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Optimal Synthesis of Thinned Arrays Utilizing Fast Fourier Transform Technique

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Abstract—This piece of work replicates on the antenna array thinning exploring the benefits of known Fourier Analysis. In this communication Fourier transform is applied to the synthesis of periodic arrays for minimizing the Peak Side Lobe (PSL) level and thereby enhancing the directivity. Furthermore, the concept of Fill Factor (degree of thinning), i.e., reduction in the number of active elements is experimented for the above said objective. The proposed methodology is worked out on periodic linear and planar arrays. Numerical study and simulation results are composed with array thinning designs from literature. The analysis demonstrates the superiority of the illustrated Fourier technique.

1. INTRODUCTION

A new method is proposed for thinning the linear periodic array of uniform spacing to minimize the side lobe level. This method is based on the concept of forward and inverse iterative Fourier transform to derive linear array element excitations from the given array factor. Array thinning is obtained by setting the amplitudes of a predetermined number of largest element excitations to unity and the others to zero during each iteration. The number of turned ON elements depends on the array filling factor and the total count of element positions [1–4]. This iterative FFT is more effective for thinning large size linear and planar arrays. Array thinning involves the removal of some radiating elements from the given array. The main motivation of array thinning is the reduction in cost and weight by getting nearly the same narrow beamwidth as for a filled array of equal size. One of the most important advantages is that all the turned on elements are operated with same amplitude, and lower side lobes can be obtained as for the same filled array illuminated with uniform weighting. Reported literature in array thinning is executed by a technique named statistical density taper technique developed by Skolnik et al. [2]. In this technique the densities of elements are made proportional to amplitudes of the aperture illumination of a conventional filled array. The selection of elements positions is evaluated statically by choosing unity or zero with probabilities proportional to the filled array, and the directivity of the antenna array is reduced because it is proportional to the number of turned on elements. The resulting thinned array is illuminated with uniform weighting for all turned ON elements [5]. In recent years, evolutionary algorithms have been used for thinning arrays featuring a specified peak side lobe. However, these techniques, all nature-inspired from algorithms [6–9] using essentially randomness, were mainly used for the synthesis of linear arrays not exceeding 200 element positions.

The concept of thinning linear array is to reduce the Peak Side lobe Level (PSL) by a method of iterative forward and reverse Fourier Transform. There exists a relation between the inverse Fourier transform Equation (3) and array factor which is given by Equation (2). This relation gives the element excitations from the given array factor iteratively. The number of elements that get the amplitude one or zero depends on the array fill factor. Array fill factor is defined as the ratio of the turned on elements to the total number of elements. The count of turned on elements is calculated by multiplying the fill

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Table 1. Nomenclature.

List of Symbols	Meaning
PSL	Peak Sidelobe Level
FFT	Fast Fourier Transform
IFT	Inverse Fourier Transform
$u = \sin\theta$	Direction cosine along Elevation angle
$F(u)$	Far field radiation pattern
A_n	Complex excitation of the nth element
d	Inter element spacing in an Array
AF	Array factor
EF	Embedded element factor
k	Wave Number
N	Number of elements in the Linear Array
T	Number of turned ON elements in the thinned array
K	Number of smaples in FFT
Fill Factor	Degree of Thinning
u, v	Direction cosines along the θ and ϕ directions in the spherical coordinates
A_{mn}	The complex excitation of the element in the planar grid

factor with the count of the array element positions. Fill factor ranges from zero to one. Table 1 gives the nomenclature of various parameters used in the article.

The organization of the paper is introduction in Section 1 followed by formulation of thinning method in Section 2. The simulation study supporting the linear array thinning is presented in Section 3 with different degrees of thinning. In Section 4 provides formulation of planar array antenna thinning with different case studies, and analysis is in Section 5. Finally, the concluding statements on the proposed procedure are given in Section 6.

2. FORMULATION OF THE THINNING METHOD FOR LINEAR ARRAY

The concept of the thinning linear array [10] is to reduce the Peak Side lobe Level (PSL) by a method of iterative forward and backward Fourier transform. A relationship is observed between the element excitation and the array factor that is the Inverse Fast Fourier Transform. There exists a relation between the inverse Fourier transform Equation (2) and array factor is given by Equation (3). This relation gives the element excitations from the given array factor iteratively. The number of elements that get the amplitude one or zero depends on the array fill factor. Array fill factor is defined as the ratio of the turned on elements to the total number of elements. The count of turned on elements is calculated by multiplying the fill factor with the count of the array element positions. Fill factor ranges from zero to one. The array factor of M uniformly spaced at distance of d of linear array is $AF(u)$. The far-field pattern of the array can be obtained by product of the array factor and embedded element pattern $EF(u)$ and is given by Equation (1).

$$F(u) = AF(u) * EF(u) \tag{1}$$

$$AF(u) = \sum_{n=0}^{N-1} A_n e^{jkndu} \tag{2}$$

where A_n is the complex excitation of the mth element, k the wave number $(2\pi/\lambda)$, λ the wavelength at which the array operates, $u = \sin(\theta)$ the direction cosine, and θ the pattern angle measured from broadside of the array. The array factor and complex excitation A_n are related by a inverse Fourier

transform. The inverse Fourier transform of the periodic signal is given by Equation (3).

$$\mathrm{IFT}[x(n)] = \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}nk} \tag{3}$$

Discrete Fourier transform is applied to the array factor to get the element excitation coefficients A_m. The number of (radiating) turned on elements in the thinned array will be obtained by a product of the fill factor and total number of elements in the original linear array. The iterative process of the Fourier transform on the array factor for the synthesis of thinned array will meet a specified PSL. The steps for the iterative process [11] are as follows.

Step 1: A random initialization with equal 0/1 probability is applied to all M element excitations in the given linear array.

Step 2: Fourier transform the element excitation A_m with K point one dimensional inverse FFT to arrive at the array Factor consisting of K samples with $K > M$ by applying zero padding.

Step 3: Match the side lobe region of AF to the side lobe requirements and the AF values left unchanged not violating the side lobe thresholds like the AF values contained in the main lobe region.

Step 4: Fourier transform the modified AF with K point one dimensional forward FFT to get a new set of element excitations.

Step 5: Truncate K samples of element excitations to M samples associated with the array elements.

Step 6: The selection of the turned ON elements takes place at each iteration after the FFT back transformation of AF, which is carried out to yield new excitations for the M elements in the given array.

Step 7: From this new excitations obtained the elements with the largest excitations are set to one, i.e., (T elements) and the remaining elements (M-T) set to zero.

Step 8: Start the new iteration if there is a difference between the present and previous selections of the turned on elements and yield the new AF, otherwise the iteration will stop.

A crude way of array thinning is by using FFT technique [12]. This technique felicitates a solution consuming less iterations for a given PSL condition. This premature termination of the synthesized thinned array distribution in general will not fit the PSL requirements exactly. However, by applying a low (a few decibels) PSL constraint there exists a good chance for the synthesized element distribution that will match the real PSL objectives. Since the initial synthesis is started with a random element distribution, each new trail will generate different distributed candidates for thinned array elements. The computational time required for a single trial is very little because the count of forward and backward FFT operations involved in the synthesis is very low. Therefore, generating large element distribution is not a time consuming process. Performing large number of trails gives the best globally optimal design, which gives a more powerful new thinned array synthesis.

3. NUMERICAL RESULTS OF LINEAR ARRAY WITH DIFFERENT FILL FACTORS

The present work is an array thinning synthesis for a 400-element linear array for various degrees of thinning. The synthesis is on both symmetrical and asymmetrical positions of turned ON element distributions. The considered linear embedded array is an isotropic pattern with inter element spacing of 0.5λ. Here 4096 point one-dimensional forward and inverse FFT techniques are used for the synthesis of array thinning in an iterative manner involving 10000 trails. The size of the FFT is decided by the PSL constraint. The number of side lobes in the given side lobe region gives the count for FFT. The far-field directions of the side lobes, which violate the PSL constraints, are corrected. Prior knowledge of far-field directions of the side lobe regions, which violate the PSL requirements, are derived from the angular position of the first null of the AF at both sides of the main beam. The greater accuracy of the angular information is obtained by computing the AF with large size FFT.

3.1. Symmetrical Linear Array

3.1.1. Case 1: Array Thinning with Degree of 77%

The first test case is demonstrated for the synthesis of the linear array of 400 elements, which is symmetric about its center. A fill factor of 77% is chosen, which means that each array half has 35 turned OFF elements. Figure 1 shows the synthesis of the array thinning with 10000 trails and the normalized far-field radiation of the best element distribution among the 10000 trails solutions featuring the maximum PSL value $\mathbf{-22.85\,dB}$. In this synthesis all the 10000 trials were subjected to the same PSL of $\mathbf{-24.80\,dB}$ (obtained by trial and error). Figure 2 shows the histogram of maximum (PSL) side lobe frequency distribution over the 10000 trails. From the histogram it is clearly shown that 4850 elements distributions have a maximum PSL of below $\mathbf{-20\,dB}$ (almost 48.50% of total population). This high percentage indicates the good demonstration of the proposed method. The maximum PSL is $\mathbf{-13.3\,dB}$ and $\mathbf{3\,dB}$ beamwidth is $\mathbf{0.521}$ deg for the original array (without thinning). The synthesis of thinning with fill factor of 77% gives a max PSL $\mathbf{-22.85\,dB}$ and 3 dB beamwidth of $\mathbf{0.603}$ deg.

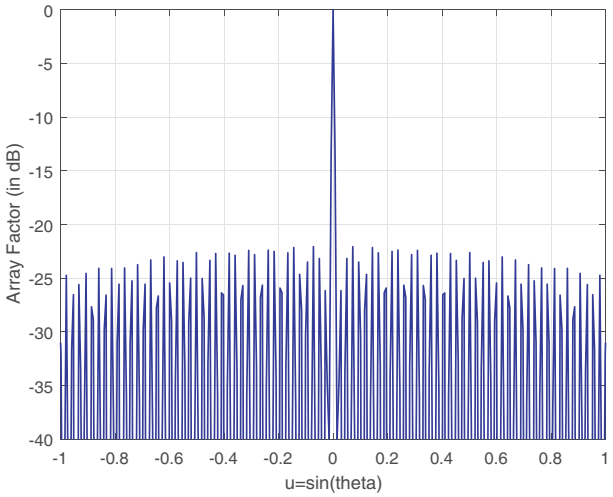

Figure 1. Symmetrical thinned linear array with fill factor of 77%.

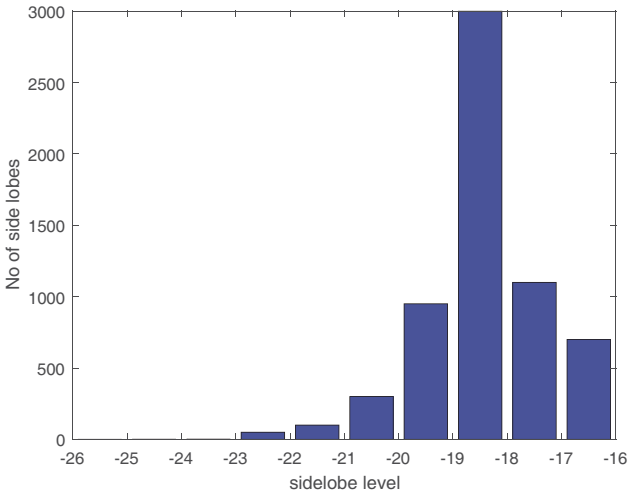

Figure 2. Count of side lobes (10000 trials) by histogram representation.

3.1.2. Case 2: Array Thinning with Degree of 63%

The synthesis is performed on the same 400-element array with a PSL requirement of $\mathbf{-25.40\,dB}$ but now with a fill factor 63% (i.e., 74 elements distributions on half side are turned OFF). The experiment in case 1 is repeated with a fill factor 63% shown in Figure 3. In this case, the best elements distribution with a maximum PSL of $\mathbf{-24.15\,dB}$ almost $\mathbf{4.25\,dB}$ is lower than $\mathbf{-19.90\,dB}$ as given the literature survey [8]. The $\mathbf{3\,dB}$ beamwidth in this case is $\mathbf{0.709}$ deg. Figure 4 shows the frequency distribution of side lobes by a histogram representation where 6438 elements of the total have maximum PSL below $\mathbf{-20\,dB}$ (almost 64).

3.1.3. Case 3: Array Thinning with Degree of 45%

The synthesis is performed on the same 400-element array with a PSL requirement of $\mathbf{-28.20\,dB}$ but now with a fill factor 45% (i.e., 110 elements distributions on half side are turned off). The experiment in case 1 is repeated with a fill factor 45% shown in Figure 5. It shows the far-field pattern of array thinning with a degree of thinning 45%. In this case, the best elements distribution is with a maximum PSL of $\mathbf{-27.15\,dB}$. The $\mathbf{3\,dB}$ beamwidth in this case is 0.789 deg. Figure 6 shows the frequency distribution of side lobes by a histogram representation where 7210 elements of the total have maximum PSL below

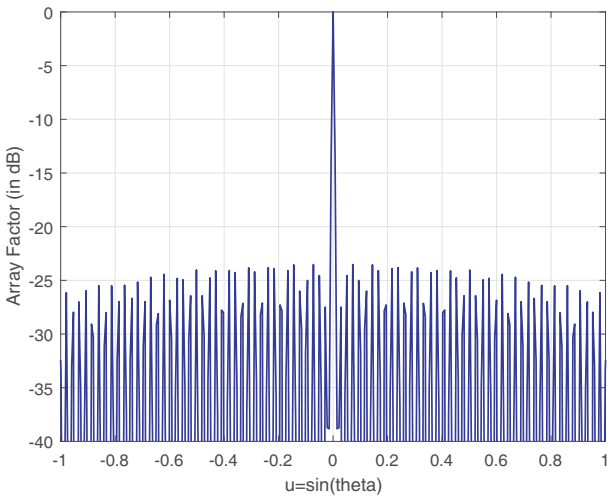

Figure 3. Symmetrical thinned linear array with fill factor of 63%.

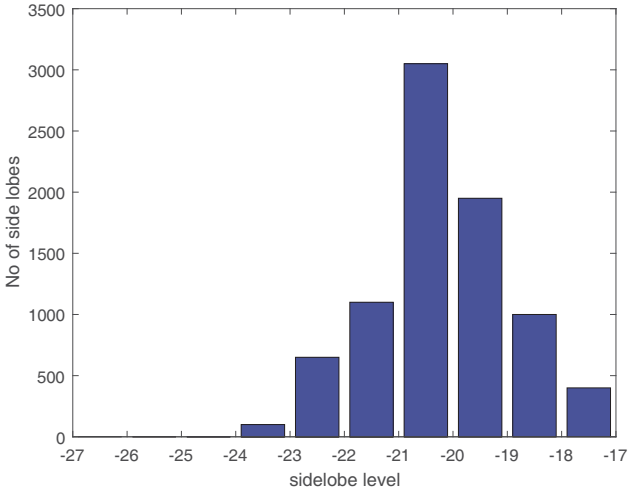

Figure 4. Count of side lobes (10000 trials) by histogram representation.

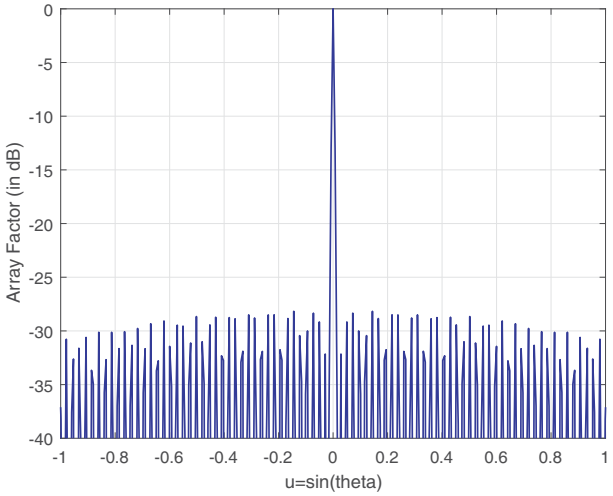

Figure 5. Symmetrical thinned linear array with fill factor of 45%.

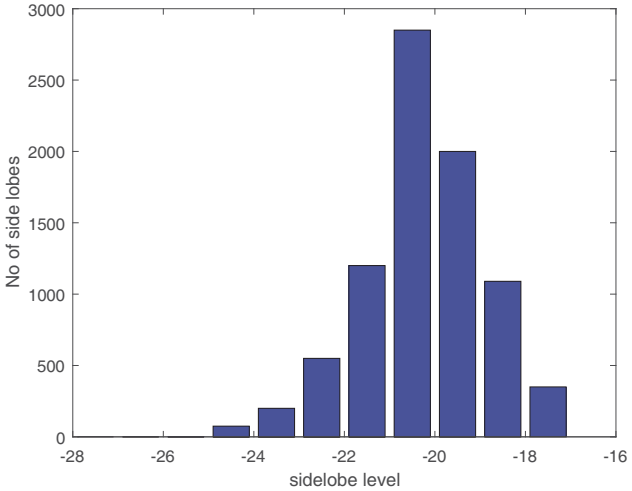

Figure 6. Count of side lobes (10000 trials) by histogram representation.

Table 2. Comparison of linear thinned array results obtained with the IFT method and the statistical density taper [2].

Fill Factor (%)	Max PSL (dB)		Directivity (dB)		Turned ON elements
	IFT	Stat [2].	IFT	Stat [2].	
77	**−22.85**	**−21.20**	**30.40**	27.90	308
63	**−26.70**	−23.95	**31.68**	29.24	252
45	**−28.20**	−25.75	**32.78**	31.90	180

−20 dB (almost 72%). The comparison table of the thinned linear array PSL with statistical density taper [11] is shown in Table 2.

3.2. Asymmetrical Linear Array

The proposed method is also applied to the synthesis of asymmetrical arrays [13], where asymmetrical array means the position of the elements in a nonuniform manner. The present work is an array thinning synthesis for a 400-element linear array for various degrees of thinning. The turned ON element is distributed in non-symmetrical positions. The considered linear embedded array is an isotropic pattern with inter element spacing of 0.5λ. Here 4096 point one-dimensional forward and inverse FFT techniques are used for the synthesis of array thinning in an iterative manner involving 10000 trails. The size of the FFT is decided by the PSL constraint. The number of side lobes in the given side lobe region gives the count for FFT. The far-field directions of the side lobes, which violate the PSL constraints, are corrected. Prior knowledge of far-field directions of the side lobe regions, which violate the PSL requirements, are derived from the angular position of the first null of the AF at both sides of the main beam. The greater accuracy of the angular information is obtained by computing the AF with large size FFT.

3.2.1. Array Thinning with Degree of 71.80%

The synthesis of the 10000 trails on a 400-element array featuring a fill factor 71.80% is as shown in Figure 7. It shows the far-field radiation of the asymmetrical array, where the thinned element distribution is also asymmetrical. This synthesis gives optimum far-field patterns having the maximum PSL below $\mathbf{-24.55\,dB}$. The maximum PSL for the same array using a hybrid approach of genetic algorithm is $\mathbf{-22.80\,dB}$ [3]. Figure 8 shows the frequency distribution of side lobes by a histogram representation. This synthesis reveals a very high number of element distributions featuring a maximum PSL close to the best obtained value of $\mathbf{-24.55\,dB}$. Performing the synthesis with a maximum requirement of $\mathbf{-24.55\,dB}$ reveals an element distribution with 53 elements turned off at the left of array and 55 elements at right side of the array. A beamwidth of 1.82 deg is obtained, which is 2.30 times the filled array (original array beam width is 0.827 deg).

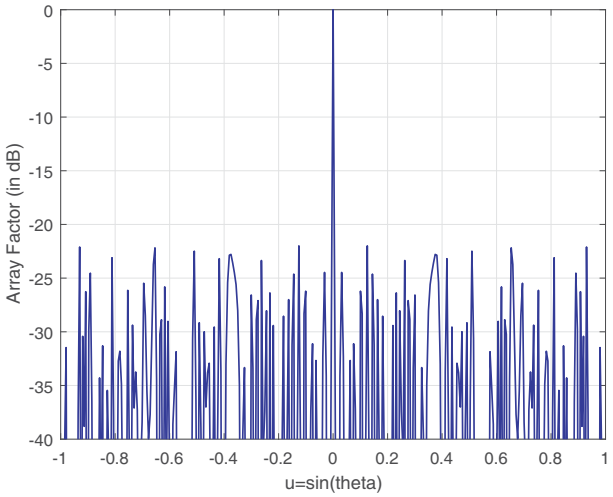

Figure 7. Asymmetrical thinned linear array with fill factor of 71.80%.

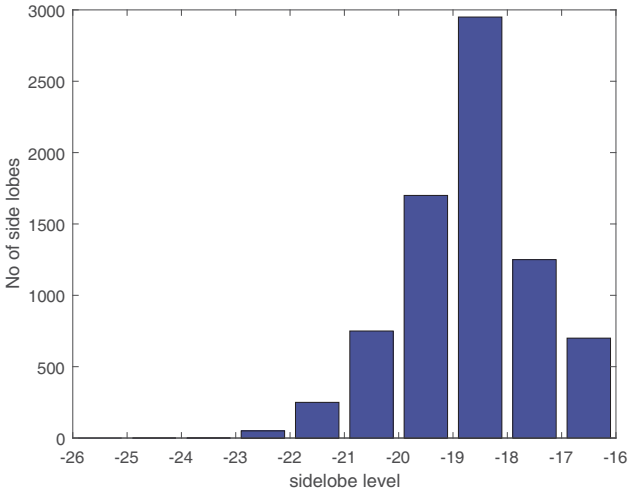

Figure 8. Count of side lobes (10000 trials) by histogram representation.

3.2.2. Array Thinning with Degree of 45%

The synthesis of the 10000 trails on a 400-element array featuring a fill factor 45% is as shown in Figure 9. It shows the far-field radiation of the asymmetrical array, where the thinned element distribution is also asymmetrical. This synthesis gives optimum far-field patterns having the maximum PSL below $\mathbf{-28.85\,dB}$. The maximum PSL for the same array using this array thinning is $\mathbf{27.10\,dB}$ [3]. Figure 10

shows the frequency distribution of side lobes by a histogram representation. This synthesis reveals a very high number of element distributions featuring a maximum PSL close to the best obtained value of $-\mathbf{28.85\,dB}$. Performing the synthesis with a maximum requirement of $-\mathbf{28.85\,dB}$ reveals an element distribution with 53 elements turned off at the left of array and 55 elements at right side of the array. A beamwidth of 1.82 deg is obtained, which is 2.30 times of the filled array (original array beamwidth is 0.827 deg). The comparison of the PSL and directivity of linear asymmetrical array obtained using IFT and statistical density taper is as shown in Table 3.

Table 3. Comparison of linear asymmetircal thinned array PSL and Directory obtained with the IFT method and the statistical density taper [2].

Fill Factor (%)	Max PSL (dB)		Directivity (dB)		Turned ON elements
	IFT	Stat [2].	IFT	Stat [2].	
71.80	$\mathbf{-22.80}$	-20.20	**29.40**	27.50	288
45	$\mathbf{-27.10}$	-24.68	**32.10**	30.30	180

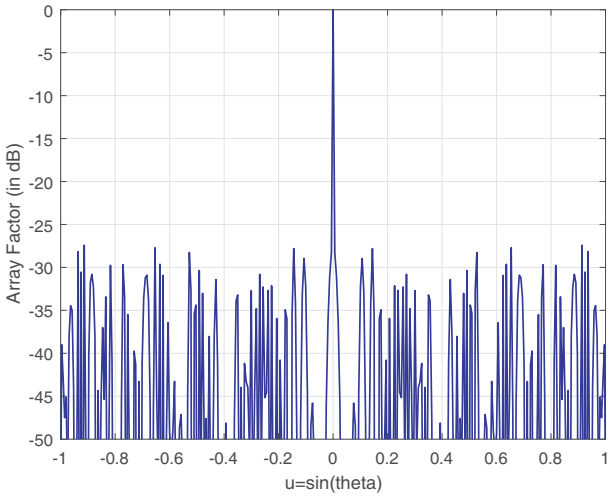

Figure 9. Asymmetrical thinned linear array with fill factor of 45%.

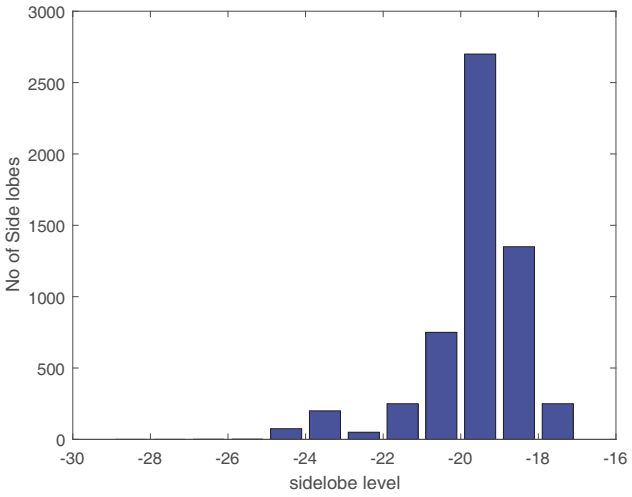

Figure 10. Count of side lobes (10000 trials) by histogram representation.

4. PLANAR ARRAY THINNING

The thinning concept is extended on planar configuration [11, 14–16]. The planar periodic array of uniform spacing with the constraint of maximum side lobe is described. This method is based on the concept of forward and inverse iterative Fourier transform [12, 13, 17] to derive the linear array element excitations from the given array factor. Array thinning is obtained by equating the maximum value of the excitation to unity. This task is applied only to predetermined array elements which are above a specified threshold limit. The rest of the elements excitations are made to zero. This procedure is repeated in each iteration. The effectiveness of the iterative Fourier Transform for thinning the planar periodic arrays is explained for a large number of arrays ($>$ 1500 element positions) [18, 19] with various degrees of thinning.

4.1. Formulation of the Planar Array Thinning

The array factor [20] of the $M \times N$ periodic planar array is arranged in the grid with a distance d along M rows and N columns as given in Equation (4), where A_{mn} is the excitation of the element positioned

at (m, n), and k the wave number $(2\pi/\lambda)$, $U = \sin(\theta)\cos(\phi)$. $V = \sin(\theta)\sin(\phi)$ are direction cosines, and theta(θ), phi(ϕ) are the spherical angular coordinates describing the field. Equation (4) can be viewed as a finite double Fourier series. It also relates the element excitation of the planar array to its array factor through the relation of inverse 2D discrete Fourier transform so that applying the inverse Fourier transform on AF gives the element excitation. The number of turned on elements $T = \text{integer}(f, M_{\text{total}})$, where f is the fill factor or degree of thinning, and M_{total} is the number of element positions of the given aperture. The IFT method for the synthesis of planar array not exceeding the maximum peak side lobe level is the same as that of thinning the linear array [21]. Implementing steps for the planar array thinning are as follows.

$$AF(u, v) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} A_{nm} e^{jkd(nu+mv)} \tag{4}$$

Steps involved in planar array thinning are as follows [22]

Step 1: Start the initial synthesis by a random initialization with equal 0/1 probability for $M_{\text{total}} = MN$ element positions.

Step 2: Obtain the AF from A_{mn} using a $K \times K$ point 2-D inverse FFT with $K > \max(M, N)$ by applying zero padding.

Step 3: Adapt AF to the prescribed maximum PSL constraints.

Step 4: Compute A_{mn} for the adapted AF using a $K \times K$ point 2-D forward FFT.

Step 5: Truncate A_{mn} from $K \times K$ samples to the $M \times N$ samples coupled to the columns and rows of the aperture.

Step 6: In the case of an aperture with a non-rectangular shape, make the $MN - M_{\text{total}}$ samples of A_{mn} located outside the aperture equal to zero.

Step 7: Set T samples of A_{mn} with the largest amplitudes equal to one (ON) and the others to zero (OFF).

Step 8: Compare the ON/OFF element distribution of the present iteration with that of the previous one. In the case of a difference, proceed with step 9, otherwise exit the iteration loop and stop the synthesis.

Step 9: Repeat steps 2 through 8 until the maximum PSL requirements are met, or the maximum number of iterations is reached.

Array thinning is obtained by step 7, i.e., setting T excitations of the planar array with the largest amplitudes to one (turned on) and remaining to zero (turned off). Before moving to the next iteration, compare the new selection of turned on elements to the previous turned on elements. Only when this comparison gives a difference then it will continue with new iteration cycle; otherwise, it will be terminated. Array thinning is completely based on the selection of the largest array element excitations and therefore largely ignoring variations of the excitations coefficients, and termination of the synthesis occurs at a quite early stage. This synthesis gives the local optimum solution and is unable to escape from it. Because of this premature termination, the resultant thinned array element excitations in general will not fit into the required PSL. By specifying the PSL constraint five dB less than the required PSL, the synthesized element distribution will fit more or less to the real PSL objective. Since the initial synthesis is started with a random element distribution, each new trail will generate a different thinned array element distribution. The global optimum solution with overall lowest PSL is obtained by applying sufficient number of trails [8, 21, 23].

5. SIMULATION RESULTS OF THINNED PLANAR ARRAY

In this section the synthesis of the 16×20 planar array thinning with different degrees of thinning is presented. The considered planar array antenna system is an isotropic pattern with inter element spacing of 0.5λ. Here 512×512 point two-dimensional forward and inverse FFT techniques are used for the synthesis of array thinning in an iterative manner involving 10000 trails. The size of the FFT is decided by the PSL constraint. The number of side lobes in the given side lobe region gives the count for FFT. The far-field directions of the side lobes, which violate the PSL constraints, are corrected.

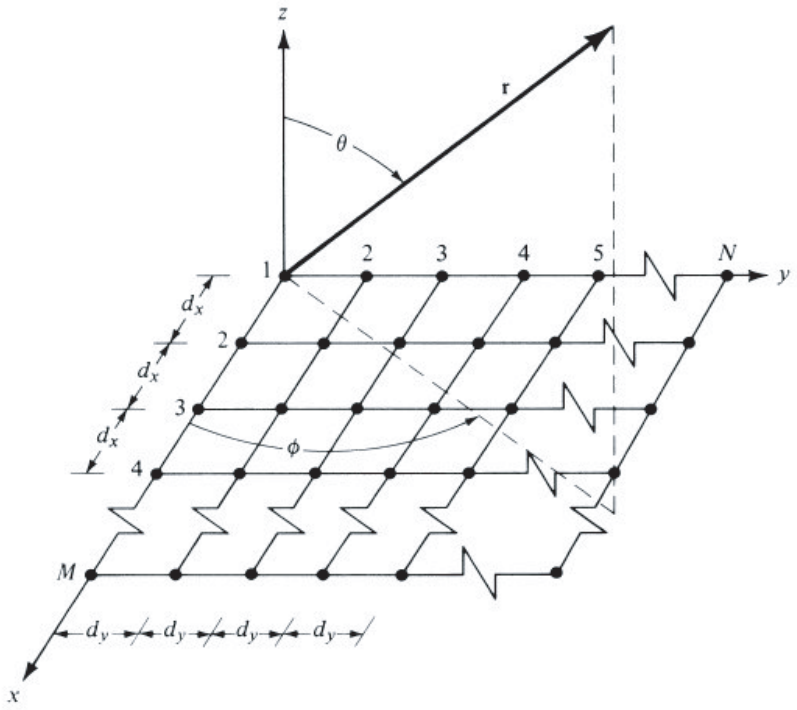

Figure 11. Arrangement of planar grid of antenna element.

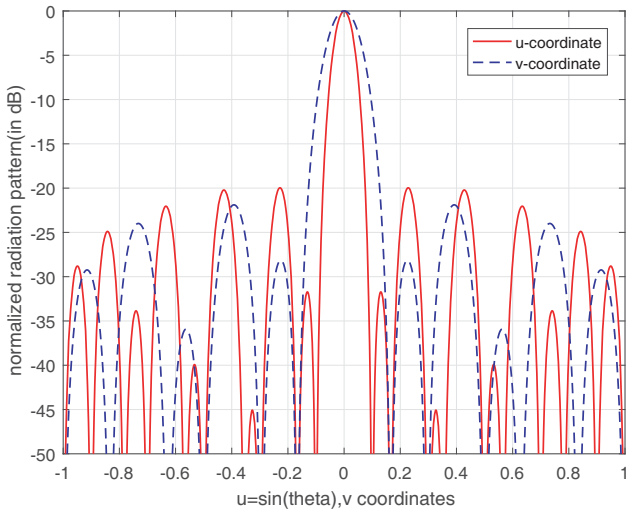

Figure 12. U-V section diagram of planar array without thinning.

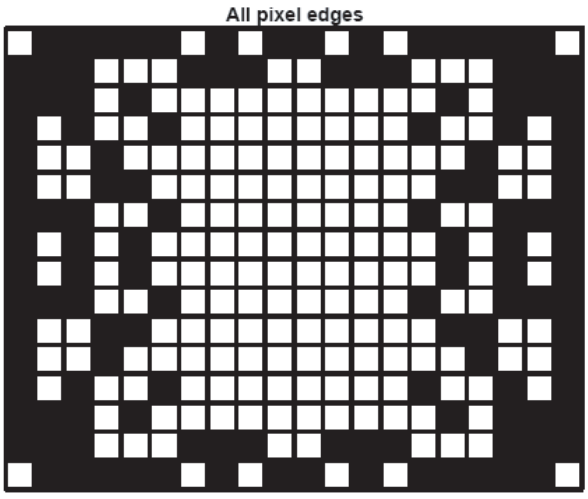

Figure 13. Contour of thinned planar array without thinning.

Prior knowledge of far-field directions of the side lobe regions, which violate the PSL requirements, are derived from the angular position of the first null of the AF at both sides of the main beam. The greater accuracy of the angular information is obtained by computing the AF with large size FFT. Synthesis of planar array without any thinning is depicted. The pictorial view of the U-V cut section diagram of planar array is shown in Figure 12. The original planar array contour without any thinning is shown in Figure 13. The method of IFT for planar array thinning was tested on a rectangular grid with fill factor of 45% and 55%.

5.1. Planar Array Thinning with a Degree of 55%

The size of rectangular grid is 16×20 as shown in Figure 11. The synthesis of thinned array is by IFT method of 512×512 ($K = 512$) samples and involving 10000 trials and $u, v\epsilon[-1, 1]$. In the whole process AF both dimensions of U-V space are subjected to the same maximum PSL requirement. All synthesized results refer to an isotropic embedded element pattern. The principal U-V cut of the synthesized array with 55% fill factor is shown in the Figure 14. In this case, the maximum PSL of the pattern is $\mathbf{-22.60\,dB}$, and the maximum required PSL is $\mathbf{-24.89\,dB}$ (obtained by trial and error) [24–26] which is the maximum requirement. The array has 320 element positions with 176 turned ON elements with 55% fill factor as shown in Figure 15.

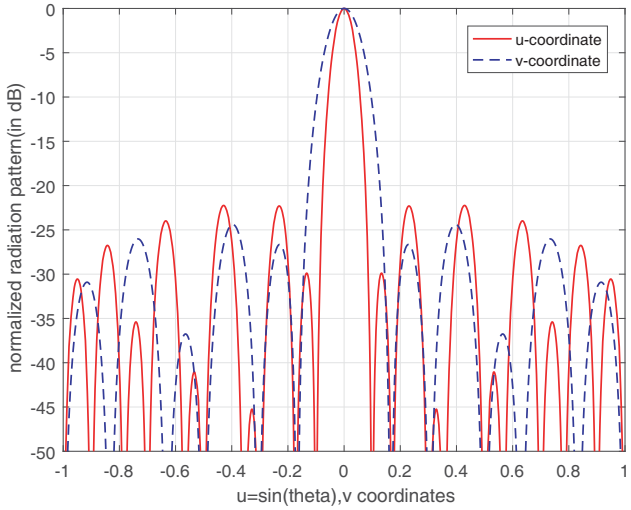

Figure 14. U-V section diagram of thinned planar array with fill factor 55%.

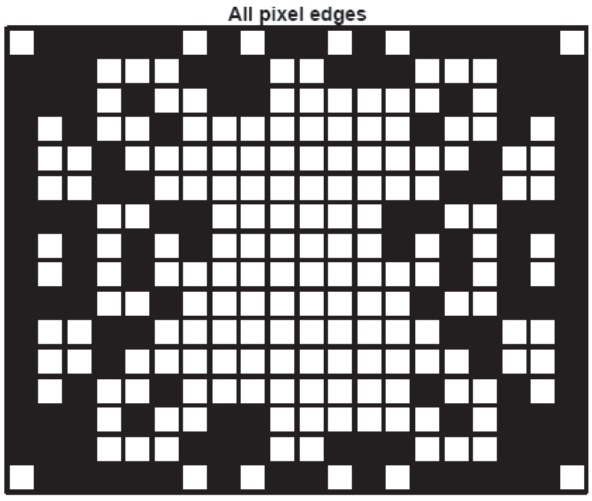

Figure 15. Contour of thinned planar array with a fill factor of 55%.

5.2. Planar Array Thinning with a Degree of 45%

The principal U-V cut of the synthesized array with 45% fill factor is as shown in Figure 16. In this case, the maximum PSL of the pattern is $\mathbf{-24.60\,dB}$, and the maximum required PSL is $\mathbf{-26.89\,dB}$ obtained by trial and error. The array has 320 elements positions with 144 turned on elements due to 45% fill factor as shown in Figure 17. These two cases reveal the high score of the thinned element distribution with low maximum PSL values. It clearly demonstrates the usefulness of the IFT method for array thinning. The corners of the planar array are not thinned. The comparison of the PSL and directovity of planar array obtained using IFT and statistical density taper is as shown in Table 4.

Table 4. Comparison of planar thinned array PSL and directory obtained with the IFT method and the statistical density taper ([11]).

Fill Factor (%)	Max PSL (dB)		Directivity (dB)		Turned ON elements
	IFT	Stat [11].	IFT	Stat [11].	
55	**−22.60**	−20.20	**30.40**	29.50	176
45	**−24.60**	−22.68	**31.10**	30.35	144

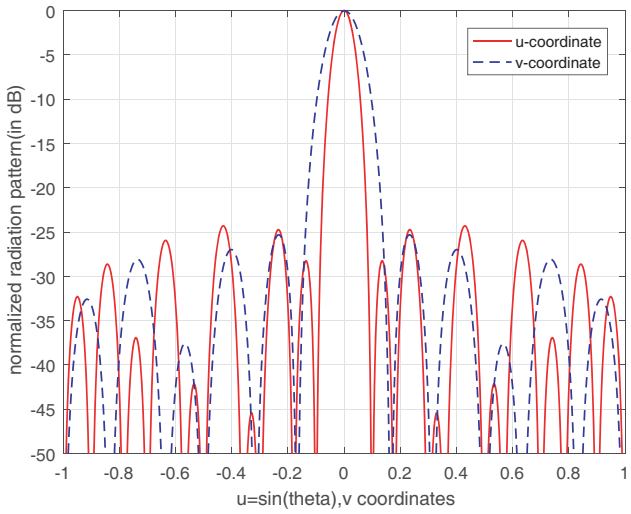

Figure 16. U-V section diagram of thinned planar array with fill factor 45%.

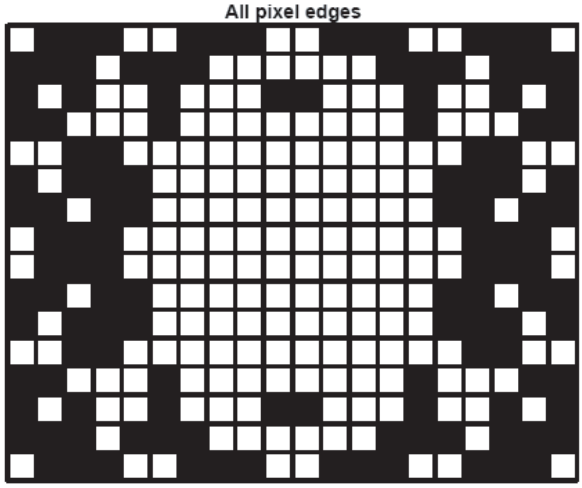

Figure 17. Contour of thinned planar array with a fill factor of 45%.

6. CONCLUSION

A concept of array thinning has been explained in this work to focus on the improvement of gain of the antenna array and to feature minimum Peak Side lobe Level. An iterative FFT procedure is applied on the Linear and Planar antenna array, and thinned beam pattern is analysed. Various simulation studies are carried out to strengthen the claim, utilizing FFT technique. Although there are thinning concepts carried out through evolutionary procedures, the proposed statistical method outperforms the random techniques. The proposed method is carried out to reach a predetermined peak side lobe level; however, there is a tradeoff between the obtained PSL and the predetermined PSL. This can prevail over by increasing the number of trial execution.

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