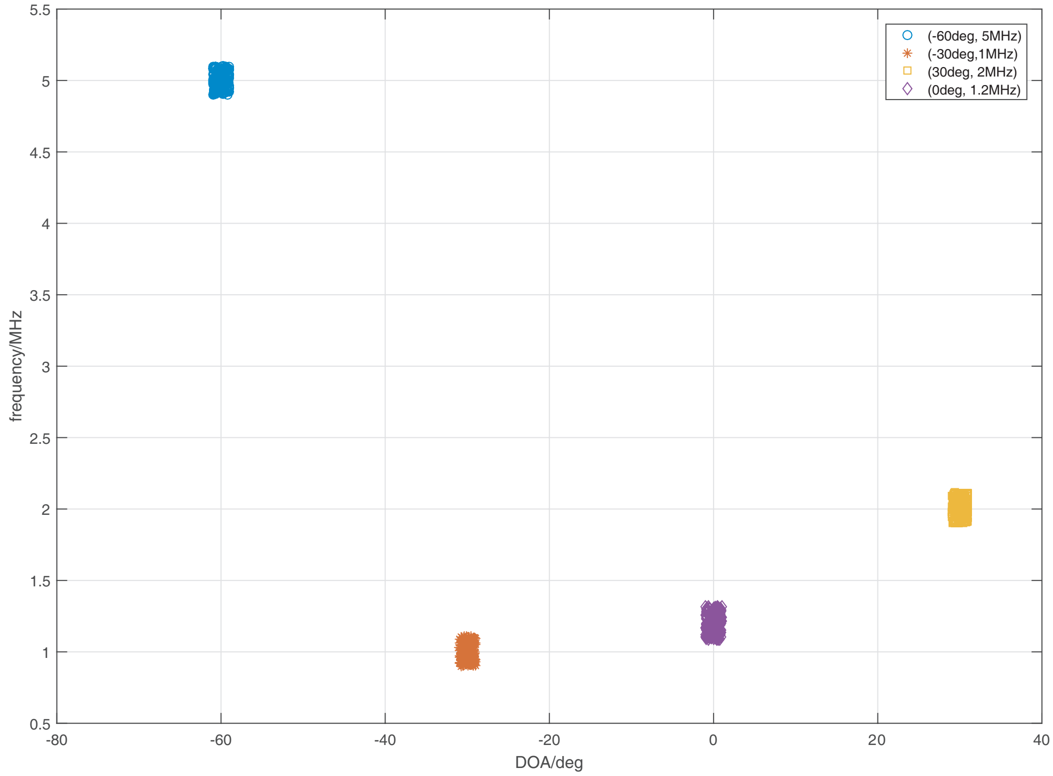

Figure 2. The scatter of the estimates based on VSTM.

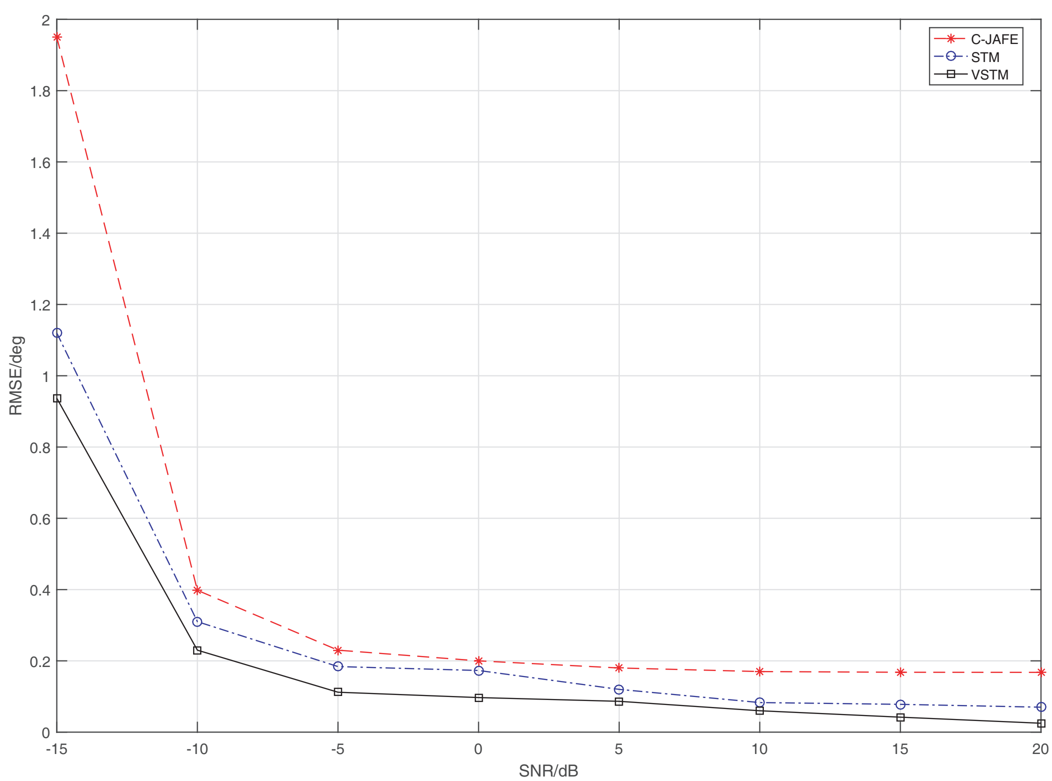

Figure 3. RMSE curves for DOA estimation.

We use root-mean-square-error (RMSE), which is defined as follows

$$\begin{cases} \mathrm{RMSE}_f = \sqrt{\mathbb{E}\left\{\sum_{k=1}^{q}\left(f_k - \hat{f}_k\right)^2\right\}} \\ \mathrm{RMSE}_\theta = \sqrt{\mathbb{E}\left\{\sum_{k=1}^{q}\left(\theta_k - \hat{\theta}_k\right)^2\right\}} \end{cases} \tag{17}$$

where $\hat{f}_k$ and $\hat{\theta}_k$ are the estimates of f_k and θ_k, for $k = 1, 2, \ldots, q$.

For comparison, three algorithms are carried out, including the proposed method, C-JAFE [4], the space-time matrix method [7]. For convenience, the proposed method and the space-time matrix method are named as VSTM and STM, respectively.

Figures 1 and 2 illustrate the scattergrams for joint DOA and frequency estimated by STM and VSTM, respectively, based on 200 independent trials under the hypothesis that signal-to-noise ratio (SNR) is equal to 5 dB. From these figures, it is clear that the proposed method provides a more precise DOA and frequency estimate than the STM.

Figures 3 and 4 give RMSE_θ and RMSE_f curves (which are computed by (17)) of the aforementioned algorithms, respectively, based on 200 independent trials under the hypothesis that SNRs range from -15 dB to 20 dB, and the number of snapshots is assigned to $N = 128$. From these figures, it is clear that the proposed VSTM outperforms C-JAFE and STM for DOA and frequency estimation. This is because the proposed virtual space-frequency matrix is estimated by using only the covariance matrices without extra noise power estimation or joint diagonalization.

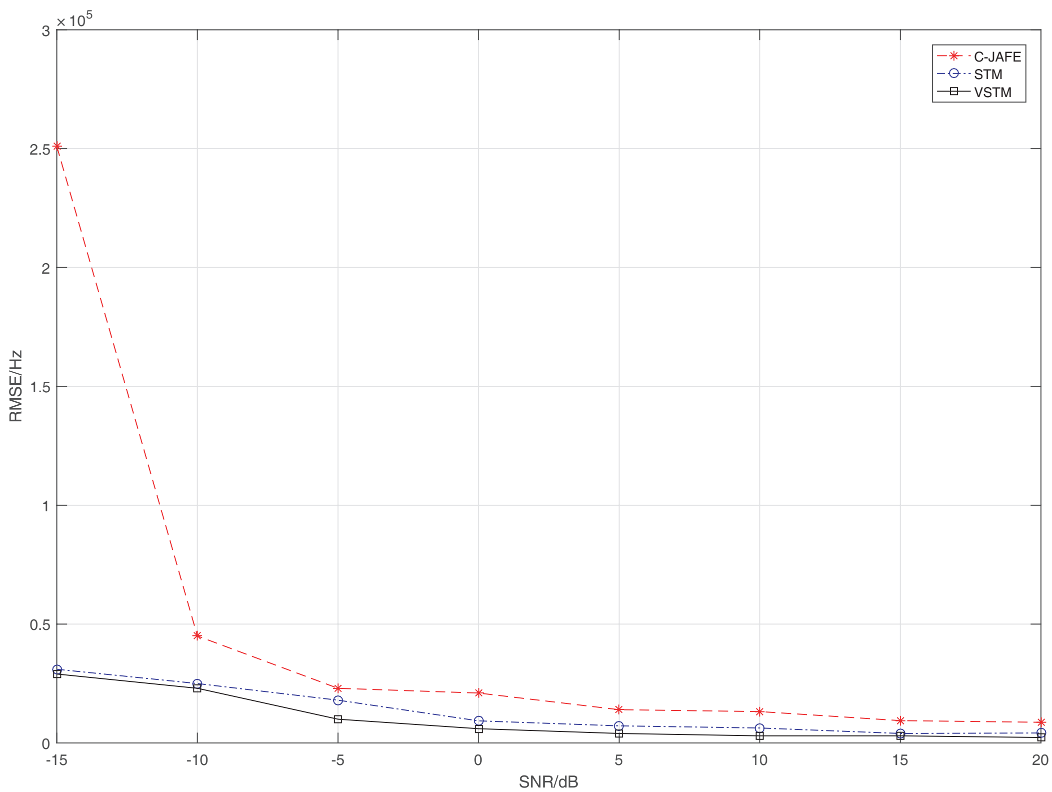

Figure 4. RMSE curves for frequency estimation.

5. CONCLUSIONS

In this paper, we present a virtual space-frequency matrix method for the joint DOA and frequency estimation problem. The temporal smoothing technique is utilized to add the structure of the received data model such that a virtual space-frequency matrix can be formed without delay taps. Additionally, the proposed approach makes use of the eigenvalues and the corresponding eigenvectors to estimate the frequencies and the DOAs, so that the pairing of the estimated DOAs and frequencies can be automatically determined. Simulation results show that the proposed method has a better performance than C-JAFE and STM.

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