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Theory of a Strip Antenna Located at the Interface of an Isotropic Medium and a Magnetoplasma

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Abstract—A study is made of the electrodynamic characteristics of an antenna having the form of a perfectly conducting, infinitesimally thin, narrow strip located at a plane interface of an isotropic medium and a cold collisionless magnetoplasma. The antenna is perpendicular to an external static magnetic field superimposed on the plasma medium and is excited by a time-harmonic given voltage. Singular integral equations for the antenna current are obtained in the case of an infinitely long strip conductor. Based on the solution of these equations, the current distribution and input impedance of the antenna are found for nonresonant and resonant frequency ranges of the magnetoplasma. The limits of applicability of an approximate approach employing the transmission line theory for determining the antenna characteristics are established. Within the framework of this approach, the results obtained are generalized to the case of a finite-length strip antenna.

1. INTRODUCTION

Electrodynamic characteristics of metal antennas in a magnetoplasma have been studied in many papers (see, e.g., [1–16] and references therein). The interest in the subject is stipulated by the wide use of such transmitters in various experiments in laboratory and space plasmas [17–22], and is stimulated continuously by the needs of practical applications such as diagnostics of plasma media, space communication, etc. In earlier theoretical works, antennas with given currents in a homogeneous magnetoplasma have been considered [1–7]. Such an approach is applicable to electrically small sources. For antennas with arbitrary sizes, the problem of finding the actual electromagnetic characteristics requires knowledge of the current distribution along the antenna wire. This problem, which turns out to be very difficult even for the simplest antennas operated in magnetized plasmas, has been solved only for some canonical antenna geometries [9–16]. Recently, increased attention has been paid to the features of excitation and propagation of electromagnetic waves in the presence of cylindrical magnetized plasma structures [23–26], and new results for loop antennas located on the surface of such structures have been obtained using the integral equation method [27, 28]. In particular, the current distribution and input impedance of a circular loop antenna located on the surface of an axially magnetized plasma column in a homogeneous dielectric medium have been found. It has been shown that the presence of a plasma column can lead to significant changes in the electrodynamic characteristics of the antenna compared with the cases of its operation in a homogeneous dielectric or plasma medium with the corresponding parameters.

Of no less interest is the problem of finding the characteristics of antennas located at a plane interface of a magnetoplasma and an isotropic medium. In particular, this problem is especially

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topical for the development of plasma diagnostic methods that employ waves guided by planar dielectric structures in a plasma medium [29, 30]. Although an attempt has recently been made towards the theory of an antenna located at the interface of such media in [31], the analysis of that work is restricted to the special case of a rather dense resonant magnetoplasma, and is inapplicable for arbitrary plasma parameters. It is the purpose of this work to generalize the approach of [31] to the case where the magnetoplasma on one side of the interface between two media may have arbitrary parameters.

In this article, using the integral equation method, we solve a model problem of the current distribution and input impedance of a strip antenna that is perpendicular to an external static magnetic field and located at a plane interface of an isotropic medium and a cold collisionless magnetoplasma. We study the antenna characteristics in the cases of both a nonresonant and resonant magnetoplasma. Recall that the magnetoplasma is nonresonant if the diagonal elements of its dielectric tensor have identical signs, and is resonant otherwise [9–11]. As is known, the refractive index surfaces of the propagating normal waves of a nonresonant magnetoplasma are closed. On the contrary, for a resonant magnetoplasma, the refractive index of one of the normal waves goes to infinity at a certain angle between the wave vector and the direction of the external magnetic field [11]. It will be shown in what follows that the antenna characteristics are essentially different in these two cases.

Our article is organized as follows. In Section 2, we present the formulation of the problem. In Section 3, we describe the salient steps of the derivation of integral equations for the antenna current. Section 4 deals with the solution of these integral equations. In Section 5, we give analytical and numerical results for the electrodynamic characteristics of an infinitely long antenna and discuss generalization of these results to the case of a finite-length antenna. Section 6 presents conclusions following from the performed analysis.

2. FORMULATION OF THE PROBLEM

Consider an infinitely long antenna, which is oriented along the x axis of a Cartesian coordinate system and has the form of a perfectly conducting, infinitesimally thin, narrow strip of width $2d$ lying in the xz plane (see Figure 1). It is assumed that this plane coincides with the interface of a magnetoplasma and an isotropic medium. The external static magnetic field $\mathbf{B}_0$ is aligned with the z axis. The half-space $y < 0$ is filled with a homogeneous cold collisionless magnetoplasma, whose dielectric permittivity tensor has the form

$$\boldsymbol{\varepsilon} = \epsilon_0 \begin{pmatrix} \varepsilon & -ig & 0 \\ ig & \varepsilon & 0 \\ 0 & 0 & \eta \end{pmatrix}, \tag{1}$$

where ϵ_0 is the permittivity of free space. Expressions for the elements of the tensor in Equation (1) are given elsewhere [32].

Homogeneous medium in the half-space $y > 0$ is isotropic and has a dielectric permittivity $\tilde{\varepsilon}_a = \epsilon_0 \varepsilon_a$. In the case where the medium in the region $y > 0$ is free space, one should put $\varepsilon_a = 1$.

The current of the antenna is excited by a time-harmonic ($\sim \exp(i\omega t)$) given voltage that creates an electric field with the component E_x^{ext}, which is nonzero for $y = 0$ and $|z| < d$ in a narrow gap $|x| \leq \Delta$:

$$E_x^{\text{ext}}(x, 0, z) = \frac{V_0}{2\Delta} \left[U(x + \Delta) - U(x - \Delta) \right] \left[U(z + d) - U(z - d) \right]. \tag{2}$$

Here, V_0 is the complex amplitude of the voltage applied to the gap, Δ is the gap half-width, and U is a Heaviside function. The distribution of E_x^{ext} for $|z| < d$ can be represented by the Fourier integral

$$E_x^{\text{ext}}(x, 0, z) = \frac{k_0}{2\pi} \int_{-\infty}^{\infty} V(n_x) \exp(-ik_0 n_x x) dn_x, \tag{3}$$

where

$$V(n_x) = V_0 \frac{\sin(k_0 n_x \Delta)}{k_0 n_x \Delta} \tag{4}$$

and k_0 is the wave number in free space.

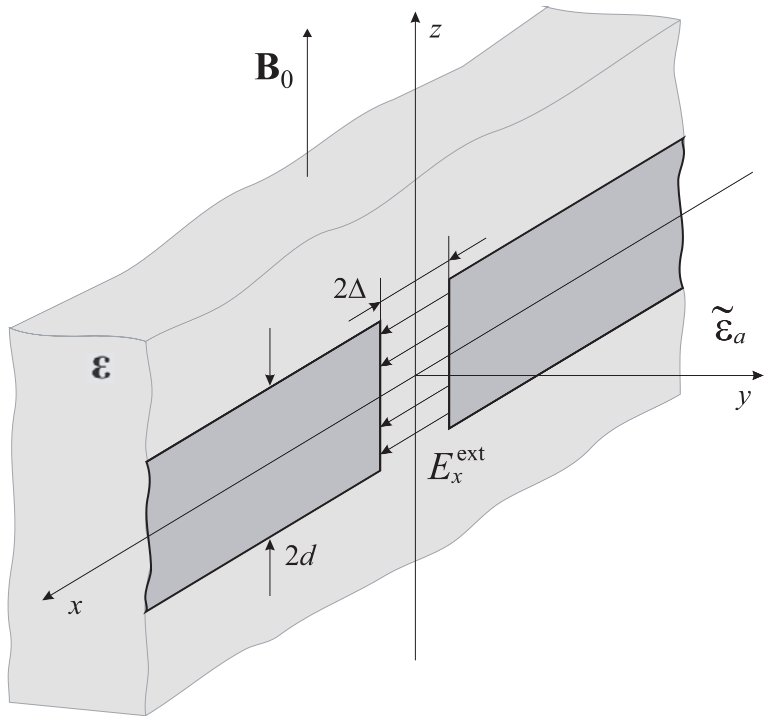

Figure 1. Geometry of the problem.

The density **J** of the electric current excited in the antenna by an external field that is given by Equation (3) can be sought in the form

$$\mathbf{J} = \mathbf{x}_0 I(x, z)\delta(y), \tag{5}$$

where $|z| < d$ and $\delta(y)$ is a Dirac function. The surface density $I(x, z)$ of the current admits the following representation:

$$I(x, z) = \frac{k_0}{2\pi} \int_{-\infty}^{\infty} \mathcal{I}(n_x, z) \exp(-ik_0 n_x x) dn_x. \tag{6}$$

To find the distribution $I(x, z)$, we express the tangential components E_x and E_z of the electric field excited by current (5) via the Fourier transform $\mathcal{I}(n_x, z)$ of the surface current density and take into account boundary conditions for the field components at the interface $y = 0$. In addition, we make use of boundary conditions for the tangential components of the electric field on the antenna surface ($y = 0$ and $|z| < d$):

$$E_x + E_x^{\text{ext}} = 0, \tag{7}$$

$$E_z = 0. \tag{8}$$

The above-described procedure yields integral equations for the unknown quantity $\mathcal{I}(n_x, z)$ and thus reduces the problem of determining the antenna current to the solution of the corresponding integral equations.

3. INTEGRAL EQUATIONS FOR THE ANTENNA CURRENT

Since the procedure of the derivation of integral equations for the antenna current was discussed in earlier work [31], we here describe only briefly the salient steps of this derivation and introduce notations that will be used in the further analysis generalizing the results of that work. We represent the antenna-excited field in the form

$$\begin{bmatrix} \mathbf{E}(x, y, z) \\ \mathbf{H}(x, y, z) \end{bmatrix} = \frac{k_0^2}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \begin{bmatrix} \mathbf{E}(n_x, y, n_z) \\ \mathbf{H}(n_x, y, n_z) \end{bmatrix} \exp[-ik_0(n_x x + n_z z)] dn_x dn_z. \tag{9}$$

It is a straightforward matter to show from the Maxwell equations that the quantities $E_{x,y}(n_x, y, n_z)$ and $H_{x,y}(n_x, y, n_z)$ can be expressed via the components $E_z(n_x, y, n_z)$ and $H_z(n_x, y, n_z)$, which satisfy the following system of equations in the region $y < 0$ [31]:

$$\frac{\partial^2 E_z}{\partial y^2} + k_0^2 \left(\eta - n_x^2 - \frac{\eta}{\varepsilon} n_z^2\right) E_z = -ik_0^2 \frac{g}{\varepsilon} n_z Z_0 H_z, \tag{10}$$

$$\frac{\partial^2 H_z}{\partial y^2} + k_0^2 \left(\frac{\varepsilon^2 - g^2}{\varepsilon} - n_x^2 - n_z^2\right) H_z = ik_0^2 \frac{g}{\varepsilon} \eta n_z Z_0^{-1} E_z, \tag{11}$$

where Z_0 is the impedance of free space. For the region $y > 0$, one should put $\varepsilon = \eta = \varepsilon_a$ and $g = 0$ in Equations (10) and (11). Solutions for the fields must satisfy the radiation condition at infinity ($|y| \to \infty$), as well as the following boundary conditions for the tangential field components at the interface $y = 0$:

$$\begin{aligned} &E_x(n_x, y-0, n_z) = E_x(n_x, y+0, n_z), \quad E_z(n_x, y-0, n_z) = E_z(n_x, y+0, n_z), \\ &H_x(n_x, y-0, n_z) = H_x(n_x, y+0, n_z), \quad H_z(n_x, y-0, n_z) = H_z(n_x, y+0, n_z) - \mathcal{I}(n_x, n_z), \end{aligned} \tag{12}$$

where

$$\mathcal{I}(n_x, n_z) = \int_{-d}^{d} \mathcal{I}(n_x, z') \exp(ik_0 n_z z') dz'. \tag{13}$$

It is seen from Equations (12) and (13) that the field components E_x, E_z, and H_x are continuous at the interface, whereas the component H_z is continuous at $y = 0$ for $|z| > d$ and undergoes a jump corresponding to surface current (6) for $|z| < d$.

Upon solution of Equations (10) and (11), the Fourier-transformed tangential field components are written as

$$\begin{aligned} E_x(n_x, y, n_z) &= i \sum_{k=1}^{2} B_k \frac{\alpha_k n_x + i n_{y,k}}{n_{\perp k}^2} \exp(ik_0 n_{y,k} y), \\ E_z(n_x, y, n_z) &= i\eta^{-1} \sum_{k=1}^{2} B_k n_k \exp(ik_0 n_{y,k} y), \\ H_x(n_x, y, n_z) &= Z_0^{-1} \sum_{k=1}^{2} B_k n_k \frac{\beta_k n_x - i n_{y,k}}{n_{\perp k}^2} \exp(ik_0 n_{y,k} y), \\ H_z(n_x, y, n_z) &= -Z_0^{-1} \sum_{k=1}^{2} B_k \exp(ik_0 n_{y,k} y) \end{aligned} \tag{14}$$

for $y < 0$, and as

$$\begin{aligned} E_x(n_x, y, n_z) &= -\frac{1}{n_\perp^2} (C_1 n_x n_z + C_2 n_y) \exp(-ik_0 n_y y), \\ E_z(n_x, y, n_z) &= C_1 \exp(-ik_0 n_y y), \\ H_x(n_x, y, n_z) &= \frac{1}{Z_0 n_\perp^2} (C_1 \varepsilon_a n_y - C_2 n_x n_z) \exp(-ik_0 n_y y), \\ H_z(n_x, y, n_z) &= Z_0^{-1} C_2 \exp(-ik_0 n_y y) \end{aligned} \tag{15}$$

for $y > 0$. Here, B_k and C_k are the coefficients determined using boundary conditions (12), while the other quantities are given by the formulas

$$\begin{aligned} n_{\perp k}^2 = (2\varepsilon)^{-1} \Big\{ &\varepsilon^2 - g^2 + \varepsilon\eta - (\eta + \varepsilon) n_z^2 + (-1)^k \big[(\eta - \varepsilon)^2 n_z^4 \\ &+ 2(g^2(\eta + \varepsilon) - \varepsilon(\eta - \varepsilon)^2) n_z^2 + (\varepsilon^2 - g^2 - \varepsilon\eta)^2 \big]^{1/2} \Big\}, \end{aligned}$$

$$
\begin{aligned}
n_k &= -\frac{\varepsilon}{n_z g}\left[n_z^2 + n_{\perp k}^2(n_z) + \frac{g^2}{\varepsilon} - \varepsilon\right], \quad n_{y,k} = \left[n_{\perp k}^2(n_z) - n_x^2\right]^{1/2}, \\
\alpha_k &= \left[n_z^2 + n_{\perp k}^2(n_z) - \varepsilon\right] g^{-1}, \quad \beta_k = n_z n_k^{-1}, \quad k = 1, 2, \\
n_\perp^2 &= \varepsilon_a - n_z^2, \quad n_y = (\varepsilon_a - n_x^2 - n_z^2)^{1/2}.
\end{aligned} \tag{16}
$$

In order to ensure the fulfillment of the radiation condition at infinity, i.e., at $|y| \to \infty$, the branches of the functions $n_{y,k}$ and n_y in Equations (14) and (15) should be chosen so as to satisfy the inequalities $\operatorname{Im} n_{y,k} < 0$ and $\operatorname{Im} n_y < 0$. If the left-hand side of either of these inequalities vanishes, one should introduce a minor loss in the corresponding medium and, upon choosing the appropriate branch, go over to the limiting case of a loss-free medium.

After some algebra, we arrive at the following expressions for the tangential electric-field components $E_x(x, y, z)$ and $E_z(x, y, z)$ at the interface $y = 0$:

$$
E_x(x, 0, z) = \frac{k_0}{2\pi}\int_{-\infty}^{\infty} dn_x \int_{-d}^{d} K_x(n_x, z - z')\mathcal{I}(n_x, z')\exp(-ik_0 n_x x)dz', \tag{17}
$$

$$
E_z(x, 0, z) = \frac{k_0}{2\pi}\int_{-\infty}^{\infty} dn_x \int_{-d}^{d} K_z(n_x, z - z')\mathcal{I}(n_x, z')\exp(-ik_0 n_x x)dz'. \tag{18}
$$

Here,

$$
K_x(n_x, \zeta) = \frac{iZ_0 k_0}{2\pi}\int_{-\infty}^{\infty}\sum_{k=1}^{2}\frac{e_k \tilde{B}_k}{D}\exp(-ik_0 n_z|\zeta|)dn_z, \tag{19}
$$

$$
K_z(n_x, \zeta) = \operatorname{sgn}\zeta\,\frac{iZ_0 k_0}{2\pi\eta}\int_{-\infty}^{\infty}\sum_{k=1}^{2}\frac{n_k \tilde{B}_k}{D}\exp(-ik_0 n_z|\zeta|)dn_z, \tag{20}
$$

where $\zeta = z - z'$. The coefficients $\tilde{B}_{1,2}$ and other quantities in Equations (19) and (20) are determined by the expressions

$$
\begin{aligned}
\tilde{B}_1 &= -e_2\frac{\eta}{\varepsilon_a}\frac{n_x n_z}{n_\perp^2} + ih_2\frac{\eta}{\varepsilon_a}\frac{n_2 n_y}{n_\perp^2} - n_2\frac{\varepsilon_a - n_x^2}{\varepsilon_a n_\perp^2}, \\
\tilde{B}_2 &= e_1\frac{\eta}{\varepsilon_a}\frac{n_x n_z}{n_\perp^2} - ih_1\frac{\eta}{\varepsilon_a}\frac{n_1 n_y}{n_\perp^2} + n_1\frac{\varepsilon_a - n_x^2}{\varepsilon_a n_\perp^2}, \\
D &= n_2\left[\frac{\eta}{\varepsilon_a}e_1 h_2 + \frac{in_y}{n_\perp^2}(e_1 + \frac{\eta}{\varepsilon_a}h_2)\right] - n_1\left[\frac{\eta}{\varepsilon_a}e_2 h_1 + \frac{in_y}{n_\perp^2}(e_2 + \frac{\eta}{\varepsilon_a}h_1)\right] \\
&\quad - (n_2 - n_1)\frac{\varepsilon_a - n_x^2}{\varepsilon_a n_\perp^2} + \frac{\eta}{\varepsilon_a}\frac{n_x n_z}{n_\perp^2}(e_1 + h_1 - e_2 - h_2), \\
e_k &= \frac{\alpha_k n_x + in_{y,k}}{n_{\perp k}^2}, \quad h_k = -\frac{\beta_k n_x - in_{y,k}}{n_{\perp k}^2}, \quad k = 1, 2.
\end{aligned} \tag{21}
$$

Since the tangential components of the electric field are continuous at the interface of a magnetoplasma and an isotropic medium, either the coefficients B_k or the coefficients C_k can be used when deriving the expressions for these field components at $y = 0$. In Equations (19) and (20), we took the coefficients B_k and made use of the fact that $B_k = Z_0\tilde{B}_k\mathcal{I}(n_x, n_z)/D$.

From boundary conditions in Equations (7) and (8) for the tangential components of the electric field on the antenna surface with allowance for Equations (17)–(20), integral equations can be obtained for the Fourier transform $\mathcal{I}(n_x, z)$ of the surface current density. From Equation (7), we have

$$
\int_{-d}^{d} K_x(z - z')\mathcal{I}(n_x, z')dz' = -V(n_x). \tag{22}
$$

The boundary condition in Equation (8) gives

$$
\int_{-d}^{d} K_z(z - z')\mathcal{I}(n_x, z')dz' = 0. \tag{23}
$$

In integral Equations (22) and (23), it is assumed that $|z| < d$, and the integrals, which turn out to be singular for $z - z' \to 0$, are understood in the sense of Cauchy's principal value.

4. SOLUTION OF INTEGRAL EQUATIONS FOR THE ANTENNA CURRENT

The behavior of solutions of the integral equations for the antenna current is determined by the properties of their kernels in Equations (19) and (20). In what follows, we show that in the case of a fairly small antenna width $2d$, where the inequalities

$$d \ll \Delta, \quad d \ll |\eta/\varepsilon|^{1/2}\Delta, \quad (k_0 d)^2 \max\{|\varepsilon_a|, |\varepsilon|, |g|, |\eta|\} \ll 1 \tag{24}$$

are fulfilled, the properties of these kernels allow one to obtain an approximate solution of Equations (22) and (23) in closed form. To this end, we represent the kernels of these equations as the sums of singular and regular terms:

$$K_x(n_x,\zeta) = K_x^{(s)}(n_x,\zeta) + K_x^{(r)}(n_x,\zeta),$$
$$K_z(n_x,\zeta) = K_z^{(s)}(n_x,\zeta) + K_z^{(r)}(n_x,\zeta),$$

where the quantities $K_x^{(s)}(n_x,\zeta)$ and $K_z^{(s)}(n_x,\zeta)$ comprise singular terms that tend to infinity at $\zeta \to 0$, whereas the quantities $K_x^{(r)}(n_x,\zeta)$ and $K_z^{(r)}(n_x,\zeta)$ remain finite (regular) in this limit. It can be shown that

$$\begin{aligned} K_x^{(s)}(n_x,\zeta) \;=\; & \frac{iZ_0k_0}{\pi}\left[\left(\frac{\beta n_x^2\varepsilon_a}{\varepsilon_a^2+|\varepsilon\eta|} + \frac{(1-\beta)n_x^2}{\varepsilon_a+|\varepsilon\eta|^{1/2}\operatorname{sgn}\varepsilon} - \frac{1}{2}\right)\int_0^\infty \frac{\cos(k_0 n_z|\zeta|)}{\sqrt{n_z^2+n_x^2}}dn_z \right.\\ & \left. +\frac{i\beta n_x^2|\varepsilon\eta|^{1/2}}{\varepsilon_a^2+|\varepsilon\eta|}\int_{\alpha|n_x|}^\infty \frac{\cos(k_0 n_z|\zeta|)}{\sqrt{n_z^2-(\alpha n_x)^2}}dn_z\right], \end{aligned} \tag{25}$$

$$\begin{aligned} K_z^{(s)}(n_x,\zeta) \;=\; & \frac{Z_0k_0n_x}{\pi}\left[\left(\frac{\beta\varepsilon_a}{\varepsilon_a^2+|\varepsilon\eta|} + \frac{1-\beta}{\varepsilon_a+|\varepsilon\eta|^{1/2}\operatorname{sgn}\varepsilon}\right)\int_0^\infty \frac{n_z\sin(k_0 n_z|\zeta|)}{\sqrt{n_z^2+n_x^2}}dn_z \right.\\ & \left. +\frac{i\beta|\varepsilon\eta|^{1/2}}{\varepsilon_a^2+|\varepsilon\eta|}\int_{\alpha|n_x|}^\infty \frac{n_z\sin(k_0 n_z|\zeta|)}{\sqrt{n_z^2-(\alpha n_x)^2}}dn_z\right]\operatorname{sgn}\zeta. \end{aligned} \tag{26}$$

Hereafter, $\beta = 0$ if $\operatorname{sgn}\varepsilon = \operatorname{sgn}\eta$, $\beta = 1$ if $\operatorname{sgn}\varepsilon \neq \operatorname{sgn}\eta$, and $\alpha = |\varepsilon/\eta|^{1/2}$.

Formulas (25) and (26) can be derived by passing to the limit $n_z \to \infty$ in the integrands of Equations (19) and (20), respectively, with allowance for the identity

$$\lim_{n_z\to\infty}\left(\frac{\sqrt{n_z^2+n_x^2}}{n_z} - \frac{n_z}{\sqrt{n_z^2+n_x^2}}\right) = 0. \tag{27}$$

It can be verified that in the case where the half-space $y < 0$ is filled with a resonant magnetoplasma, for which $\operatorname{sgn}\varepsilon \neq \operatorname{sgn}\eta$, Equations (25) and (26) are reduced to the results of [31] if the plasma is sufficiently dense such that $|\varepsilon\eta| \gg \varepsilon_a^2$. In contrast to [31], the representations in Equations (25) and (26) turn out to be valid for arbitrary plasma parameters, regardless of the signs and values of ε and η.

The regular parts $K_x^{(r)}(n_x,\zeta)$ and $K_z^{(r)}(n_x,\zeta)$ of the kernels are found by subtracting the limiting quantities, which are obtained in the above-described way, from the corresponding integrands of Equations (19) and (20). We do not present very cumbersome formulas for $K_x^{(r)}(n_x,\zeta)$ and $K_z^{(r)}(n_x,\zeta)$ here, because they are derived straightforwardly using the above explanations. Note that in the case of a narrow strip, i.e., under conditions in Equation (24), one can put $\zeta = 0$ when calculating the quantities $K_x^{(r)}(n_x,\zeta)$ and $K_z^{(r)}(n_x,\zeta)$ in view of their regularity. In this case, the properties of the function $K_z^{(r)}(n_x,\zeta)$ make it possible to write $K_z^{(r)}(n_x,0) = 0$ (see [27]). In turn, the quantity $K_x^{(r)}(n_x,0)$ can generally be calculated only numerically.

The integrals in Equations (25) and (26) can be evaluated analytically [33, 34] as follows:

$$\begin{aligned}
\int_0^\infty \frac{\cos(k_0 n_z|\zeta|)}{\sqrt{n_z^2+n_x^2}}dn_z &= K_0(k_0|n_x\zeta|),\\
\int_{\alpha|n_x|}^\infty \frac{\cos(k_0 n_z|\zeta|)}{\sqrt{n_z^2-(\alpha n_x)^2}}\,dn_z &= -\frac{\pi}{2}Y_0(k_0\alpha|n_x\zeta|),\\
\int_0^\infty \frac{n_z\sin(k_0 n_z|\zeta|)}{\sqrt{n_z^2+n_x^2}}dn_z &= |n_x|K_1(k_0|n_x\zeta|),\\
\int_{\alpha|n_x|}^\infty \frac{n_z\sin(k_0 n_z|\zeta|)}{\sqrt{n_z^2-(\alpha n_x)^2}}\,dn_z &= -\frac{\pi}{2}\alpha|n_x|Y_1(k_0\alpha|n_x\zeta|).
\end{aligned} \tag{28}$$

Here, K_m and Y_m are modified Bessel functions of the second kind and Bessel functions of the second kind of order m, respectively, and the evaluation of the last integral in Equation (28) was performed within the framework of the theory of tempered distributions. With allowance for the well-known small-argument approximations of cylindrical functions, in the limit $\zeta \to 0$ we have

$$K_x^{(s)}(n_x,\zeta) \;=\; -\frac{iZ_0k_0}{2\pi\chi}\left(\ln\frac{k_0|\zeta|}{2}+\ln|n_x|+\gamma+F\right), \tag{29}$$

$$K_z^{(s)}(n_x,\zeta) \;=\; \frac{Z_0}{2\pi\varepsilon_{\text{eff}}}\frac{n_x}{\zeta}, \tag{30}$$

where

$$\chi = \frac{\varepsilon_{\text{eff}}}{n_x^2-\varepsilon_{\text{eff}}}, \tag{31}$$

$\gamma = 0.5772\ldots$ is Euler's constant, and the quantity F for $\operatorname{sgn}\varepsilon \neq \operatorname{sgn}\eta$ is defined as

$$F = i\chi\frac{n_x^2|\varepsilon\eta|^{1/2}}{\varepsilon_a^2+|\varepsilon\eta|}\ln\frac{|\varepsilon|}{|\eta|}. \tag{32}$$

In the opposite case where $\operatorname{sgn}\varepsilon = \operatorname{sgn}\eta$, one should put $F = 0$. The quantity ε_{eff} is determined by the expression

$$\varepsilon_{\text{eff}} = \frac{\varepsilon_{\text{p}}+\varepsilon_a}{2}, \tag{33}$$

where

$$\varepsilon_{\text{p}} = \begin{cases} (\varepsilon\eta)^{1/2}\operatorname{sgn}\varepsilon & \text{if} \quad \operatorname{sgn}\varepsilon = \operatorname{sgn}\eta,\\ -i|\varepsilon\eta|^{1/2} & \text{if} \quad \operatorname{sgn}\varepsilon \neq \operatorname{sgn}\eta. \end{cases} \tag{34}$$

It should be noted that rigorously speaking, approximate expressions (29) and (30) cease to be valid for sufficiently large values of $|n_x|$. However, as is evident from what follows, the $|n_x|$ values significantly exceeding $(k_0\Delta)^{-1}$ affect only slightly the results of calculating the antenna current. Hence, the fulfillment of the first two inequalities in Equation (24) ensures the applicability of the used approximations.

After the above algebra, integral Equations (22) and (23) are rewritten for $|z| < d$ as

$$\int_{-d}^{d}\mathcal{I}(n_x,z')\ln\frac{k_0|z-z'|}{2}dz' \;=\; -\frac{i2\pi\chi}{Z_0k_0}V(n_x) - S(n_x)\int_{-d}^{d}\mathcal{I}(n_x,z')dz', \tag{35}$$

$$\int_{-d}^{d} n_x\frac{\mathcal{I}(n_x,z')}{z-z'}dz' \;=\; 0, \tag{36}$$

where

$$S(n_x) = i2\pi\chi(Z_0k_0)^{-1}K_x^{(r)}(n_x,0)+\ln|n_x|+\gamma+F. \tag{37}$$

It can be shown [13, 16] that the solutions of Equations (35) and (36) are the main terms of asymptotics of the solutions of the initial integral Equations (22) and (23) when inequalities (24) are fulfilled. In what follows, we restrict ourselves to analyzing only Equations (35) and (36).

The solution of Equation (35) with a logarithmic kernel, which also satisfies Equation (36) with Cauchy's kernel [35], can be obtained in closed form as

$$\mathcal{I}(n_x, z) = \frac{2i}{Z_0 k_0 \sqrt{d^2 - z^2}} \frac{\chi V_0}{\ln(4/k_0 d) - S(n_x)} \frac{\sin(k_0 n_x \Delta)}{k_0 n_x \Delta}. \tag{38}$$

Substituting Equation (38) into Equation (6), we arrive at the formula for the surface current density $I(x, z)$. Integrating $I(x, z)$ over z from $-d$ to d yields the total current $I_\Sigma(x)$ of the antenna in the cross section $x = \text{const}$:

$$I_\Sigma(x) = \frac{iV_0}{Z_0} \int_{-\infty}^{\infty} \frac{\sin(k_0 n_x \Delta)}{k_0 n_x \Delta} \frac{\chi \exp(-ik_0 n_x x)}{\ln(4/k_0 d) - S(n_x)} dn_x. \tag{39}$$

Note that the singularity of the function $I(x, z)$ at $|z| \to d$, which corresponds to the Meixner condition at the edge [36], turns out to be integrable, so that the total current of the antenna is finite.

5. CURRENT DISTRIBUTION AND INPUT IMPEDANCE OF THE ANTENNA

The integral representation in Equation (39) admits only a numerical study in the general case. However, if the condition $\ln(4/k_0 d) \gg |S(n_x)|$ holds for the values $|n_x| < (k_0 \Delta)^{-1}$, which give the main contribution to the integral in Equation (39), this integral can be evaluated analytically and the antenna current takes the following form for $|x| > \Delta$:

$$I_\Sigma(x) = \frac{V_0}{Z_0} \frac{k_0 \varepsilon_{\text{eff}}}{h} \frac{\pi}{\ln(4/k_0 d)} \exp(-ih|x|), \tag{40}$$

where

$$h = k_0 \varepsilon_{\text{eff}}^{1/2}. \tag{41}$$

In the case where the current-distribution constant h is complex-valued, it is assumed that $\operatorname{Im} h < 0$.

An approximate representation of Equation (40) corresponds to the transmission line theory. Accordingly, the conditions under which this representation was derived determine the limits of applicability of this theory for a strip antenna located at the interface of the media considered. It is evident that if the magnetoplasma on one side of the interface $y = 0$ is nonresonant, i.e., $\operatorname{sgn} \varepsilon = \operatorname{sgn} \eta$, and $(\varepsilon\eta)^{1/2} \operatorname{sgn} \varepsilon + \varepsilon_a > 0$, then the current behavior is the same as that for an antenna in a homogeneous transparent medium with the dielectric permittivity ε_{eff}. However, in the case $(\varepsilon\eta)^{1/2} \operatorname{sgn} \varepsilon + \varepsilon_a < 0$, which is possible for the nonresonant magnetoplasma with $\varepsilon < 0$ and $\eta < 0$, the quantity h turns out to be purely imaginary and the antenna current exponentially decays with distance from the excitation gap. If the magnetoplasma is resonant such that $\operatorname{sgn} \varepsilon \neq \operatorname{sgn} \eta$, the quantity ε_{eff} and hence the current-distribution constant h are complex, so that the current shape is characterized by spatial oscillations whose amplitude decays along the antenna conductor with distance from the antenna input.

Using the current distribution $I_\Sigma(x)$, we can find the input impedance Z of the antenna using the formula $Z = V_0 / I_\Sigma(\Delta)$. Within the framework of the approximation in Equation (40) for the current under the additional condition $|h|\Delta \ll 1$, we obtain

$$Z = \frac{Z_0}{\pi} \frac{k_0}{h} \ln\left(\frac{4}{k_0 d}\right). \tag{42}$$

It is important that the above results can be extended to the case of a finite-length antenna if we represent it as a transmission line of length $2L$. Following the standard approach [15], one can find the current of such an antenna in the form

$$I_\Sigma(x) = \frac{I_0}{\sin(hL)} \sin\left[h(L - |x|)\right], \tag{43}$$

where $|x| < L$, $I_0 = I_\Sigma(0)$ is the current at the antenna input, and h is determined by Equation (41). The quantity I_0 is found as $I_0 = V_0 / Z_L$, where Z_L is the input impedance of the finite-length antenna. For the known current shape $I_\Sigma(x)/I_0$, the impedance Z_L can be calculated using the induced EMF method [37]. In the case of an electrically short antenna where $|h|L \ll 1$, Equation (43) yields a "triangular" distribution of current along the antenna conductor ($|x| < L$):

$$I_\Sigma(x) = I_0(1 - |x|/L). \tag{44}$$

Since detailed calculations of the antenna characteristics for all possible cases would take up much space, we now dwell on the most interesting examples of the antenna-current behavior. Namely, we will discuss the distribution of the antenna current if the quantity h is purely imaginary or complex. We assume that the angular frequency ω is much higher than the lower hybrid frequency of a magnetoplasma [23]. In this case, we can neglect contribution of the ion motion to the elements of the plasma dielectric tensor in Equation (1) and represent them as follows [32]:

$$\varepsilon = 1 + \frac{\omega_p^2}{\omega_H^2 - \omega^2}, \quad g = -\frac{\omega_p^2 \omega_H}{(\omega_H^2 - \omega^2)\omega}, \quad \eta = 1 - \frac{\omega_p^2}{\omega^2}, \tag{45}$$

where ω_H and ω_p are the gyrofrequency and the plasma frequency of electrons, respectively. The calculations were performed for the plasma parameters corresponding to the laboratory conditions: $\omega_H = 3.5 \times 10^9\,\mathrm{s}^{-1}$ (external static magnetic field $B_0 = 200\,\mathrm{G}$) and $4 \times 10^{10}\,\mathrm{s}^{-1} \leq \omega_p \leq 8 \times 10^{10}\,\mathrm{s}^{-1}$ (the plasma density varies in the interval between $5 \times 10^{11}\,\mathrm{cm}^{-3}$ and $2 \times 10^{12}\,\mathrm{cm}^{-3}$). The isotropic medium in the region $y > 0$ is free space, i.e., $\varepsilon_a = 1$.

First, we consider the frequency $\omega = 5 \times 10^9\,\mathrm{s}^{-1}$ lying in the range $\omega_H < \omega < \omega_p$, for which $\varepsilon < 0$ and $\eta < 0$. In this case, the magnetoplasma in the half-space $y < 0$ is nonresonant and, moreover, $\varepsilon_{\text{eff}} < 0$. Figure 2 shows the snapshots of the distributions of the antenna current, normalized to its value at $x = 0$, along the infinitely long antenna at the indicated frequency for $k_0 d = 1.67 \times 10^{-3}$, $\Delta = 5d$, and three values of the plasma frequency $\omega_p = 4 \times 10^{10}\,\mathrm{s}^{-1}$ ($\varepsilon_{\text{eff}} = -43.75$), $\omega_p = 5.64 \times 10^{10}\,\mathrm{s}^{-1}$ ($\varepsilon_{\text{eff}} = -88.5$), and $\omega_p = 8 \times 10^{10}\,\mathrm{s}^{-1}$ ($\varepsilon_{\text{eff}} = -178.1$), which correspond to dashed curves *1*, *2* and *3*, respectively. Note that for the chosen parameters, the results of calculations by Equation (39) and approximate formula (40) coincide with graphical accuracy. This fact implies that the off-diagonal element g of the plasma dielectric tensor affects the current distribution only slightly. Indeed, this element contributes only to the regular parts of the kernels of integral equations for the antenna current. These parts are not taken into account within the framework of the transmission line theory. Hence, the results yielded by this theory can be obtained even easier, namely, by using the uniaxial tensor with $g = 0$ instead of general tensor (1). The exponential decay of current with distance from the antenna input is explained by the fact that the quantity h is purely imaginary in the case considered. The solid lines with respective labels in Figure 2, which correspond to the chosen values of the plasma density ω_p, present the results of calculations by formula (43) for a finite antenna with the dimensionless half-length $k_0 L = 0.33$. This value is marked by the vertical dash-dot line in the figure. Note that for solid curves *1*, *2*, and *3* in Figure 2, $|\mathrm{Im}\, h| L = 2.2$, $|\mathrm{Im}\, h| L = 3.14$, and $|\mathrm{Im}\, h| L = 4.45$, respectively.

Figure 3 shows the current distributions of the antenna in the case of a resonant magnetoplasma where $\mathrm{sgn}\,\varepsilon \neq \mathrm{sgn}\,\eta$ at the frequency $\omega = 10^9\,\mathrm{s}^{-1}$. In this case, $k_0 d = 3.33 \times 10^{-4}$. We used the previous value of ω_H, but put $k_0 L = 0.167$ for a finite-length antenna. Curves *1*, *2*, and *3*, which are plotted for the same plasma frequencies as those in Figure 2, correspond to $\varepsilon_{\text{eff}} = 0.5 - i2.37 \times 10^2$, $\varepsilon_{\text{eff}} = 0.5 - i4.73 \times 10^2$, and $\varepsilon_{\text{eff}} = 0.5 - i9.45 \times 10^2$, respectively. Since the quantity ε_{eff} is now complex-valued, the snapshots of the antenna current have oscillations that exponentially decay with distance from the antenna input.

It is seen in Figures 2 and 3 that the current distribution of the infinitely long antenna satisfactorily approximates the current behavior of the finite-length antenna for $|\mathrm{Im}\, h| L > 3$, excepting small regions near the ends $z = \pm L$. In this case, the input impedance of an infinitely long antenna can be used as a good approximation for the impedance Z_L of the antenna of finite length. However, one should bear in mind that in the case of a purely imaginary h, the impedance given by Equation (42) has a zero real part. To determine $\mathrm{Re}\, Z$, the regular part of kernel (19) must be taken into account. In contrast to this, in the case of a resonant magnetoplasma where h is complex, Equation (42) yields both the real and imaginary parts of the antenna impedance. This is due to the fact that the transmission line theory for a resonant magnetoplasma accounts for the excitation of quasielectrostatic waves in the plasma, which are known to predominantly determine the radiation resistance of a thin-wire antenna [3, 11, 16].

Finally, we note that the current distribution in Equation (43) goes over to Equation (40) in the limit $|\mathrm{Im}\, h| L \gg 1$. Hence, increasing the antenna length up to values much greater than the scale $|\mathrm{Im}\, h|^{-1}$ of the current decay along the antenna wire is not expedient since this is no more accompanied by a change in the antenna characteristics.

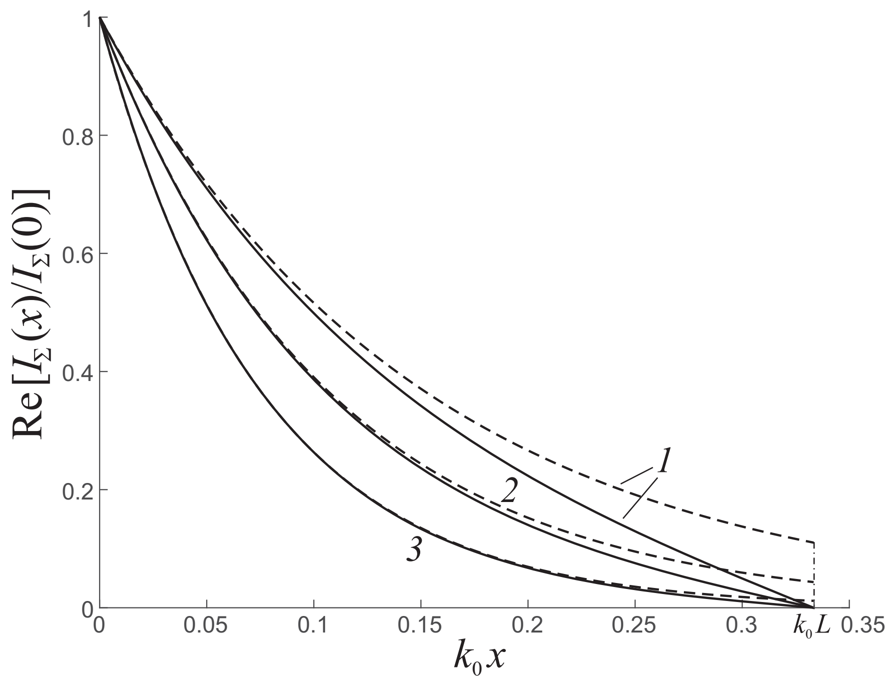

Figure 2. Current distributions along the infinitely long antenna (dashed lines) and the finite-length antenna (solid lines) for $\omega = 5 \times 10^9\,\mathrm{s}^{-1}$ and $\omega_H = 3.5 \times 10^9\,\mathrm{s}^{-1}$ in the cases where $\omega_p = 4 \times 10^{10}\,\mathrm{s}^{-1}$ and $|\mathrm{Im}\,h|L = 2.2$ (curves *1*), $\omega_p = 5.64 \times 10^{10}\,\mathrm{s}^{-1}$ and $|\mathrm{Im}\,h|L = 3.14$ (curves *2*), and $\omega_p = 8 \times 10^{10}\,\mathrm{s}^{-1}$ and $|\mathrm{Im}\,h|L = 4.45$ (curves *3*). The vertical dash-dot line indicates the k_0L value on the horizontal axis.

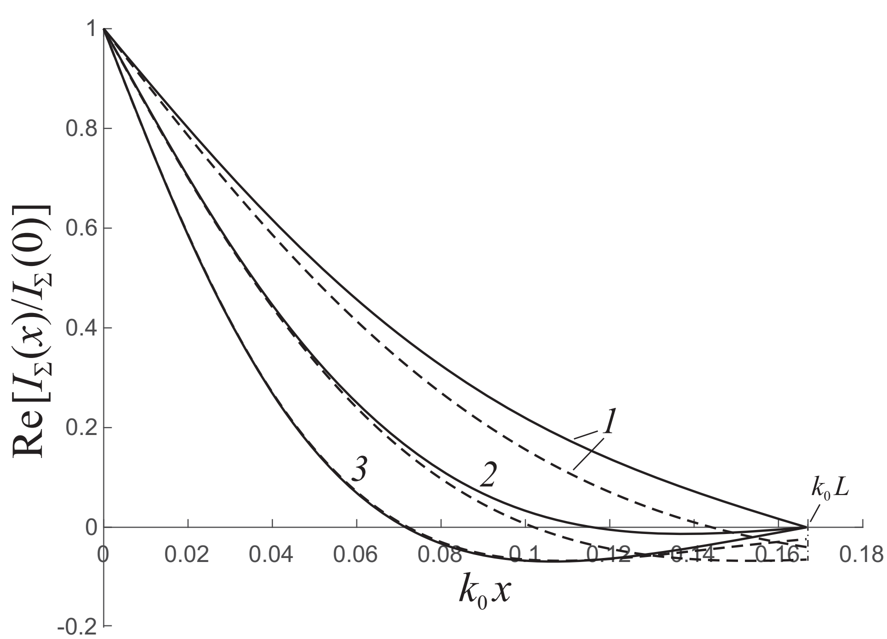

Figure 3. The same as in Figure 2 but for $\omega = 10^9\,\mathrm{s}^{-1}$ in the cases where $\omega_p = 4 \times 10^{10}\,\mathrm{s}^{-1}$ and $|\mathrm{Im}\,h|L = 1.81$ (curves *1*), $\omega_p = 5.64 \times 10^{10}\,\mathrm{s}^{-1}$ and $|\mathrm{Im}\,h|L = 2.56$ (curves *2*), and $\omega_p = 8 \times 10^{10}\,\mathrm{s}^{-1}$ and $|\mathrm{Im}\,h|L = 3.62$ (curves *3*).

6. CONCLUSION

In this work, through the use of the theory of singular integral equations, we have considered the problem of finding the electrodynamic characteristics of a perfectly conducting, narrow strip antenna that is perpendicular to the external static magnetic field and located at a plane interface of a magnetoplasma and an isotropic medium. The cases of both a resonant and nonresonant plasma occupying the half-space on one side of the interface have been analyzed. For an infinitely long strip, we have obtained the current distribution and input impedance of such an antenna and established conditions under which these characteristics admit relatively simple closed-form representations corresponding to the transmission line theory. Within the framework of this theory, the current distribution and input impedance of the antenna coincide with the corresponding characteristics of a certain equivalent transmission line. We have also discussed the possibility to construct approximately the current distribution for a finite-length antenna. In the cases where the imaginary part of the current-distribution constant is nonzero and the antenna is not too short, the results obtained for an infinitely long antenna are shown to be applicable for a finite antenna.

Another important implication of the performed analysis is that the current-distribution constant derived within the framework of the transmission line theory turns out to be independent of the off-diagonal element of the plasma dielectric tensor. This fact allows one to employ the uniaxial model of a magnetoplasma when determining the antenna characteristics in a first approximation. A similar approach can evidently be used for finding the characteristics of an antenna located at an interface of more complex media described by permittivity or permeability tensors of arbitrary form.

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