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Dielectric Slab Reflection/Transmission as a Self-Consistent Radiation Phenomenon

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Abstract—We revisit the standard electromagnetic problem wherein wave propagation within a uniform, lossless dielectric is interrupted by a dissipative slab of finite thickness. While such a problem is easily solved on the basis of interface field continuity, we proceed to treat it here under the viewpoint of radiative self-consistency, with effective current sources resident only within the slab interior and gauged by ohmic/polarization parameter comparisons against those of the reference, exterior medium. Radiative self-consistency finds its natural expression as an integral equation over the slab interior field which, once solved, permits a direct, fully constructive buildup, both up and down, of the reflected/transmitted field contributions, without any need for ascertaining such quantities implicitly via the enforcement of boundary conditions. The persistent cadence of solution steps in such integral-equation problems asserts itself here, too, in the sense that it leads, first, to an exact cancellation, left and right, of that interior, unknown field, and second, that it brings in still other contributions of a reference medium variety, of which it is required that they, and only they, balance the incoming excitation. Balancing of this latter sort provides indeed the linear conditions for slab field determination. The two-step solution pattern thus described may be regarded as a manifestation at some remove of Ewald-Oseen extinction, even though the analytic framework now on view differs fundamentally from proofs elsewhere available. We go on to solve the several balancing equations by direct, vector manipulation avoiding all recourse to large, unwieldy determinants, and then offer a partial confirmation by exhibiting a canonical, boundary value counterpart in the special case of perpendicular incidence. Following all of this, in an appendix, we allow the receptor, downstream half space to differ from that wherein the excitation had been launched and which continues to serve as the reference medium. Effective currents are now found not only within the slab proper, but also throughout an entire half space, necessitating a suitable generalization of the underlying integral equation, and a provision, during its solution, of cross-talk, both up and down, between slab and the half space now contributing as a radiation source. We provide in this appendix a fairly accelerated presentation of these generalized features, but with all logical details nevertheless fully displayed in plain view. The integral equation radiative self-consistency method is, to our way of thinking, physically far more satisfying than the prevailing method of scattered fields guessed as to their structure and then fixed by boundary conditions. Its analytic themes, moreover, are far, far more elegant.

1. INTRODUCTION

We wish here to revisit from a radically fresh viewpoint the standard problem of plane-wave electromagnetic reflection from, and transmission across, a lossy dielectric slab. Our motivation for doing so is rooted in methodological, pedagogical, and, above all else, in physical considerations. Indeed, we shall show that both the so-called reflected and transmitted fields arise as the cumulative radiation from the self-consistent ohmic/dielectric polarization currents excited within the slab interior. In particular, under this perspective, boundary matching conditions of the standard sort will never need to be invoked.

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This, at least from our vantage point upon reality, provides a physically much more satisfying picture of, say, the reflected wave, which, in normal, glib parlance, embodies an implied behavior resembling some type of interface bounce, and never mind that the ensuing boundary matching program does in fact yield correct results. A bounce there indeed is, but in the developments that follow we, so to speak, dissect its radiative anatomy, its radiative genesis.

Pedagogy and methodology are of course inextricably intertwined, and here both benefit in conviction by being allowed to unfold in a setting whose resolution is available by other means. The problem that we envision involves a lossy dielectric slab of thickness a, permittivity ϵ_2, conductivity σ_2, sandwiched between two identical, nondissipative dielectric half spaces, reference permittivity $\epsilon_1 \neq \epsilon_2$, reference conductivity $\sigma_1 = 0$.† A plane wave propagates downward from above, traversing the slab, and continuing into the half space below. But, because the slab differs from the reference medium, a current density $(\sigma_2 - i\omega\{\epsilon_2 - \epsilon_1\})\mathbf{E}_{\text{tot}}$, with $\mathbf{E}_{\text{tot}}$ being the local self-consistent, *total* electric field, incident plus radiated, is set up throughout its interior.‡ Radiation from these currents is then superimposed upon the primary in such a way as to establish that self-consistent field $\mathbf{E}_{\text{tot}}$ on both interior and exterior, and thus to modify as needed the plane wave vector amplitude in the half space below, and, of course, to assemble the plane-wave echo, due to slab presence, in the half space of wave origin above. When duly asserted within the slab interior, this statement of radiative self-consistency takes the form of an integral equation, easily derived and recorded as Eq. (9) below.

We shall move at once to set down the interior slab field in terms of up/down quasi plane waves and their vector amplitudes $\mathbf{E}^{\pm}_{\text{slab}}$, momentarily unknown, and to delineate the *bona fide* plane-wave aspect of the fields radiated into both enclosing half spaces. When framing the structure of the self-consistent interior field we shall naturally incorporate as much *a priori* information as the underlying Maxwell differential equations permit, but will of course stop short of reciting such equations anew. Following this, we shall turn to the main task of actually solving Eq. (9) for $\mathbf{E}^{\pm}_{\text{slab}}$ by methods which, while still intricate, are free from all allusion to boundary matching and the machinery of determinants. This work, in particular, will proceed without any orientational constraint upon the incoming plane wave vector amplitude $\mathbf{E}_{\text{inc}}$, apart, of course, from its obligatory orthogonality to the sense of propagation $\hat{\mathbf{n}}$, $\hat{\mathbf{n}} \cdot \mathbf{E}_{\text{inc}} = 0$. We will then pause momentarily to specialize our formulae so as to recover in every detail the classical results, usually associated with the name of Augustin-Jean Fresnel, at least in the simplest, default situation of perpendicular incidence. A concluding appendix, of considerable size, will generalize this work so as to allow for dissimilar enclosing half spaces, with nonvanishing ohmic dissipation available as an option on the downstream, receptor side.

2. PROBLEM SETUP

Origin $\mathbf{0}$ of a right-handed Cartesian coordinate system is taken to lie on the upper, wave incidence slab side, with unit vertical vector $\hat{\mathbf{e}}_z$ pointing upward, against the sense of arrival, and both x and y axes thus contained in the material interface.§ The incident plane wave is considered to propagate along $\hat{\mathbf{n}} = \hat{\mathbf{n}}_{\|} + \hat{n}_z \hat{\mathbf{e}}_z$, with its vertical component $\hat{n}_z$ negative, $\hat{n}_z < 0$, and $\hat{\mathbf{n}}_{\|}$ being its x-y projection.

The impinging wave has a propagation constant $k_1 = \omega\sqrt{\epsilon_1 \mu}$ and a spatial behavior governed by a phase factor $\exp(ik_1 \hat{\mathbf{n}} \cdot \mathbf{r})$, whose horizontal evolution in accordance with $\exp(ik_1 \hat{\mathbf{n}}_{\|} \cdot \boldsymbol{\rho})$ is imprinted upon all participating field categories, both within and exterior to the slab. Such commonality responds to the patent need to maintain a compatible horizontal phase progression everywhere, and is in fact quite spontaneously reaffirmed by our integral, radiative framework.

And so, if, for $-a < z < 0$, we write the total self-consistent slab field as

$$\mathbf{E}_{\text{tot}}(\mathbf{r}) = \sum_{\pm} \mathbf{E}^{\pm}_{\text{slab}} \exp\left(ik_1 \hat{\mathbf{n}}_{\|} \cdot \boldsymbol{\rho} + p_{\pm} z\right), \tag{1}$$

† In an appendix we set out at some length the type of analytic complications which ensue should dissimilar exterior half spaces be admitted into the discussion. Although not insurmountable, such complications are best avoided in an initial, proof-of-principle discussion such as that which now confronts us. Moreover, so as not to unduly complicate our slab problem, we further insist upon strict magnetic uniformity, with a common magnetic permeability $\mu_2 = \mu_1 = \mu$.

‡ A simple harmonic time dependence $\exp(-i\omega t)$, ω being of either sign, is assumed throughout and simply removed from all equations as a non-participating factor.

§ The corresponding basis vectors in the x-y plane are $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$, with vector $\boldsymbol{\rho} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y$ designating horizontal displacement from a vertical axis passing through origin $\mathbf{0}$. A full three-dimensional displacement $\mathbf{r}$ from $\mathbf{0}$ is then rendered as $\mathbf{r} = \boldsymbol{\rho} + z\hat{\mathbf{e}}_z$.

then a demand that it conforms to an underlying Helmholtz equation requires that vertical propagation parameters $p_\pm$ obey

$$p_\pm^2 - k_1^2 \hat{\mathbf{n}}_\parallel \cdot \hat{\mathbf{n}}_\parallel = -\omega^2 \mu \epsilon_2 \left(1 + i\sigma_2/\omega\epsilon_2\right). \tag{2}$$

On further setting $\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_z = \hat{n}_z = -\cos\vartheta$, with incidence angle ϑ bounded from above and below by $-\pi/2 < \vartheta < \pi/2$, we find that $\hat{\mathbf{n}}_\parallel \cdot \hat{\mathbf{n}}_\parallel = \sin^2\vartheta$ and hence

$$\begin{aligned} p_\pm \;=\; & \pm k_1 \sqrt{\sin^2\vartheta - (\epsilon_2/\epsilon_1)\left(1 + i\sigma_2/\omega\epsilon_2\right)} = \pm\frac{|k_1|}{\sqrt{2}}\left\{\sqrt{\sqrt{\left(\sin^2\vartheta - \epsilon_2/\epsilon_1\right)^2 + \left(\sigma_2/\omega\epsilon_1\right)^2} + \sin^2\vartheta - \epsilon_2/\epsilon_1}\right. \\ & \left. -i\frac{\omega}{|\omega|}\sqrt{\sqrt{\left(\sin^2\vartheta - \epsilon_2/\epsilon_1\right)^2 + \left(\sigma_2/\omega\epsilon_1\right)^2} - \sin^2\vartheta + \epsilon_2/\epsilon_1}\right\}, \end{aligned} \tag{3}$$

which latter pays due deference to the sign of angular frequency ω, all real square roots being taken positive. It then follows that index $+$ in Eq. (1) selects a wave component whose amplitude diminishes with penetration into the slab, whereas the complementary index $-$ accompanies amplitude growth. Observe that expression (3) accepts as valid the ordering $0 < \epsilon_2 < \epsilon_1$, so that our slab could in fact represent a thinning, a soft gap, in some sense a dilution, an evacuation of material in comparison with its two abutting half spaces. The half-angle trigonometric formulae which underlie the nested roots in Eq. (3) must track their angular arguments with great care when arriving at the stated expression.

Field in Eq. (1) must in addition be solenoidal, with vanishing divergence, circumstance which requires that vector amplitudes $\mathbf{E}^\pm_{\text{slab}}$ submit to the null dot products

$$\mathbf{P}^\pm \cdot \mathbf{E}^\pm_{\text{slab}} = 0, \tag{4}$$

with

$$\mathbf{P}^\pm = ik_1 \hat{\mathbf{n}}_\parallel + p_\pm \hat{\mathbf{e}}_z. \tag{5}$$

3. BASIC INTEGRAL EQUATION

We advocate the elementary viewpoint that the ϵ_1 medium of the enclosing half spaces provides a standard wave propagation environment disrupted only by the ohmic/excess polarization current density $(\sigma_2 - i\omega\{\epsilon_2 - \epsilon_1\})\mathbf{E}_{\text{tot}}(\mathbf{r})$ encountered across the vertical slot $0 > z > -a$. Such currents then radiate *everywhere* a scattered magnetic field

$$\mathbf{B}_{\text{scatt}}(\mathbf{r}) = \frac{\mu(\sigma_2 - i\omega\{\epsilon_2 - \epsilon_1\})}{4\pi}\sum_\pm \boldsymbol{\nabla}\times\left\{\mathbf{E}^\pm_{\text{slab}}\int_{\text{slab}}\frac{e^{ik_1|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\exp\left(ik_1\hat{\mathbf{n}}_\parallel\cdot\boldsymbol{\rho}' + p_\pm z'\right)dx'dy'dz'\right\} \tag{6}$$

which combines linearly with that incident, $\mathbf{B}_{\text{inc}}(\mathbf{r}) = (k_1/\omega)(\hat{\mathbf{n}} \times \mathbf{E}_{\text{inc}})\exp(ik_1\hat{\mathbf{n}}\cdot\mathbf{r})$, to yield a total

$$\begin{aligned} \mathbf{B}_{\text{tot}}(\mathbf{r}) \;=\; & \frac{k_1}{\omega}\hat{\mathbf{n}}\times\mathbf{E}_{\text{inc}}e^{ik_1\hat{\mathbf{n}}\cdot\mathbf{r}} \\ & +\frac{\mu(\sigma_2 - i\omega\{\epsilon_2 - \epsilon_1\})}{4\pi}\sum_\pm\boldsymbol{\nabla}\times\left\{\mathbf{E}^\pm_{\text{slab}}\int_{\text{slab}}\frac{e^{ik_1|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\exp\left(ik_1\hat{\mathbf{n}}_\parallel\cdot\boldsymbol{\rho}' + p_\pm z'\right)dx'dy'dz'\right\}. \end{aligned} \tag{7}$$

If we next define an effective dielectric permeability $\hat{\epsilon}(z)$ by setting

$$\hat{\epsilon}(z) = \begin{cases} \epsilon_1; & z > 0 \\ \epsilon_2 + i\sigma_2/\omega; & 0 > z > -a \\ \epsilon_1; & -a > z \end{cases}, \tag{8}$$

we find that the total electric field $\mathbf{E}_{\text{tot}}(\mathbf{r})$ accompanying (7) is gotten as

$$\begin{aligned} \hat{\epsilon}(z)\mathbf{E}_{\text{tot}}(\mathbf{r}) = & \;\epsilon_1\mathbf{E}_{\text{inc}}e^{ik_1\hat{\mathbf{n}}\cdot\mathbf{r}} \\ & +\frac{i(\sigma_2 - i\omega\{\epsilon_2-\epsilon_1\})}{4\pi\omega}\sum_\pm\boldsymbol{\nabla}\times\left[\boldsymbol{\nabla}\times\left\{\mathbf{E}^\pm_{\text{slab}}\int_{\text{slab}}\frac{e^{ik_1|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\exp\left(ik_1\hat{\mathbf{n}}_\parallel\cdot\boldsymbol{\rho}' + p_\pm z'\right)dx'dy'dz'\right\}\right]. \end{aligned} \tag{9}$$

Equations (7) and (9) hold everywhere. Evaluated respectively for $z > 0$ and $z < -a$, they provide a direct buildup of both reflected and transmitted fields, whereas enforcement across the slab interior $0 > z > -a$ determines the key interior amplitudes $\mathbf{E}^{\pm}_{\text{slab}}$. Total field buildup as a sum of incident plus scattered/radiated contributions, while assuredly not a novel concept, receives a concise, integral-equation statement, akin to Eq. (9), in [1] and [2], and a multitude of sources elsewhere, most notably in quantum mechanics/quantum field theory, where it spawns, *inter multa alia*, an entire universe of iterative approximations, the lowest among them associated with the name of Born.

4. PRELIMINARY CALCULATIONS

The slab integral on the right in Eq. (9) is easily performed with the aid of well-known results concerning Bessel function J_0. Thus a temporary shift of the horizontal origin to $\boldsymbol{\rho}$, followed by quadrature across the full, 0 to 2π azimuthal range, gives[‖]

$$\begin{aligned}
&\int_{\text{slab}} \frac{e^{ik_1|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \exp\left(ik_1\hat{\mathbf{n}}_\parallel \cdot \boldsymbol{\rho}' + p_\pm z'\right) dx'dy'dz' \\
&= 2\pi \exp\left(ik_1\hat{\mathbf{n}}_\parallel \cdot \boldsymbol{\rho}\right) \int_{-a}^{0} dz' \exp\left(p_\pm z'\right) \int_0^\infty \frac{e^{ik_1\sqrt{\rho'^2+(z-z')^2}}}{\sqrt{\rho'^2+(z-z')^2}} \rho' J_0(k_1\rho'\sin\vartheta)d\rho' \\
&= \frac{2\pi}{ik_1\hat{n}_z} \exp\left(ik_1\hat{\mathbf{n}}_\parallel \cdot \boldsymbol{\rho}\right) \int_{-a}^{0} \exp\left(p_\pm z' - ik_1\hat{n}_z|z-z'|\right) dz'.
\end{aligned} \tag{10}$$

The final integral on the right in Eq. (10) must be individually adapted to the slab *per se* and to each of the enclosing half spaces, its respective values being denoted as $I_1^\pm(z)$ for $z > 0$, $I_2^\pm(z)$ when $0 > z > -a$, and finally $I_3^\pm(z)$ throughout the lower half space with $-a > z$. We get:

<u>$\boldsymbol{I_1^\pm(z),\ z > 0}$</u>:

$$\begin{aligned}
I_1^\pm(z) &= \exp\left(-ik_1\hat{n}_z z\right) \int_{-a}^{0} \exp\left(\{ik_1\hat{n}_z + p_\pm\} z'\right) dz' \\
&= \exp\left(-ik_1\hat{n}_z z\right) \{ik_1\hat{n}_z + p_\pm\}^{-1} \left[1 - \exp\left(-\{ik_1\hat{n}_z + p_\pm\} a\right)\right];
\end{aligned} \tag{11}$$

<u>$\boldsymbol{I_3^\pm(z),\ -a > z}$</u>:

$$\begin{aligned}
I_3^\pm(z) &= \exp\left(ik_1\hat{n}_z z\right) \int_{-a}^{0} \exp\left(-\{ik_1\hat{n}_z - p_\pm\} z'\right) dz' \\
&= -\exp\left(ik_1\hat{n}_z z\right) \{ik_1\hat{n}_z - p_\pm\}^{-1} \left[1 - \exp\left(\{ik_1\hat{n}_z - p_\pm\} a\right)\right];
\end{aligned} \tag{12}$$

and then

<u>$\boldsymbol{I_2^\pm(z),\ 0 > z > -a}$</u>:

$$\begin{aligned}
I_2^\pm(z) &= \exp\left(-ik_1\hat{n}_z z\right) \int_{-a}^{z} \exp\left(\{ik_1\hat{n}_z + p_\pm\} z'\right) dz' + \exp\left(ik_1\hat{n}_z z\right) \int_{z}^{0} \exp\left(-\{ik_1\hat{n}_z - p_\pm\} z'\right) dz' \\
&= \exp\left(-ik_1\hat{n}_z z\right) \{ik_1\hat{n}_z + p_\pm\}^{-1} \left[\exp\left(\{ik_1\hat{n}_z + p_\pm\} z\right) - \exp\left(-\{ik_1\hat{n}_z + p_\pm\} a\right)\right] \\
&\quad + \exp\left(ik_1\hat{n}_z z\right) \{ik_1\hat{n}_z - p_\pm\}^{-1} \left[\exp\left(-\{ik_1\hat{n}_z - p_\pm\} z\right) - 1\right].
\end{aligned} \tag{13}$$

On collating Eq. (12) with Eq. (10) we see that $I_3^\pm(z)$ automatically assembles a phase progression $\exp(ik_1\hat{\mathbf{n}} \cdot \mathbf{r})$ befitting wave penetration into the subjacent half space. Structure $I_1^\pm(z)$ similarly provides for the emission of a reflected wave governed by $\exp(ik_1\{\hat{\mathbf{n}}_\parallel \cdot \boldsymbol{\rho} - \hat{n}_z z\})$ as its phase factor, a traditional

‖ We call attention to the fact that $\hat{n}_z$, here and in all that follows, is negative. That the third line of Eq. (10) duly accommodates this attribute is guaranteed by meticulous checks against the Bessel quadrature formulae found in [3]. It is further reaffirmed, and much more directly so, by the elementary quadrature which is made available on setting $\vartheta = 0$. The very convergence of this latter quadrature, and its succinct, convenient outcome, are predicated as always on k_1 having a small, residual dissipative part, regardless of the sign of angular frequency ω.

outcome which we formalize by setting $\hat{\mathbf{n}}' = \hat{\mathbf{n}}_{\parallel} - \hat{n}_z \hat{\mathbf{e}}_z$ so as to arrive finally at its abbreviated form $\exp(ik_1 \hat{\mathbf{n}}' \cdot \mathbf{r})$. In both cases the indicated wave progression is of a pure unidirectional type.

By contrast, intermediate structure $I_2^{\pm}(z)$ from Eq. (13) mixes the inherent slab propagation according to $\exp(ik_1 \hat{\mathbf{n}}_{\parallel} \cdot \boldsymbol{\rho} + p_{\pm} z)$ with both exterior progression types found under Eqs. (11) and (12). We make this segregation into propagation types more explicit by rewriting Eq. (13) as

$\boldsymbol{I_2^{\pm}(z),\ 0 > z > -a}$:

$$I_2^{\pm}(z) = \left\{ \frac{1}{ik_1\hat{n}_z + p_{\pm}} + \frac{1}{ik_1\hat{n}_z - p_{\pm}} \right\} e^{p_{\pm} z} - \frac{1}{ik_1\hat{n}_z - p_{\pm}} e^{ik_1\hat{n}_z z} - \frac{e^{-\{ik_1\hat{n}_z + p_{\pm}\}a}}{ik_1\hat{n}_z + p_{\pm}} e^{-ik_1\hat{n}_z z} \tag{14}$$

wherein, for imminent use in Eq. (19), we note that

$$\frac{1}{ik_1\hat{n}_z + p_{\pm}} + \frac{1}{ik_1\hat{n}_z - p_{\pm}} = -\frac{2ik_1\hat{n}_z}{k_1^2\hat{n}_z^2 + p_{\pm}^2} = \frac{2k_1\hat{n}_z}{\omega\mu\left(\sigma_2 - i\omega\{\epsilon_2 - \epsilon_1\}\right)}, \tag{15}$$

the final member on the right emerging after appeal to Eq. (2). Imminent too in Eq. (20) is a need for the two statements that

$$\mathbf{P}^{\pm} \cdot \mathbf{P}^{\pm} = p_{\pm}^2 - k_1^2 \hat{\mathbf{n}}_{\parallel} \cdot \hat{\mathbf{n}}_{\parallel} = -\omega^2 \mu \epsilon_2 \left(1 + i\sigma_2/\omega\epsilon_2\right), \tag{16}$$

statements already anticipated by Eq. (2).

5. UP/DOWN HALF SPACE FIELD STRUCTURE

With the tools in Eqs. (10)–(12) in hand, we can at once state the total half space fields respectively above and below in the highly symmetric forms

$$\mathbf{E}_{\text{tot}}(\mathbf{r}) = \mathbf{E}_{\text{inc}} e^{ik_1\hat{\mathbf{n}}\cdot\mathbf{r}} - \frac{k_1(\sigma_2 - i\omega\{\epsilon_2 - \epsilon_1\})}{2\hat{n}_z\epsilon_1\omega} e^{ik_1\hat{\mathbf{n}}'\cdot\mathbf{r}} \hat{\mathbf{n}}' \times \left[\hat{\mathbf{n}}' \times \sum_{\pm} \left\{ \frac{1 - \exp\left(-\{ik_1\hat{n}_z + p_{\pm}\} a\right)}{ik_1\hat{n}_z + p_{\pm}} \right\} \mathbf{E}_{\text{slab}}^{\pm} \right] \tag{17}$$

and

$$\mathbf{E}_{\text{tot}}(\mathbf{r}) = e^{ik_1\hat{\mathbf{n}}\cdot\mathbf{r}} \left(\mathbf{E}_{\text{inc}} + \frac{k_1(\sigma_2 - i\omega\{\epsilon_2 - \epsilon_1\})}{2\hat{n}_z\epsilon_1\omega} \hat{\mathbf{n}} \times \left[\hat{\mathbf{n}} \times \sum_{\pm} \left\{ \frac{1 - \exp\left(\{ik_1\hat{n}_z - p_{\pm}\} a\right)}{ik_1\hat{n}_z - p_{\pm}} \right\} \mathbf{E}_{\text{slab}}^{\pm} \right] \right). \tag{18}$$

Not to be overlooked is the automatic transversality to propagation directions $\hat{\mathbf{n}}'$ and $\hat{\mathbf{n}}$ with which Eqs. (17) and (18) endow the up/down radiated field contributions. At the same time the information which Eqs. (17) and (18) provide remains incomplete until such time as vector amplitudes $\mathbf{E}_{\text{slab}}^{\pm}$ have been fully ascertained. Sections 7 and 8, Eqs. (30), (31), (33), and (37) below provide the requisite completion.

6. SOLVING THE SLAB INTEGRAL EQUATION: STEP 1

With reference to Eqs. (1), (5), (10), (14)–(15), Eq. (9), when stated within the slab interior $0 > z > -a$, can now be written as

$$\begin{aligned}
&\left(\frac{\epsilon_2 + i\sigma_2/\omega}{\epsilon_1} \right) \sum_{\pm} \mathbf{E}_{\text{slab}}^{\pm} \exp\left(\mathbf{P}^{\pm} \cdot \mathbf{r}\right) \\
&= \mathbf{E}_{\text{inc}} e^{ik_1\hat{\mathbf{n}}\cdot\mathbf{r}} + k_1^{-2} \sum_{\pm} \mathbf{P}^{\pm} \times \left\{ \mathbf{P}^{\pm} \times \mathbf{E}_{\text{slab}}^{\pm} \right\} \exp\left(\mathbf{P}^{\pm} \cdot \mathbf{r}\right) \\
&\quad + \left(\frac{k_1\left(\sigma_2 - i\omega\left\{\epsilon_2 - \epsilon_1\right\}\right)}{2\hat{n}_z\epsilon_1\omega} \right) e^{ik_1\hat{\mathbf{n}}\cdot\mathbf{r}} \hat{\mathbf{n}} \times \left[\hat{\mathbf{n}} \times \sum_{\pm} \left\{ \frac{1}{ik_1\hat{n}_z - p_{\pm}} \right\} \mathbf{E}_{\text{slab}}^{\pm} \right] \\
&\quad + \left(\frac{k_1\left(\sigma_2 - i\omega\left\{\epsilon_2 - \epsilon_1\right\}\right)}{2\hat{n}_z\epsilon_1\omega} \right) e^{ik_1\hat{\mathbf{n}}'\cdot\mathbf{r}} \hat{\mathbf{n}}' \times \left[\hat{\mathbf{n}}' \times \sum_{\pm} \left\{ \frac{\exp\left(-\left\{ik_1\hat{n}_z + p_{\pm}\right\} a\right)}{ik_1\hat{n}_z + p_{\pm}} \right\} \mathbf{E}_{\text{slab}}^{\pm} \right].
\end{aligned} \tag{19}$$

And then, since

$$\mathbf{P}^{\pm} \times \left\{\mathbf{P}^{\pm} \times \mathbf{E}_{\text{slab}}^{\pm}\right\} = \left(\mathbf{P}^{\pm} \cdot \mathbf{E}_{\text{slab}}^{\pm}\right) \mathbf{P}^{\pm} - \left(\mathbf{P}^{\pm} \cdot \mathbf{P}^{\pm}\right) \mathbf{E}_{\text{slab}}^{\pm} = k_1^2 \left(\frac{\epsilon_2 + i\sigma_2/\omega}{\epsilon_1}\right) \mathbf{E}_{\text{slab}}^{\pm}, \tag{20}$$

its second entry holding in consequence of Eqs. (4) and (16), we see that the slab field $\sum_{\pm} \mathbf{E}_{\text{slab}}^{\pm} \exp(\mathbf{P}^{\pm} \cdot \mathbf{r})$ *per se* cancels¶ identically from both sides of Eq. (19), leaving us to contend with just

$$\mathbf{0} \;=\; \mathbf{E}_{\text{inc}} e^{ik_1 \hat{\mathbf{n}}\cdot\mathbf{r}} + \left(\frac{k_1 \left(\sigma_2 - i\omega\left\{\epsilon_2 - \epsilon_1\right\}\right)}{2\hat{n}_z \epsilon_1 \omega}\right) e^{ik_1 \hat{\mathbf{n}}\cdot\mathbf{r}} \hat{\mathbf{n}} \times \left[\hat{\mathbf{n}} \times \sum_{\pm} \left\{\frac{1}{ik_1 \hat{n}_z - p_{\pm}}\right\} \mathbf{E}_{\text{slab}}^{\pm}\right]$$
$$+ \left(\frac{k_1 \left(\sigma_2 - i\omega\left\{\epsilon_2 - \epsilon_1\right\}\right)}{2\hat{n}_z \epsilon_1 \omega}\right) e^{ik_1 \hat{\mathbf{n}}'\cdot\mathbf{r}} \hat{\mathbf{n}}' \times \left[\hat{\mathbf{n}}' \times \sum_{\pm} \left\{\frac{\exp\left(-\left\{ik_1 \hat{n}_z + p_{\pm}\right\} a\right)}{ik_1 \hat{n}_z + p_{\pm}}\right\} \mathbf{E}_{\text{slab}}^{\pm}\right]. \tag{21}$$

But, as the functions $e^{\pm ik_1 \hat{n}_z z}$ are linearly independent, it follows that Eq. (21) decouples into the two individual vector statements

$$\hat{\mathbf{n}} \times \left[\hat{\mathbf{n}} \times \sum_{\pm} \left\{\frac{1}{ik_1 \hat{n}_z - p_{\pm}}\right\} \mathbf{E}_{\text{slab}}^{\pm}\right] = -\left(\frac{2\hat{n}_z \epsilon_1 \omega}{k_1 \left(\sigma_2 - i\omega\left\{\epsilon_2 - \epsilon_1\right\}\right)}\right) \mathbf{E}_{\text{inc}} \tag{22}$$

and

$$\hat{\mathbf{n}}' \times \left[\hat{\mathbf{n}}' \times \sum_{\pm} \left\{\frac{\exp\left(-\left\{ik_1 \hat{n}_z + p_{\pm}\right\} a\right)}{ik_1 \hat{n}_z + p_{\pm}}\right\} \mathbf{E}_{\text{slab}}^{\pm}\right] = \mathbf{0}, \tag{23}$$

fully adequate to determine all six components+ of vectors $\mathbf{E}_{\text{slab}}^{\pm}$. A like linear independence had already permitted us to assert the individual orthogonalities subsumed under Eq. (4), orthogonalities of critical importance throughout the work that follows.

7. SOLVING THE SLAB INTEGRAL EQUATION: STEP 2

We tackle first the homogeneous, subsidiary condition in Eq. (23). But, before doing so, we ease and systematize somewhat the ensuing notation by setting

$$\begin{cases} \mathbf{P}^{\nu} \cdot \hat{\mathbf{n}} = ik_1 \sin^2 \vartheta - p_{\nu} \cos \vartheta \\ \mathbf{P}^{\nu} \cdot \hat{\mathbf{n}}' = ik_1 \sin^2 \vartheta + p_{\nu} \cos \vartheta \\ \gamma_{\nu} = -ik_1 \cos \vartheta + p_{\nu} \\ \nu = \pm. \end{cases} \tag{24}$$

In particular, on collating the last two denominators from Eq. (15), we observe at once that

$$\omega\mu\left(\sigma_2 - i\omega\left\{\epsilon_2 - \epsilon_1\right\}\right) = i\left(k_1^2 \hat{n}_z^2 + p_{\pm}^2\right) = i\left(k_1^2 \cos^2 \vartheta + p_{\pm}^2\right) = -i\gamma_{\mp}\gamma_{\pm}. \tag{25}$$

And, suffer though it may from the self-evident redundancies $\mathbf{P}^{\nu} \cdot \hat{\mathbf{n}}' = \mathbf{P}^{-\nu} \cdot \hat{\mathbf{n}}$, tableau of Eq. (24) will nevertheless be immediately seen to bestow a much valued notational compression. The redundancies

¶ Cancellation of this sort is a persistent analytic phenomenon, verily a hallmark *leitmotif*, uniformly encountered whenever analogues of integral equation (9) are brought to bear on other geometries. In particular, it occurs in the case of a dissipative sphere exposed to a punctual electric dipole radiating in close proximity. Solution details, albeit inherently intricate, are, from a conceptual viewpoint, no more involved than those here undertaken. A presentation, complete with a Poynting/Joule energy budget analysis, is set out in two unpublished internal memoranda, proprietary to Allwave Corporation and dated on April 29 and May 19, 2009.

It would be tempting to muster out similar energy assessments in the present planar context. But we shall refrain from doing so, first in the interest of maintaining some semblance of brevity, and second, since, in fact, such assessments, at base, have nothing whatsoever to do with the electromagnetic thesis at hand, a thesis that centers on radiative self-consistency.

Moreover, the fact that integration over slab polarization/ohmic sources not only reproduces the internal field exactly, without in any way subjecting it directly to additional conditions, a field further accompanied by type 1 contributions which latter must then neutralize the incoming, type 1 excitation(s), as conveyed by Eqs. (22)–(23) and (73)–(74), can be regarded as a manifestation of the Ewald-Oseen extinction theorem, even though the present analysis differs essentially from the proof on view in [4].

+ In point of fact, Eqs. (22)–(23), being stated *ipso facto* in planes respectively perpendicular to propagation vectors $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}'$, amount to just *four* equations. But this number suffices to underpin a fully determinate linear system since solenoidal constraints in Eq. (4) similarly reduce to four the number of independent Cartesian components. In what follows, however, we bypass completely all effort to solve this four-by-four system along any such routine path.

themselves will play decisive roles in settling critical arguments further along. In any event, it now follows from Eq. (23), once its repeated vector product has been duly unraveled, that

$$\sum_{\nu} \gamma_{\nu}^{-1} e^{-\gamma_{\nu} a} \mathbf{E}_{\text{slab}}^{\nu} = \alpha \hat{\mathbf{n}}', \tag{26}$$

with coefficient α yet to be determined. And, with a view to Eq. (4), we then further get

$$\mathbf{P}^{\nu} \cdot \mathbf{E}_{\text{slab}}^{-\nu} = \alpha\, \gamma_{-\nu} e^{\gamma_{-\nu} a} \left(\mathbf{P}^{\nu} \cdot \hat{\mathbf{n}}'\right) = \alpha \gamma_{-\nu} e^{\gamma_{-\nu} a} \left(\mathbf{P}^{-\nu} \cdot \hat{\mathbf{n}}\right). \tag{27}$$

In similar fashion, scalar multiplication by $\mathbf{P}^{\nu}$ of the inhomogeneous relation in Eq, (22), taking advantage once again of Eq. (4), yields

$$\sum_{\nu'} \gamma_{-\nu'}^{-1} \left(\hat{\mathbf{n}} \cdot \mathbf{E}_{\text{slab}}^{\nu'}\right) = \left(\mathbf{P}^{\nu} \cdot \hat{\mathbf{n}}\right)^{-1} \left\{ \gamma_{\nu}^{-1} \left(\mathbf{P}^{\nu} \cdot \mathbf{E}_{\text{slab}}^{-\nu}\right) - \left(\frac{2\hat{n}_z \epsilon_1 \omega}{k_1 \left(\sigma_2 - i\omega \left\{\epsilon_2 - \epsilon_1\right\}\right)} \right) \left(\mathbf{P}^{\nu} \cdot \mathbf{E}_{\text{inc}}\right) \right\}. \tag{28}$$

We subtract next the two versions of Eq. (28) at $\nu = \pm$ so as to remove its left member, and we bring in the information already acquired in Eq. (27), producing thereby an explicit, if seemingly unwieldy determination of coefficient α. Thus

$$\alpha \sum_{\nu} \nu \frac{\gamma_{-\nu} e^{\gamma_{-\nu} a} \left(\mathbf{P}^{-\nu} \cdot \hat{\mathbf{n}}\right)}{\gamma_{\nu} \left(\mathbf{P}^{\nu} \cdot \hat{\mathbf{n}}\right)} = \left(\frac{2\hat{n}_z \epsilon_1 \omega}{k_1 \left(\sigma_2 - i\omega \left\{\epsilon_2 - \epsilon_1\right\}\right)} \right) \sum_{\nu} \nu \frac{\left(\mathbf{P}^{\nu} \cdot \mathbf{E}_{\text{inc}}\right)}{\left(\mathbf{P}^{\nu} \cdot \hat{\mathbf{n}}\right)}. \tag{29}$$

This determination emerges finally in the more streamlined form

$$\alpha \sum_{\nu} \nu \gamma_{-\nu}^2 e^{\gamma_{-\nu} a} \frac{\left(\mathbf{P}^{-\nu} \cdot \hat{\mathbf{n}}\right)}{\left(\mathbf{P}^{\nu} \cdot \hat{\mathbf{n}}\right)} = 2ik_1 \hat{n}_z \sum_{\nu} \nu \frac{\left(\mathbf{P}^{\nu} \cdot \mathbf{E}_{\text{inc}}\right)}{\left(\mathbf{P}^{\nu} \cdot \hat{\mathbf{n}}\right)} \tag{30}$$

when further use is made of Eq. (25).

8. SOLVING THE SLAB INTEGRAL EQUATION: STEP 3

Since α is now known, we can navigate freely between Eqs. (22) and (26) so as to finalize an explicit buildup of both slab amplitudes $\mathbf{E}_{\text{slab}}^{\pm}$. For instance, setting, on the basis of Eq. (26),

$$\mathbf{E}_{\text{slab}}^{-} = \gamma_{-} e^{\gamma_{-} a} \left\{ \alpha \hat{\mathbf{n}}' - \gamma_{+}^{-1} e^{-\gamma_{+} a} \mathbf{E}_{\text{slab}}^{+} \right\}, \tag{31}$$

its subsequent introduction into Eq. (22) yields

$$\hat{\mathbf{n}} \times \left[\hat{\mathbf{n}} \times \left(\alpha \gamma_{+} \gamma_{-}^2 e^{\gamma_{-} a} \hat{\mathbf{n}}' + \left\{ \gamma_{+}^2 - \gamma_{-}^2 e^{(\gamma_{-} - \gamma_{+}) a} \right\} \mathbf{E}_{\text{slab}}^{+} \right) \right] = -2ik_1 \hat{n}_z \gamma_{+} \mathbf{E}_{\text{inc}}. \tag{32}$$

And then, on segregating the several vectorial terms of Eq. (32) into those already known

$$\mathbf{\Omega} = i \left\{ \gamma_{-}^2 e^{(\gamma_{-} - \gamma_{+}) a} - \gamma_{+}^2 \right\}^{-1} \left[2k_1 \hat{n}_z \gamma_{+} \mathbf{E}_{\text{inc}} - i\alpha \gamma_{+} \gamma_{-}^2 e^{\gamma_{-} a} \hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \hat{\mathbf{n}}'\right) \right], \tag{33}$$

and those *a priori* presumed to be still unknown, *viz.*, $\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{E}_{\text{slab}}^{+})$, we nonetheless find

$$\hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \mathbf{E}_{\text{slab}}^{+}\right) = \mathbf{\Omega}, \tag{34}$$

or else

$$\hat{\mathbf{n}} \left(\hat{\mathbf{n}} \cdot \mathbf{E}_{\text{slab}}^{+}\right) - \mathbf{E}_{\text{slab}}^{+} = \mathbf{\Omega}, \tag{35}$$

whereupon an appeal yet again to Eq. (4) now yields

$$\left(\mathbf{P}^{+} \cdot \hat{\mathbf{n}}\right) \left(\hat{\mathbf{n}} \cdot \mathbf{E}_{\text{slab}}^{+}\right) = \left(\mathbf{P}^{+} \cdot \mathbf{\Omega}\right) \tag{36}$$

and thus fixes

$$\mathbf{E}_{\text{slab}}^{+} = \hat{\mathbf{n}} \left(\mathbf{P}^{+} \cdot \mathbf{\Omega}\right) / \left(\mathbf{P}^{+} \cdot \hat{\mathbf{n}}\right) - \mathbf{\Omega} \tag{37}$$

in its entirety. A retrospective glance at Eq. (31) confirms that full knowledge regarding $\mathbf{E}_{\text{slab}}^{+}$ fixes likewise partner $\mathbf{E}_{\text{slab}}^{-}$, and in principle, we are done.

9. A PARTIAL RECONCILIATION WITH BOUNDARY MATCHING

As initially announced, we have now succeeded in solving the entire slab reflection/transmission problem without ever once allowing our pen to stray in the direction of boundary matching and its determinantal support apparatus. Still, it behooves us to demonstrate some level of concordance, however minimal.

We propose indeed to base a technique comparison in the very simplest context, that of normal wave incidence. While admittedly this will not exercise the full scope of either apparatus, avoidance of a potential failure should weaken any *a priori* urge to condemn. But, as will happily turn out, our limited comparison is destined to emerge unscathed by any such failure.

At normal incidence $\vartheta = 0$, $\hat{\mathbf{n}} = -\hat{\mathbf{e}}_z$, $\hat{n}_z = -1$, so that $\mathbf{P}^{\pm} \cdot \mathbf{E}_{\text{inc}} = 0$, whence $\alpha = 0$ and thus

$$\mathbf{E}^{-}_{\text{slab}} = -\gamma_{-}\gamma_{+}^{-1} e^{(\gamma_{-}-\gamma_{+})a} \mathbf{E}^{+}_{\text{slab}}. \tag{38}$$

As a replacement of Eq. (3) we now have

$$p_{\pm} = \pm \frac{|k_1|}{\sqrt{2\epsilon_1}} \left\{ \sqrt{\sqrt{\epsilon_2^2 + (\sigma_2/\omega)^2} - \epsilon_2} - i\frac{\omega}{|\omega|}\sqrt{\sqrt{\epsilon_2^2 + (\sigma_2/\omega)^2} + \epsilon_2} \right\} \tag{39}$$

and $\gamma_{\pm} = -ik_1 + p_{\pm}$. Moreover $\hat{\mathbf{n}} \times \hat{\mathbf{n}}' = \mathbf{0}$ so that

$$\mathbf{\Omega} = 2ik_1\gamma_{+} \left\{ \gamma_{+}^2 - \gamma_{-}^2 e^{(\gamma_{-}-\gamma_{+})a} \right\}^{-1} \mathbf{E}_{\text{inc}}, \tag{40}$$

whereas

$$\mathbf{E}^{+}_{\text{slab}} = \hat{\mathbf{n}} \left(\mathbf{P}^{+} \cdot \mathbf{\Omega}\right) / \left(\mathbf{P}^{+} \cdot \hat{\mathbf{n}}\right) - \mathbf{\Omega} = -\mathbf{\Omega}. \tag{41}$$

Turning first to the reflected vector amplitude $\mathbf{E}_{\text{ref}}$, we have from Eq. (17)

$$\begin{aligned} \mathbf{E}_{\text{ref}} &= -\frac{k_1(\sigma_2 - i\omega\{\epsilon_2 - \epsilon_1\})}{2\hat{n}_z\epsilon_1\omega} \hat{\mathbf{n}}' \times \left[\hat{\mathbf{n}}' \times \sum_{\pm} \left\{ \frac{1 - \exp\left(-\{ik_1\hat{n}_z + p_{\pm}\}\, a\right)}{ik_1\hat{n}_z + p_{\pm}} \right\} \mathbf{E}^{\pm}_{\text{slab}} \right] \\ &= -i\frac{\gamma_{-}\gamma_{+}}{2k_1} \hat{\mathbf{n}}' \times \left[\hat{\mathbf{n}}' \times \sum_{\pm} \left\{ \frac{1}{ik_1\hat{n}_z + p_{\pm}} \right\} \mathbf{E}^{\pm}_{\text{slab}} \right], \end{aligned} \tag{42}$$

the latter simplification gotten by virtue of Eq. (23). But now, since clearly $\hat{\mathbf{n}}' = -\hat{\mathbf{n}}$ whereas Eqs. (40), (41), and (38) insist that $\mathbf{E}^{\pm}_{\text{slab}}$ be both aligned along $\mathbf{E}_{\text{inc}}$, we further get

$$\mathbf{E}_{\text{ref}} = i\frac{\gamma_{-}\gamma_{+}}{2k_1} \sum_{\pm} \left\{ \frac{1}{ik_1\hat{n}_z + p_{\pm}} \right\} \mathbf{E}^{\pm}_{\text{slab}} = \frac{i}{2k_1} \sum_{\pm} \gamma_{\mp} \mathbf{E}^{\pm}_{\text{slab}} \tag{43}$$

which, written somewhat more explicitly, reads

$$\mathbf{E}_{\text{ref}} = \gamma_{-}\gamma_{+} \left(\frac{1 - e^{(\gamma_{-}-\gamma_{+})a}}{\gamma_{+}^2 - \gamma_{-}^2 e^{(\gamma_{-}-\gamma_{+})a}} \right) \mathbf{E}_{\text{inc}}. \tag{44}$$

From Eqs. (18) and (22) combined we similarly find the total transmitted field vector amplitude $\mathbf{E}_{\text{trans}}$ in the form

$$\begin{aligned} \mathbf{E}_{\text{trans}} &= \mathbf{E}_{\text{inc}} + \frac{k_1(\sigma_2 - i\omega\{\epsilon_2 - \epsilon_1\})}{2\hat{n}_z\epsilon_1\omega} \hat{\mathbf{n}} \times \left[\hat{\mathbf{n}} \times \sum_{\pm} \left\{ \frac{1 - \exp\left(\{ik_1\hat{n}_z - p_{\pm}\}\, a\right)}{ik_1\hat{n}_z - p_{\pm}} \right\} \mathbf{E}^{\pm}_{\text{slab}} \right] \\ &= \frac{i}{2k_1} \sum_{\pm} \gamma_{\pm} e^{\gamma_{\mp} a} \mathbf{E}^{\pm}_{\text{slab}} = \left(\frac{\gamma_{+}^2 - \gamma_{-}^2}{\gamma_{+}^2 - \gamma_{-}^2 e^{(\gamma_{-}-\gamma_{+})a}} \right) e^{\gamma_{-} a} \mathbf{E}_{\text{inc}}. \end{aligned} \tag{45}$$

We pause to remark, by way of a partial check, that slab removal, signaled by having $a \downarrow 0+$, implies the requisite limits $\mathbf{E}_{\text{ref}} \to \mathbf{0}$ and $\mathbf{E}_{\text{trans}} \to \mathbf{E}_{\text{inc}}$.

The standard treatment of this scattering problem insists upon interface continuity at $z = 0$ and $z = -a$ of both electric and magnetic tangential field components, yielding altogether four linear equations for the four vector amplitudes $\mathbf{E}_{\text{ref}}$, $\mathbf{E}_{\text{trans}}$, and $\mathbf{E}^{\pm}_{\text{slab}}$, all four of them coplanar and perpendicular to propagation vector $\hat{\mathbf{n}}$. Faraday's equation selectively invoked according to vertical coordinate z provides the magnetic field $\mathbf{B}(z)$ in the form

$$\mathbf{B}(z)=\begin{cases}-(k_1/\omega)\hat{\mathbf{e}}_z\times\left(e^{-ik_1z}\mathbf{E}_{\rm inc}-e^{ik_1z}\mathbf{E}_{\rm ref}\right); & z>0\\ -i(p_+/\omega)\hat{\mathbf{e}}_z\times\left(e^{p_+z}\mathbf{E}^+_{\rm slab}-e^{-p_+z}\mathbf{E}^-_{\rm slab}\right); & 0>z>-a\\ -(k_1/\omega)\hat{\mathbf{e}}_z\times\left(e^{-ik_1z}\mathbf{E}_{\rm trans}\right); & -a>z\end{cases}. \tag{46}$$

Enforcement across the board of one additional cross product with vector $\hat{\mathbf{e}}_z$ rotates $\mathbf{B}(\mathbf{r})$ into the common plane of electric vectors and supplies a vector quantity $\hat{\mathbf{e}}_z\times\mathbf{B}(z)$ whose interface continuity serves here just as well as that of $\mathbf{B}(z)$ *per se*. Thus

$$\hat{\mathbf{e}}_z\times\mathbf{B}(z)=\begin{cases}(k_1/\omega)\left(e^{-ik_1z}\mathbf{E}_{\rm inc}-e^{ik_1z}\mathbf{E}_{\rm ref}\right); & z>0\\ i(p_+/\omega)\left(e^{p_+z}\mathbf{E}^+_{\rm slab}-e^{-p_+z}\mathbf{E}^-_{\rm slab}\right); & 0>z>-a\\ (k_1/\omega)\left(e^{-ik_1z}\mathbf{E}_{\rm trans}\right); & -a>z\end{cases}. \tag{47}$$

The requisite electric interface continuity equations can now be written as

$$\begin{cases}\mathbf{E}_{\rm inc}+\mathbf{E}_{\rm ref}=\mathbf{E}^+_{\rm slab}+\mathbf{E}^-_{\rm slab}\\ e^{-p_+a}\mathbf{E}^+_{\rm slab}+e^{p_+a}\mathbf{E}^-_{\rm slab}=e^{ik_1a}\mathbf{E}_{\rm trans}\end{cases} \tag{48}$$

and are joined as

$$\begin{cases}k_1\left(\mathbf{E}_{\rm inc}-\mathbf{E}_{\rm ref}\right)=ip_+\left(\mathbf{E}^+_{\rm slab}-\mathbf{E}^-_{\rm slab}\right)\\ ip_+\left(e^{-p_+a}\mathbf{E}^+_{\rm slab}-e^{p_+a}\mathbf{E}^-_{\rm slab}\right)=k_1\left(e^{ik_1a}\mathbf{E}_{\rm trans}\right)\end{cases} \tag{49}$$

by their magnetic partners.

While linear system in Eqs. (48)–(49) can of course be solved *en masse* for the unknowns $\mathbf{E}_{\rm ref}$, $\mathbf{E}_{\rm trans}$, and $\mathbf{E}^{\pm}_{\rm slab}$ through recourse to a four-by-four determinant, we prefer to straddle it instead via a cascade of two-by-two linear systems. Thus, on grouping the first and third lines we obtain a two-by-two system for $\mathbf{E}^{\pm}_{\rm slab}$ having its source built up from $\mathbf{E}_{\rm inc}$ and $\mathbf{E}_{\rm ref}$. When instead the second and fourth lines are so grouped, the source becomes controlled by $\mathbf{E}_{\rm trans}$. A final demand that the seemingly discordant solutions for $\mathbf{E}^{\pm}_{\rm slab}$ so gotten be in fact identical will then emerge as a two-by-two linear system for $\mathbf{E}_{\rm ref}$ and $\mathbf{E}_{\rm trans}$ alone, allowing us to recover outcomes in Eqs. (44) and (45) as previously encountered.

And so, from

$$\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\mathbf{E}^+_{\rm slab}\\ \mathbf{E}^-_{\rm slab}\end{bmatrix}=\begin{bmatrix}\mathbf{E}_{\rm inc}+\mathbf{E}_{\rm ref}\\ -ik_1/p_+\left(\mathbf{E}_{\rm inc}-\mathbf{E}_{\rm ref}\right)\end{bmatrix} \tag{50}$$

we obtain

$$\begin{bmatrix}\mathbf{E}^+_{\rm slab}\\ \mathbf{E}^-_{\rm slab}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}(1-ik_1/p_+)\,\mathbf{E}_{\rm inc}+(1+ik_1/p_+)\,\mathbf{E}_{\rm ref}\\ (1+ik_1/p_+)\,\mathbf{E}_{\rm inc}+(1-ik_1/p_+)\,\mathbf{E}_{\rm ref}\end{bmatrix}, \tag{51}$$

whereas

$$\begin{bmatrix}e^{-p_+a} & e^{p_+a}\\ e^{-p_+a} & -e^{p_+a}\end{bmatrix}\begin{bmatrix}\mathbf{E}^+_{\rm slab}\\ \mathbf{E}^-_{\rm slab}\end{bmatrix}=\begin{bmatrix}1\\ -ik_1/p_+\end{bmatrix}e^{ik_1a}\mathbf{E}_{\rm trans} \tag{52}$$

similarly gives

$$\begin{bmatrix}\mathbf{E}^+_{\rm slab}\\ \mathbf{E}^-_{\rm slab}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}e^{p_+a} & e^{p_+a}\\ e^{-p_+a} & -e^{-p_+a}\end{bmatrix}\begin{bmatrix}1\\ -ik_1/p_+\end{bmatrix}e^{ik_1a}\mathbf{E}_{\rm trans}, \tag{53}$$

or else

$$\begin{bmatrix}\mathbf{E}^+_{\rm slab}\\ \mathbf{E}^-_{\rm slab}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}(1-ik_1/p_+)\,e^{p_+a}\\ (1+ik_1/p_+)\,e^{-p_+a}\end{bmatrix}e^{ik_1a}\mathbf{E}_{\rm trans}. \tag{54}$$

A final demand that Eqs. (51) and (54) stand in agreement amounts to the statement that

$$\begin{bmatrix}(1+ik_1/p_+) & -(1-ik_1/p_+)\,e^{(ik_1+p_+)a}\\ (1-ik_1/p_+) & -(1+ik_1/p_+)\,e^{(ik_1-p_+)a}\end{bmatrix}\begin{bmatrix}\mathbf{E}_{\rm ref}\\ \mathbf{E}_{\rm trans}\end{bmatrix}=-\begin{bmatrix}(1-ik_1/p_+)\\ (1+ik_1/p_+)\end{bmatrix}\mathbf{E}_{\rm inc}. \tag{55}$$

Recalling next the existing abbreviations $-ik_1 + p_\pm = \gamma_\pm$, Eq, (55) can be condensed into

$$\begin{bmatrix} -\gamma_- & -\gamma_+ e^{-\gamma_- a} \\ \gamma_+ & \gamma_- e^{-\gamma_+ a} \end{bmatrix} \begin{bmatrix} \mathbf{E}_{\mathrm{ref}} \\ \mathbf{E}_{\mathrm{trans}} \end{bmatrix} = \begin{bmatrix} -\gamma_+ \\ \gamma_- \end{bmatrix} \mathbf{E}_{\mathrm{inc}} \tag{56}$$

whose solution

$$\begin{bmatrix} \mathbf{E}_{\mathrm{ref}} \\ \mathbf{E}_{\mathrm{trans}} \end{bmatrix} = \frac{e^{\gamma_- a}}{\gamma_+^2 - \gamma_-^2 e^{(\gamma_- - \gamma_+)a}} \begin{bmatrix} \gamma_- e^{-\gamma_+ a} & \gamma_+ e^{-\gamma_- a} \\ -\gamma_+ & -\gamma_- \end{bmatrix} \begin{bmatrix} -\gamma_+ \\ \gamma_- \end{bmatrix} \mathbf{E}_{\mathrm{inc}} \tag{57}$$

recovers Eq. (44) and the second line from Eq. (45) as to their every detail. $\mathbf{E}^{\pm}_{\mathrm{slab}}$ can of course now simply be read off from either Eq. (51) or (53). This labyrinthine odyssey is thus graced by success and so bestows at least a partial nod of approval upon our radiative self-consistency program.

10. APPENDIX: RADIATIVE SELF-CONSISTENCY WITH DISSIMILAR HALF SPACES

Should we allow the lower half space to differ from its upper counterpart, having now an arbitrary dielectric constant ϵ_3 and an arbitrary conductivity σ_3, it being understood however that at least one of ϵ_3 and σ_3 differs respectively from ϵ_2 and σ_2, and likewise from ϵ_1 and 0+, then a corresponding provision would have to be made for additional radiative sources seen, as before, from the perspective of the nondissipative medium 1.

10.1. Revised Integral Equation

The total electric field in the lower half space can now be taken as

$$\mathbf{E}_{\mathrm{tot}}(\mathbf{r}) = \mathbf{E}^{+}_{\mathrm{half-low}} \exp\left(ik_1 \hat{\mathbf{n}}_\parallel \cdot \boldsymbol{\rho} + q_+ z\right) \tag{58}$$

with

$$q_+^2 - k_1^2 \hat{\mathbf{n}}_\parallel \cdot \hat{\mathbf{n}}_\parallel = -\omega^2 \mu \epsilon_3 \left(1 + i\sigma_3/\omega\epsilon_3\right) \tag{59}$$

and hence

$$\begin{aligned} q_+ &= k_1 \sqrt{\sin^2\vartheta - (\epsilon_3/\epsilon_1)\left(1 + i\sigma_3/\omega\epsilon_3\right)} = \frac{|k_1|}{\sqrt{2}} \left\{ \sqrt{\sqrt{\left(\sin^2\vartheta - \epsilon_3/\epsilon_1\right)^2 + \left(\sigma_3/\omega\epsilon_1\right)^2} + \sin^2\vartheta - \epsilon_3/\epsilon_1} \right. \\ &\left. -i\frac{\omega}{|\omega|} \sqrt{\sqrt{\left(\sin^2\vartheta - \epsilon_3/\epsilon_1\right)^2 + \left(\sigma_3/\omega\epsilon_1\right)^2} - \sin^2\vartheta + \epsilon_3/\epsilon_1} \right\}. \end{aligned} \tag{60}$$

Moreover, if one sets

$$\mathbf{Q}^+ = ik_1 \hat{\mathbf{n}}_\parallel + q_+ \hat{\mathbf{e}}_z, \tag{61}$$

then the requisite divergenceless nature of field in Eq, (58), seen now in the form $\mathbf{E}^{+}_{\mathrm{half-low}} \exp(\mathbf{Q}^+ \cdot \mathbf{r})$, is assured by having

$$\mathbf{Q}^+ \cdot \mathbf{E}^{+}_{\mathrm{half-low}} = 0. \tag{62}$$

As the counterpart to Eq. (7), the total magnetic field now becomes

$$\begin{aligned} \mathbf{B}_{\mathrm{tot}}(\mathbf{r}) &= \frac{k_1}{\omega} \hat{\mathbf{n}} \times \mathbf{E}_{\mathrm{inc}} e^{ik_1 \hat{\mathbf{n}} \cdot \mathbf{r}} \\ &+ \frac{\mu(\sigma_2 - i\omega\{\epsilon_2 - \epsilon_1\})}{4\pi} \sum_{\pm} \boldsymbol{\nabla} \times \left\{ \mathbf{E}^{\pm}_{\mathrm{slab}} \int_{0>z'>-a} \frac{e^{ik_1|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \exp\left(ik_1 \hat{\mathbf{n}}_\parallel \cdot \boldsymbol{\rho}' + p_\pm z'\right) dx'dy'dz' \right\} \\ &+ \frac{\mu(\sigma_3 - i\omega\{\epsilon_3 - \epsilon_1\})}{4\pi} \boldsymbol{\nabla} \times \left\{ \mathbf{E}^{+}_{\mathrm{half-low}} \int_{-a>z'} \frac{e^{ik_1|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \exp\left(ik_1 \hat{\mathbf{n}}_\parallel \cdot \boldsymbol{\rho}' + q_+ z'\right) dx'dy'dz' \right\} \end{aligned} \tag{63}$$

and leads, just as before, to the revised integral equation

$$\hat{\epsilon}(z)\mathbf{E}_{\text{tot}}(\mathbf{r}) = \epsilon_1 \mathbf{E}_{\text{inc}} e^{ik_1 \hat{\mathbf{n}}\cdot\mathbf{r}}$$
$$+\frac{i(\sigma_2 - i\omega\{\epsilon_2-\epsilon_1\})}{4\pi\omega}\sum_{\pm}\boldsymbol{\nabla}\times\left[\boldsymbol{\nabla}\times\left\{\mathbf{E}^{\pm}_{\text{slab}}\int_{0>z'>-a}\frac{e^{ik_1|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\exp\left(ik_1\hat{\mathbf{n}}_{\|}\cdot\boldsymbol{\rho}'+p_{\pm}z'\right)dx'dy'dz'\right\}\right]$$
$$+\frac{i(\sigma_3 - i\omega\{\epsilon_3-\epsilon_1\})}{4\pi\omega}\boldsymbol{\nabla}\times\left[\boldsymbol{\nabla}\times\left\{\mathbf{E}^{+}_{\text{half}-\text{low}}\int_{-a>z'}\frac{e^{ik_1|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\exp\left(ik_1\hat{\mathbf{n}}_{\|}\cdot\boldsymbol{\rho}'+q_{+}z'\right)dx'dy'dz'\right\}\right] \quad (64)$$

valid everywhere without exception, provided only that we modify the last entry in Eq. (8) so as to read $\hat{\epsilon}(z) = \epsilon_3 + i\sigma_3/\omega$ whenever $-a > z$.

10.2. Revised Preliminaries

The half space source additions to Eqs. (63)–(64) entail a fresh calculation

$$\int_{-a>z'}\frac{e^{ik_1|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\exp\left(ik_1\hat{\mathbf{n}}_{\|}\cdot\boldsymbol{\rho}'+q_{+}z'\right)dx'dy'dz'$$
$$= 2\pi\exp\left(ik_1\hat{\mathbf{n}}_{\|}\cdot\boldsymbol{\rho}\right)\int_{-\infty}^{-a}dz'\exp\left(q_{+}z'\right)\int_0^{\infty}\frac{e^{ik_1\sqrt{\rho'^2+(z-z')^2}}}{\sqrt{\rho'^2+(z-z')^2}}\rho' J_0(k_1\rho'\sin\vartheta)d\rho'$$
$$= \frac{2\pi}{ik_1\hat{n}_z}\exp\left(ik_1\hat{\mathbf{n}}_{\|}\cdot\boldsymbol{\rho}\right)\int_{-\infty}^{-a}\exp\left(q_{+}z'-ik_1\hat{n}_z|z-z'|\right)dz' \quad (65)$$

whose final integral over z' must be separately considered for $z > -a$ (values $K_1(z)$) and $-a > z$ (values $K_2(z)$). We find:

$K_1(z)$, $z > -a$:

$$K_1(z) = \exp\left(-ik_1\hat{n}_z z\right)\int_{-\infty}^{-a}\exp\left(\{ik_1\hat{n}_z+q_+\}z'\right)dz'$$
$$= \exp\left(-ik_1\hat{n}_z z\right)\{ik_1\hat{n}_z+q_+\}^{-1}\exp\left(-\{ik_1\hat{n}_z+q_+\}a\right); \quad (66)$$

and

$K_2(z)$, $-a > z$:

$$K_2(z) = \exp\left(-ik_1\hat{n}_z z\right)\int_{-\infty}^{z}\exp\left(\{ik_1\hat{n}_z+q_+\}z'\right)dz' + \exp\left(ik_1\hat{n}_z z\right)\int_{z}^{-a}\exp\left(-\{ik_1\hat{n}_z-q_+\}z'\right)dz'$$
$$= \left\{\frac{1}{ik_1\hat{n}_z+q_+}+\frac{1}{ik_1\hat{n}_z-q_+}\right\}e^{q_+z} - \frac{e^{\{ik_1\hat{n}_z-q_+\}a}}{ik_1\hat{n}_z-q_+}e^{ik_1\hat{n}_z z}. \quad (67)$$

10.3. Revised Reflected Field

From Eqs. (64)–(66) we now find that the total electric field $\mathbf{E}_{\text{tot}}(\mathbf{r})$ as first encountered in Eq. (17) for $z > 0$, must be augmented so as to read

$$\mathbf{E}_{\text{tot}}(\mathbf{r}) = \mathbf{E}_{\text{inc}}e^{ik_1\hat{\mathbf{n}}\cdot\mathbf{r}} - \frac{k_1(\sigma_2-i\omega\{\epsilon_2-\epsilon_1\})}{2\hat{n}_z\epsilon_1\omega}e^{ik_1\hat{\mathbf{n}}'\cdot\mathbf{r}}\hat{\mathbf{n}}'\times\left[\hat{\mathbf{n}}'\times\sum_{\pm}\left\{\frac{1-\exp\left(-\{ik_1\hat{n}_z+p_\pm\}a\right)}{ik_1\hat{n}_z+p_\pm}\right\}\mathbf{E}^{\pm}_{\text{slab}}\right]$$
$$-\frac{k_1(\sigma_3-i\omega\{\epsilon_3-\epsilon_1\})}{2\hat{n}_z\epsilon_1\omega}e^{ik_1\hat{\mathbf{n}}'\cdot\mathbf{r}}\hat{\mathbf{n}}'\times\left[\hat{\mathbf{n}}'\times\left\{\frac{\exp\left(-\{ik_1\hat{n}_z+q_+\}a\right)}{ik_1\hat{n}_z+q_+}\right\}\mathbf{E}^{+}_{\text{half}-\text{low}}\right]. \quad (68)$$

10.4. Electromagnetic Cross Talk, Up and Down, between Slab and Subjacent Half Space

Turning attention with the aid of Eq. (67) to the lower half space $-a > z$ as such, we compute, on the basis of Eq. (60), that

$$\left\{\frac{1}{ik_1\hat{n}_z + q_+} + \frac{1}{ik_1\hat{n}_z - q_+}\right\} = -\frac{2ik_1\hat{n}_z}{k_1^2\hat{n}_z^2 + q_+^2} = \frac{2k_1\hat{n}_z}{\omega\mu\left(\sigma_3 - i\omega\left\{\epsilon_3 - \epsilon_1\right\}\right)}, \tag{69}$$

akin to what had been earlier found in Eq. (15). The associated cluster from Eq. (67) thus contributes

$$\left(\epsilon_3 + i\sigma_3/\omega\right)\mathbf{E}^+_{\mathrm{half-low}}\exp\left(\mathbf{Q}^+\cdot\mathbf{r}\right) \tag{70}$$

to the third line of integral equation (64), a contribution which, in the spirit of Ewald-Oseen (*cf.* the closing comments in the footnote accompanying Eq. (19)), cancels exactly once again the term $\hat{\epsilon}(z)\mathbf{E}_{\mathrm{tot}}(\mathbf{r})$ on its left. Orthogonality (62) participates decisively in the intervening algebra, as do Eqs. (59) and (65).

The second cluster in the second line from Eq. (67) produces by contrast a contribution whose phase progression $e^{ik_1\hat{\mathbf{n}}\cdot\mathbf{r}}$ mimics that of the incoming excitation. After some algebra we find this contribution to be

$$\frac{k_1(\sigma_3 - i\omega\{\epsilon_3 - \epsilon_1\})}{2\hat{n}_z\omega}e^{ik_1\hat{\mathbf{n}}\cdot\mathbf{r}}\hat{\mathbf{n}}\times\left[\hat{\mathbf{n}}\times\left\{\frac{e^{\{ik_1\hat{n}_z - q_+\}a}}{ik_1\hat{n}_z - q_+}\right\}\mathbf{E}^+_{\mathrm{half-low}}\right]. \tag{71}$$

This latter must of course be compounded with ϵ_1 times the total transmitted structure from Eq. (18) so as to yield a net result of **0**. Taking note of Eq. (22), which retains validity unimpaired, and the abbreviations introduced under Eqs. (24)–(25), ϵ_1 times the content of Eq. (18) on its right can be rendered as

$$\frac{ik_1}{2\hat{n}_z\omega^2\mu}e^{ik_1\hat{\mathbf{n}}\cdot\mathbf{r}}\hat{\mathbf{n}}\times\left[\hat{\mathbf{n}}\times\sum_\pm\gamma_\pm e^{\gamma_\mp a}\mathbf{E}^\pm_{\mathrm{slab}}\right] \tag{72}$$

and hence leads us to demand that

$$\hat{\mathbf{n}}\times\left[\hat{\mathbf{n}}\times\sum_\pm\gamma_\pm e^{\gamma_\mp a}\mathbf{E}^\pm_{\mathrm{slab}}\right] = i\omega\mu\left(\sigma_3 - i\omega\{\epsilon_3 - \epsilon_1\}\right)\hat{\mathbf{n}}\times\left[\hat{\mathbf{n}}\times\left\{\frac{e^{\{ik_1\hat{n}_z - q_+\}a}}{ik_1\hat{n}_z - q_+}\right\}\mathbf{E}^+_{\mathrm{half-low}}\right] \tag{73}$$

as the first amplitude condition which brings in the participation of $\mathbf{E}^+_{\mathrm{half-low}}$.

Yet another appearance of $\mathbf{E}^+_{\mathrm{half-low}}$ arises by virtue of cross-talk in reverse between the lower half space and the slab as expressed by the second line from (68), the structure of that line being uniformly applicable whenever $z > -a$, and not merely $z > 0$ when subordinated as a contributor to Eq. (68). While Eq. (22) remains unaffected, reference to Eq. (21) shows that Eq. (23) now blends into

$$\begin{aligned}&\hat{\mathbf{n}}'\times\left[\hat{\mathbf{n}}'\times\sum_\pm\gamma_\mp e^{-\gamma_\pm a}\mathbf{E}^\pm_{\mathrm{slab}}\right]\\ &= i\omega\mu\left(\sigma_3 - i\omega\{\epsilon_3 - \epsilon_1\}\right)\hat{\mathbf{n}}'\times\left[\hat{\mathbf{n}}'\times\left\{\frac{\exp\left(-\left\{ik_1\hat{n}_z + q_+\right\}a\right)}{ik_1\hat{n}_z + q_+}\right\}\mathbf{E}^+_{\mathrm{half-low}}\right].\end{aligned} \tag{74}$$

10.5. Radiating Amplitudes $E^\pm_{slab}$ and $E_{half-low}$ Solved for Invariantively

In principle, the triplet of vector Eqs. (22), (73)–(74), reinforced as before by solenoidal constraints in Eqs. (4) and (62), suffices to determine in full, as a six-by-six linear system subject to routine linear algebra methods, the vector amplitudes $\mathbf{E}^\pm_{\mathrm{slab}}$ and $\mathbf{E}^+_{\mathrm{half-low}}$. Nevertheless we would like to follow here an invariantive, vector solution path, resembling that previously elaborated in Eq. (26) onward. As before we can ease the notation somewhat by writing $\zeta_\pm = ik_1\hat{n}_z \pm q_+$, so that $\zeta_\pm\zeta_\mp = i\omega\mu(\sigma_3 - i\omega\{\epsilon_3 - \epsilon_1\})$. This allows Eqs. (73)–(74) to be restated more concisely as

$$\hat{\mathbf{n}}\times\left[\hat{\mathbf{n}}\times\sum_\pm\gamma_\pm e^{\gamma_\mp a}\mathbf{E}^\pm_{\mathrm{slab}}\right] = \zeta_+ e^{\zeta_- a}\hat{\mathbf{n}}\times\left(\hat{\mathbf{n}}\times\mathbf{E}^+_{\mathrm{half-low}}\right) \tag{75}$$

and

$$\hat{\mathbf{n}}' \times \left[\hat{\mathbf{n}}' \times \sum_{\pm} \gamma_{\mp} e^{-\gamma_{\pm} a} \mathbf{E}_{\text{slab}}^{\pm}\right] = \zeta_{-} e^{-\zeta_{+} a} \hat{\mathbf{n}}' \times \left(\hat{\mathbf{n}}' \times \mathbf{E}_{\text{half-low}}^{+}\right). \tag{76}$$

Since, say

$$\mathbf{E}_{\text{half-low}}^{+} = \hat{\mathbf{n}}\left(\hat{\mathbf{n}} \cdot \mathbf{E}_{\text{half-low}}^{+}\right) - \hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \mathbf{E}_{\text{half-low}}^{+}\right), \tag{77}$$

it follows from Eq. (75) that

$$\hat{\mathbf{n}} \cdot \mathbf{E}_{\text{half-low}}^{+} = e^{-\zeta_{-} a} \frac{\left(\mathbf{Q}^{+} \cdot \left\{\hat{\mathbf{n}} \times \left[\hat{\mathbf{n}} \times \sum_{\pm} \gamma_{\pm} e^{\gamma_{\mp} a} \mathbf{E}_{\text{slab}}^{\pm}\right]\right\}\right)}{\zeta_{+}\left(\mathbf{Q}^{+} \cdot \hat{\mathbf{n}}\right)} \tag{78}$$

and then

$$\begin{aligned}
\mathbf{E}_{\text{half-low}}^{+} &= \zeta_{+}^{-1} e^{-\zeta_{-} a}\left[\hat{\mathbf{n}}\left\{\frac{\left(\mathbf{Q}^{+} \cdot \left\{\hat{\mathbf{n}} \times \left[\hat{\mathbf{n}} \times \sum_{\pm} \gamma_{\pm} e^{\gamma_{\mp} a} \mathbf{E}_{\text{slab}}^{\pm}\right]\right\}\right)}{\left(\mathbf{Q}^{+} \cdot \hat{\mathbf{n}}\right)}\right\} - \hat{\mathbf{n}} \times \left\{\hat{\mathbf{n}} \times \sum_{\pm} \gamma_{\pm} e^{\gamma_{\mp} a} \mathbf{E}_{\text{slab}}^{\pm}\right\}\right] \\
&= \zeta_{+}^{-1} e^{-\zeta_{-} a}\left[\hat{\mathbf{n}}\left\{\frac{\left(\mathbf{Q}^{+} \cdot \hat{\mathbf{n}}\right) \sum_{\nu} \gamma_{\nu} e^{\gamma_{-\nu} a}\left(\hat{\mathbf{n}} \cdot \mathbf{E}_{\text{slab}}^{\nu}\right) - \sum_{\nu} \gamma_{\nu} e^{\gamma_{-\nu} a}\left(\mathbf{Q}^{+} \cdot \mathbf{E}_{\text{slab}}^{\nu}\right)}{\left(\mathbf{Q}^{+} \cdot \hat{\mathbf{n}}\right)}\right\}\right. \\
&\quad \left. -\hat{\mathbf{n}}\left\{\sum_{\nu} \gamma_{\nu} e^{\gamma_{-\nu} a}\left(\hat{\mathbf{n}} \cdot \mathbf{E}_{\text{slab}}^{\nu}\right)\right\} + \sum_{\nu} \gamma_{\nu} e^{\gamma_{-\nu} a} \mathbf{E}_{\text{slab}}^{\nu}\right] \\
&= -\zeta_{+}^{-1} e^{-\zeta_{-} a}\left[\hat{\mathbf{n}}\left\{\sum_{\nu} \gamma_{\nu} e^{\gamma_{-\nu} a}\left(\mathbf{Q}^{+} \cdot \mathbf{E}_{\text{slab}}^{\nu}\right) / \left(\mathbf{Q}^{+} \cdot \hat{\mathbf{n}}\right)\right\} - \sum_{\nu} \gamma_{\nu} e^{\gamma_{-\nu} a} \mathbf{E}_{\text{slab}}^{\nu}\right].
\end{aligned} \tag{79}$$

In an entirely similar fashion it likewise follows from (76) that

$$\mathbf{E}_{\text{half-low}}^{+} = -\zeta_{-}^{-1} e^{\zeta_{+} a}\left[\hat{\mathbf{n}}'\left\{\sum_{\nu} \gamma_{-\nu} e^{-\gamma_{\nu} a}\left(\mathbf{Q}^{+} \cdot \mathbf{E}_{\text{slab}}^{\nu}\right) / \left(\mathbf{Q}^{+} \cdot \hat{\mathbf{n}}'\right)\right\} - \sum_{\nu} \gamma_{-\nu} e^{-\gamma_{\nu} a} \mathbf{E}_{\text{slab}}^{\nu}\right]. \tag{80}$$

Representations in Eqs. (79)–(80) must clearly agree, and the demand for such agreement suffices to fix both scalar products $(\mathbf{Q}^{+} \cdot \mathbf{E}_{\text{slab}}^{\pm})$ and thus $\mathbf{E}_{\text{half-low}}^{+}$, once slab amplitudes $\mathbf{E}_{\text{slab}}^{\pm}$ themselves have been brought into sharper focus. Determination of this sort proceeds by projecting the common value on the right in both Eqs. (79) and (80) onto $\mathbf{P}^{\pm}$, with note taken once more of the two orthogonalities listed under Eq. (4). And so we find

$$\begin{aligned}
&\zeta_{+}^{-1} e^{-\zeta_{-} a}\left[\left(\mathbf{P}^{+} \cdot \hat{\mathbf{n}}\right)\left\{\sum_{\nu} \gamma_{\nu} e^{\gamma_{-\nu} a}\left(\mathbf{Q}^{+} \cdot \mathbf{E}_{\text{slab}}^{\nu}\right) / \left(\mathbf{Q}^{+} \cdot \hat{\mathbf{n}}\right)\right\} - \gamma_{-} e^{\gamma_{+} a}\left(\mathbf{P}^{+} \cdot \mathbf{E}_{\text{slab}}^{-}\right)\right] \\
&= \zeta_{-}^{-1} e^{\zeta_{+} a}\left[\left(\mathbf{P}^{+} \cdot \hat{\mathbf{n}}'\right)\left\{\sum_{\nu} \gamma_{-\nu} e^{-\gamma_{\nu} a}\left(\mathbf{Q}^{+} \cdot \mathbf{E}_{\text{slab}}^{\nu}\right) / \left(\mathbf{Q}^{+} \cdot \hat{\mathbf{n}}'\right)\right\} - \gamma_{+} e^{-\gamma_{-} a}\left(\mathbf{P}^{+} \cdot \mathbf{E}_{\text{slab}}^{-}\right)\right]
\end{aligned} \tag{81}$$

and

$$\begin{aligned}
&\zeta_{+}^{-1} e^{-\zeta_{-} a}\left[\left(\mathbf{P}^{-} \cdot \hat{\mathbf{n}}\right)\left\{\sum_{\nu} \gamma_{\nu} e^{\gamma_{-\nu} a}\left(\mathbf{Q}^{+} \cdot \mathbf{E}_{\text{slab}}^{\nu}\right) / \left(\mathbf{Q}^{+} \cdot \hat{\mathbf{n}}\right)\right\} - \gamma_{+} e^{\gamma_{-} a}\left(\mathbf{P}^{-} \cdot \mathbf{E}_{\text{slab}}^{+}\right)\right] \\
&= \zeta_{-}^{-1} e^{\zeta_{+} a}\left[\left(\mathbf{P}^{-} \cdot \hat{\mathbf{n}}'\right)\left\{\sum_{\nu} \gamma_{-\nu} e^{-\gamma_{\nu} a}\left(\mathbf{Q}^{+} \cdot \mathbf{E}_{\text{slab}}^{\nu}\right) / \left(\mathbf{Q}^{+} \cdot \hat{\mathbf{n}}'\right)\right\} - \gamma_{-} e^{-\gamma_{+} a}\left(\mathbf{P}^{-} \cdot \mathbf{E}_{\text{slab}}^{+}\right)\right].
\end{aligned} \tag{82}$$

Rearrangement then gives

$$\begin{bmatrix} \Gamma & \Lambda \\ \Psi & \Theta \end{bmatrix}\begin{bmatrix} \mathbf{Q}^{+} \cdot \mathbf{E}_{\text{slab}}^{+} \\ \mathbf{Q}^{+} \cdot \mathbf{E}_{\text{slab}}^{-} \end{bmatrix} = \begin{bmatrix} \Upsilon \\ \Phi \end{bmatrix}, \tag{83}$$

or else

$$\begin{bmatrix} \mathbf{Q}^{+} \cdot \mathbf{E}_{\text{slab}}^{+} \\ \mathbf{Q}^{+} \cdot \mathbf{E}_{\text{slab}}^{-} \end{bmatrix} = \frac{1}{\Gamma\Theta - \Lambda\Psi}\begin{bmatrix} \Theta & -\Lambda \\ -\Psi & \Gamma \end{bmatrix}\begin{bmatrix} \Upsilon \\ \Phi \end{bmatrix} = \frac{1}{\Gamma\Theta - \Lambda\Psi}\begin{bmatrix} \Theta\Upsilon - \Lambda\Phi \\ -\Psi\Upsilon + \Gamma\Phi \end{bmatrix}, \tag{84}$$

with

$$\Gamma = (\zeta_-/\zeta_+)\gamma_+ e^{-(\zeta_+ + \zeta_- - \gamma_-)a} \left(\mathbf{P}^+ \cdot \hat{\mathbf{n}}\right) / \left(\mathbf{Q}^+ \cdot \hat{\mathbf{n}}\right) - \gamma_- e^{-\gamma_+ a} \left(\mathbf{P}^+ \cdot \hat{\mathbf{n}}'\right) / \left(\mathbf{Q}^+ \cdot \hat{\mathbf{n}}'\right), \tag{85}$$

$$\Lambda = (\zeta_-/\zeta_+)\gamma_- e^{-(\zeta_+ + \zeta_- - \gamma_+)a} \left(\mathbf{P}^+ \cdot \hat{\mathbf{n}}\right) / \left(\mathbf{Q}^+ \cdot \hat{\mathbf{n}}\right) - \gamma_+ e^{-\gamma_- a} \left(\mathbf{P}^+ \cdot \hat{\mathbf{n}}'\right) / \left(\mathbf{Q}^+ \cdot \hat{\mathbf{n}}'\right), \tag{86}$$

$$\Psi = (\zeta_-/\zeta_+)\gamma_+ e^{-(\zeta_+ + \zeta_- - \gamma_-)a} \left(\mathbf{P}^- \cdot \hat{\mathbf{n}}\right) / \left(\mathbf{Q}^+ \cdot \hat{\mathbf{n}}\right) - \gamma_- e^{-\gamma_+ a} \left(\mathbf{P}^- \cdot \hat{\mathbf{n}}'\right) / \left(\mathbf{Q}^+ \cdot \hat{\mathbf{n}}'\right), \tag{87}$$

$$\Theta = (\zeta_-/\zeta_+)\gamma_- e^{-(\zeta_+ + \zeta_- - \gamma_+)a} \left(\mathbf{P}^- \cdot \hat{\mathbf{n}}\right) / \left(\mathbf{Q}^+ \cdot \hat{\mathbf{n}}\right) - \gamma_+ e^{-\gamma_- a} \left(\mathbf{P}^- \cdot \hat{\mathbf{n}}'\right) / \left(\mathbf{Q}^+ \cdot \hat{\mathbf{n}}'\right), \tag{88}$$

$$\Upsilon = \left\{(\zeta_-/\zeta_+)\gamma_- e^{-(\zeta_+ + \zeta_- - \gamma_+)a} - \gamma_+ e^{-\gamma_- a}\right\} \left(\mathbf{P}^+ \cdot \mathbf{E}^-_{\text{slab}}\right), \tag{89}$$

$$\Phi = \left\{(\zeta_-/\zeta_+)\gamma_+ e^{-(\zeta_+ + \zeta_- - \gamma_-)a} - \gamma_- e^{-\gamma_+ a}\right\} \left(\mathbf{P}^- \cdot \mathbf{E}^+_{\text{slab}}\right). \tag{90}$$

And, with a view to imminent developments, it is useful to further abbreviate by introducing a two-by-two matrix $A_{\nu,\nu'}$ and setting its entries in accordance with

$$A_{+,+} = \frac{\Theta}{\Gamma\Theta - \Lambda\Psi} \left\{(\zeta_-/\zeta_+)\gamma_- e^{-(\zeta_+ + \zeta_- - \gamma_+)a} - \gamma_+ e^{-\gamma_- a}\right\}, \tag{91}$$

$$A_{+,-} = -\frac{\Lambda}{\Gamma\Theta - \Lambda\Psi} \left\{(\zeta_-/\zeta_+)\gamma_+ e^{-(\zeta_+ + \zeta_- - \gamma_-)a} - \gamma_- e^{-\gamma_+ a}\right\}, \tag{92}$$

$$A_{-,+} = -\frac{\Psi}{\Gamma\Theta - \Lambda\Psi} \left\{(\zeta_-/\zeta_+)\gamma_- e^{-(\zeta_+ + \zeta_- - \gamma_+)a} - \gamma_+ e^{-\gamma_- a}\right\}, \tag{93}$$

$$A_{-,-} = \frac{\Gamma}{\Gamma\Theta - \Lambda\Psi} \left\{(\zeta_-/\zeta_+)\gamma_+ e^{-(\zeta_+ + \zeta_- - \gamma_-)a} - \gamma_- e^{-\gamma_+ a}\right\}. \tag{94}$$

Such abbreviation allows Eq. (84) to be restated as

$$\begin{bmatrix} \mathbf{Q}^+ \cdot \mathbf{E}^+_{\text{slab}} \\ \mathbf{Q}^+ \cdot \mathbf{E}^-_{\text{slab}} \end{bmatrix} = \begin{bmatrix} A_{+,+}\left(\mathbf{P}^+ \cdot \mathbf{E}^-_{\text{slab}}\right) + A_{+,-}\left(\mathbf{P}^- \cdot \mathbf{E}^+_{\text{slab}}\right) \\ A_{-,+}\left(\mathbf{P}^+ \cdot \mathbf{E}^-_{\text{slab}}\right) + A_{-,-}\left(\mathbf{P}^- \cdot \mathbf{E}^+_{\text{slab}}\right) \end{bmatrix}. \tag{95}$$

10.5.1. Final Solution Assault

The apparatus now assembled in Eqs. (81)–(95) narrows all attention to just the slab amplitudes $\mathbf{E}^{\pm}_{\text{slab}}$. When advancing toward this ultimate goal we must adjoin to Eq. (22) just one of Eq. (75) or (76) and, in our choice thereof, we must toggle between use of recipes in Eqs. (79)–(80) for $\mathbf{E}^+_{\text{half}-\text{low}}$. For instance, insertion of the third line from Eq. (79) into Eq. (75) amounts to an empty tautology, strictly not incorrect, but something clearly to be avoided. We can and do instead couple that last line with Eq. (76). But we could, equally well, utilize Eq. (75) in conjunction with Eq. (80). We adopt the pairing between Eqs. (76) and (79) merely in order to make the ensuing manipulations resemble as closely as possible their much simpler antecedents from Sections 7 and 8.

From Eq. (76) and the third line in Eq. (79) we thus get

$$\sum_{\nu} \left\{\gamma_{-\nu} e^{-\gamma_\nu a} - (\zeta_-/\zeta_+)\gamma_\nu e^{-(\zeta_+ + \zeta_- - \gamma_{-\nu})a}\right\} \mathbf{E}^\nu_{\text{slab}}$$
$$+\hat{\mathbf{n}} \left\{(\zeta_-/\zeta_+) e^{-(\zeta_+ + \zeta_-)a} \sum_{\nu} \gamma_\nu e^{\gamma_{-\nu} a} \left(\mathbf{Q}^+ \cdot \mathbf{E}^\nu_{\text{slab}}\right) / \left(\mathbf{Q}^+ \cdot \hat{\mathbf{n}}\right)\right\} = \tilde{\alpha}\hat{\mathbf{n}}', \tag{96}$$

with some number $\tilde{\alpha}$ soon to be determined.* But, before embarking on such determination, we avail ourselves of the indicated apparatus to rewrite Eq. (96) as

$$\sum_{\nu} \left\{\gamma_{-\nu} e^{-\gamma_\nu a} - (\zeta_-/\zeta_+)\gamma_\nu e^{-(\zeta_+ + \zeta_- - \gamma_{-\nu})a}\right\} \mathbf{E}^\nu_{\text{slab}}$$
$$+\hat{\mathbf{n}} \left[(\zeta_-/\zeta_+) e^{-(\zeta_+ + \zeta_-)a} \sum_{\nu,\nu'} \gamma_\nu e^{\gamma_{-\nu} a} A_{\nu,\nu'} \left(\mathbf{P}^{\nu'} \cdot \mathbf{E}^{-\nu'}_{\text{slab}}\right) / \left(\mathbf{Q}^+ \cdot \hat{\mathbf{n}}\right)\right] = \tilde{\alpha}\hat{\mathbf{n}}'. \tag{97}$$

* We remark, in passing, that $\tilde{\alpha}$ differs as to dimension from its precursor α as introduced in (26), that previous dimension being lower by distance raised to the second power.

On projecting Eq. (97) sequentially upon $\mathbf{P}^{\pm}$ we then find

$$\left\{\gamma_\nu e^{-\gamma_{-\nu}a} - (\zeta_-/\zeta_+)\gamma_{-\nu} e^{-(\zeta_+ + \zeta_- - \gamma_\nu)a}\right\}\left(\mathbf{P}^\nu \cdot \mathbf{E}_{\text{slab}}^{-\nu}\right)$$
$$+\left(\mathbf{P}^\nu \cdot \hat{\mathbf{n}}\right)\left[(\zeta_-/\zeta_+)e^{-(\zeta_+ + \zeta_-)a}\sum_{\nu',\nu''}\gamma_{\nu'} e^{\gamma_{-\nu'}a} A_{\nu',\nu''}\left(\mathbf{P}^{\nu''} \cdot \mathbf{E}_{\text{slab}}^{-\nu''}\right)/\left(\mathbf{Q}^+ \cdot \hat{\mathbf{n}}\right)\right] = \tilde{\alpha}\left(\mathbf{P}^\nu \cdot \hat{\mathbf{n}}'\right), \quad (98)$$

yet another two-by-two system

$$\begin{bmatrix} F & G \\ L & M \end{bmatrix}\begin{bmatrix} \mathbf{P}^+ \cdot \mathbf{E}_{\text{slab}}^- \\ \mathbf{P}^- \cdot \mathbf{E}_{\text{slab}}^+ \end{bmatrix} = \tilde{\alpha}\begin{bmatrix} \mathbf{P}^+ \cdot \hat{\mathbf{n}}' \\ \mathbf{P}^- \cdot \hat{\mathbf{n}}' \end{bmatrix}, \quad (99)$$

now with matricial entries

$$F = \left\{\gamma_+ e^{-\gamma_- a} - (\zeta_-/\zeta_+)\gamma_- e^{-(\zeta_+ + \zeta_- - \gamma_+)a}\right\} + \left(\mathbf{P}^+ \cdot \hat{\mathbf{n}}\right)(\zeta_-/\zeta_+)e^{-(\zeta_+ + \zeta_-)a}\sum_\nu \gamma_\nu e^{\gamma_{-\nu}a} A_{\nu,+}/\left(\mathbf{Q}^+ \cdot \hat{\mathbf{n}}\right), \quad (100)$$

$$G = \left(\mathbf{P}^+ \cdot \hat{\mathbf{n}}\right)(\zeta_-/\zeta_+)e^{-(\zeta_+ + \zeta_-)a}\sum_\nu \gamma_\nu e^{\gamma_{-\nu}a} A_{\nu,-}/\left(\mathbf{Q}^+ \cdot \hat{\mathbf{n}}\right), \quad (101)$$

$$L = \left(\mathbf{P}^- \cdot \hat{\mathbf{n}}\right)(\zeta_-/\zeta_+)e^{-(\zeta_+ + \zeta_-)a}\sum_\nu \gamma_\nu e^{\gamma_{-\nu}a} A_{\nu,+}/\left(\mathbf{Q}^+ \cdot \hat{\mathbf{n}}\right), \quad (102)$$

$$M = \left\{\gamma_- e^{-\gamma_+ a} - (\zeta_-/\zeta_+)\gamma_+ e^{-(\zeta_+ + \zeta_- - \gamma_-)a}\right\} + \left(\mathbf{P}^- \cdot \hat{\mathbf{n}}\right)(\zeta_-/\zeta_+)e^{-(\zeta_+ + \zeta_-)a}\sum_\nu \gamma_\nu e^{\gamma_{-\nu}a} A_{\nu,-}/\left(\mathbf{Q}^+ \cdot \hat{\mathbf{n}}\right), \quad (103)$$

and the explicit solution

$$\begin{bmatrix} \mathbf{P}^+ \cdot \mathbf{E}_{\text{slab}}^- \\ \mathbf{P}^- \cdot \mathbf{E}_{\text{slab}}^+ \end{bmatrix} = \frac{\tilde{\alpha}}{FM - GL}\begin{bmatrix} M\left(\mathbf{P}^+ \cdot \hat{\mathbf{n}}'\right) - G\left(\mathbf{P}^- \cdot \hat{\mathbf{n}}'\right) \\ -L\left(\mathbf{P}^+ \cdot \hat{\mathbf{n}}'\right) + F\left(\mathbf{P}^- \cdot \hat{\mathbf{n}}'\right) \end{bmatrix}. \quad (104)$$

Past this point the final denouement becomes somewhat anticlimactic as we revert to a near replica of material already found under Sections 7 and 8. And so, since Eq. (22) remains in force, we simply echo Eq. (28) in the form

$$\sum_{\nu'}\gamma_{-\nu'}^{-1}\left(\hat{\mathbf{n}} \cdot \mathbf{E}_{\text{slab}}^{\nu'}\right) = \left(\mathbf{P}^\nu \cdot \hat{\mathbf{n}}\right)^{-1}\left\{\gamma_\nu^{-1}\left(\mathbf{P}^\nu \cdot \mathbf{E}_{\text{slab}}^{-\nu}\right) - \left(\frac{2ik_1\hat{n}_z}{\gamma_+\gamma_-}\right)\left(\mathbf{P}^\nu \cdot \mathbf{E}_{\text{inc}}\right)\right\}. \quad (105)$$

Then, as before in connection with Eqs. (29)–(30), subtraction of the individual, $\nu = \pm$ realizations removes the left-hand side and, with the intervention of Eq. (104), permits the formerly unknown parameter $\tilde{\alpha}$ to rise to center stage. We find in

$$\tilde{\alpha}\left\{\gamma_-\left(\frac{M\left(\mathbf{P}^+ \cdot \hat{\mathbf{n}}'\right) - G\left(\mathbf{P}^- \cdot \hat{\mathbf{n}}'\right)}{\left(\mathbf{P}^+ \cdot \hat{\mathbf{n}}\right)}\right) - \gamma_+\left(\frac{-L\left(\mathbf{P}^+ \cdot \hat{\mathbf{n}}'\right) + F\left(\mathbf{P}^- \cdot \hat{\mathbf{n}}'\right)}{\left(\mathbf{P}^- \cdot \hat{\mathbf{n}}\right)}\right)\right\}$$
$$= 2ik_1\hat{n}_z\left(FM - GL\right)\sum_\nu \nu\frac{\left(\mathbf{P}^\nu \cdot \mathbf{E}_{\text{inc}}\right)}{\left(\mathbf{P}^\nu \cdot \hat{\mathbf{n}}\right)} \quad (106)$$

its explicit determination.

It remains now only to emulate Eqs. (31)–(37) from Section 8. But, once again, before doing so, it is useful to bolster our confidence in these developments by demonstrating that $\tilde{\alpha}$ does indeed regress to $\gamma_+\gamma_-\alpha$ in the limit $\sigma_3 \to 0+$, $\epsilon_3 \to \epsilon_1$, at which point $\zeta_- \to 0$ on the strength of Eq. (60). From Eqs. (101)–(102) we then have $G = L = 0$, whereas $F = \gamma_+ e^{-\gamma_- a}$, $M = \gamma_- e^{-\gamma_+ a}$ respectively from (100) and (103). All of this has the effect of converting Eq. (106) into

$$\tilde{\alpha}\left\{\gamma_-^2 e^{-\gamma_+ a}\frac{\left(\mathbf{P}^- \cdot \hat{\mathbf{n}}\right)}{\left(\mathbf{P}^+ \cdot \hat{\mathbf{n}}\right)} - \gamma_+^2 e^{-\gamma_- a}\frac{\left(\mathbf{P}^+ \cdot \hat{\mathbf{n}}\right)}{\left(\mathbf{P}^- \cdot \hat{\mathbf{n}}\right)}\right\} = 2ik_1\hat{n}_z\gamma_+\gamma_- e^{-(\gamma_+ + \gamma_-)a}\sum_\nu \nu\frac{\left(\mathbf{P}^\nu \cdot \mathbf{E}_{\text{inc}}\right)}{\left(\mathbf{P}^\nu \cdot \hat{\mathbf{n}}\right)}, \quad (107)$$

whence comparison against Eq. (30) indeed confirms that $\tilde{\alpha} = \gamma_+\gamma_-\alpha$, consistent with a like comparison of (26) against (96). In arriving at Eq. (107), use has been made of the convenient identities $\mathbf{P}^{\nu} \cdot \hat{\mathbf{n}}' = \mathbf{P}^{-\nu} \cdot \hat{\mathbf{n}}$ as previously noted in the wake of Eq. (24).

Fixing attention on Eq. (97), we pause only to remind ourselves that its second and third terms, respectively proportional to unit vectors $\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}'$, are now fully known, as is of course their common multiplier $\tilde{\alpha}$. We then have

$$\begin{aligned} \mathbf{E}^-_{\text{slab}} = & \left(\frac{1}{\gamma_+ e^{-\gamma_- a} - (\zeta_-/\zeta_+)\gamma_- e^{-(\zeta_+ + \zeta_- - \gamma_+)a}}\right)\left\{\tilde{\alpha}\hat{\mathbf{n}}' - \left\{\gamma_- e^{-\gamma_+ a} - (\zeta_-/\zeta_+)\gamma_+ e^{-(\zeta_+ + \zeta_- - \gamma_-)a}\right\}\mathbf{E}^+_{\text{slab}}\right. \\ & \left. -\hat{\mathbf{n}}\left[(\zeta_-/\zeta_+)e^{-(\zeta_+ + \zeta_-)a}\sum_{\nu,\nu'}\gamma_\nu e^{\gamma_{-\nu} a} A_{\nu,\nu'}\left(\mathbf{P}^{\nu'} \cdot \mathbf{E}^{-\nu'}_{\text{slab}}\right) / \left(\mathbf{Q}^+ \cdot \hat{\mathbf{n}}\right)\right]\right\}. \end{aligned} \tag{108}$$

Once again we introduce this structure on the left in Eq. (22), observing of course that the lengthy term in Eq. (108) proportional to $\hat{\mathbf{n}}$ is sheared away under cross product multiplication with itself. We thus get, as a counterpart of Eq. (32),

$$\begin{aligned} & \hat{\mathbf{n}} \times \left[\hat{\mathbf{n}} \times \left(\gamma_-\tilde{\alpha}\hat{\mathbf{n}}' + \left\{\left(\gamma_+^2 e^{-\gamma_- a} - \gamma_-^2 e^{-\gamma_+ a}\right) - (\zeta_-/\zeta_+)\gamma_-\gamma_+ e^{-(\zeta_+ + \zeta_-)a}\left(e^{\gamma_+ a} - e^{\gamma_- a}\right)\right\}\mathbf{E}^+_{\text{slab}}\right)\right] \\ & = -2ik_1\hat{n}_z\left(\gamma_+ e^{-\gamma_- a} - (\zeta_-/\zeta_+)\gamma_- e^{-(\zeta_+ + \zeta_- - \gamma_+)a}\right)\mathbf{E}_{\text{inc}}, \end{aligned} \tag{109}$$

whence there emerges the known vector quantity

$$\begin{aligned} \tilde{\mathbf{\Omega}} = & \left\{\left(\gamma_-^2 e^{-\gamma_+ a} - \gamma_+^2 e^{-\gamma_- a}\right) - (\zeta_-/\zeta_+)\gamma_-\gamma_+ e^{-(\zeta_+ + \zeta_-)a}\left(e^{\gamma_- a} - e^{\gamma_+ a}\right)\right\}^{-1} \\ & \times\left[2ik_1\hat{n}_z\left(\gamma_+ e^{-\gamma_- a} - (\zeta_-/\zeta_+)\gamma_- e^{-(\zeta_+ + \zeta_- - \gamma_+)a}\right)\mathbf{E}_{\text{inc}} + \gamma_-\tilde{\alpha}\hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \hat{\mathbf{n}}'\right)\right] \end{aligned} \tag{110}$$

and the penultimate clinching statement

$$\hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \mathbf{E}^+_{\text{slab}}\right) = \tilde{\mathbf{\Omega}} \tag{111}$$

which, just as before in Eqs. (34)–(37), sequentially leads to

$$\hat{\mathbf{n}}\left(\hat{\mathbf{n}} \cdot \mathbf{E}^+_{\text{slab}}\right) - \mathbf{E}^+_{\text{slab}} = \tilde{\mathbf{\Omega}}, \tag{112}$$

$$\left(\mathbf{P}^+ \cdot \hat{\mathbf{n}}\right)\left(\hat{\mathbf{n}} \cdot \mathbf{E}^+_{\text{slab}}\right) = \left(\mathbf{P}^+ \cdot \tilde{\mathbf{\Omega}}\right), \tag{113}$$

and finally

$$\mathbf{E}^+_{\text{slab}} = \hat{\mathbf{n}}\left(\mathbf{P}^+ \cdot \tilde{\mathbf{\Omega}}\right) / \left(\mathbf{P}^+ \cdot \hat{\mathbf{n}}\right) - \tilde{\mathbf{\Omega}}, \tag{114}$$

at which point slab amplitude $\mathbf{E}^+_{\text{slab}}$ has been fully disclosed in terms of known quantities, knowledge which dispels all mystery regarding its partner $\mathbf{E}^-_{\text{slab}}$ from Eq. (108), the half space amplitude $\mathbf{E}_{\text{half}-\text{low}}$ as found in Eqs. (79)–(80), and, indeed, the echo field as contained on the right in Eq. (68), the latter being the quantity of paramount operational interest. And this entire process of discovery has advanced along concrete steps of vector algebra, essentially bypassing an otherwise arid appeal to a six-by-six determinantal solution. To be sure, much algebraic intricacy does remain, enough indeed to make the prospect of numerical transcription into computer code itself a formidable task. All in all, perhaps the qualitative, cautionary lesson to be learned here is that the slightest generalization of any given physical landscape will almost inevitably elicit a disproportionate inflation of its analytic embodiment.

10.5.2. Half Space Similarity Restored

The two-step, Oseen-type solution cadence becomes moot and void when the lower enveloping half space reverts to its reference status, $\sigma_3 \to 0+$, $\epsilon_3 \to \epsilon_1$, should one nevertheless still insist upon enforcing it for $z < -a$ in the enlarged context of Eqs. (63)–(64). Indeed, the latter two revert then to their simpler precursors from Eqs. (7) and (9).♯ But, by way of a sanity check, we should demand at least that, in that limit, $\mathbf{E}_{\text{half}-\text{low}}$ recover the total transmitted electric field amplitude from Eq. (18), ϵ_1 times whose value has already been encapsulated as Eq. (72), *viz.*,

$$\mathbf{E}_{\text{half}-\text{low}} = \frac{i}{2k_1\hat{n}_z}\hat{\mathbf{n}} \times \left\{\hat{\mathbf{n}} \times \sum_{\pm} \gamma_\pm e^{\gamma_\mp a}\mathbf{E}^{\pm}_{\text{slab}}\right\}, \tag{115}$$

and not merely the somewhat less restrictive balance in Eq. (75). The required verification here is really quite simple, requiring of us only that we enforce the indicated limit in (60), which yields in the first instance

$$q_+ = -ik_1\cos\vartheta = ik_1\hat{n}_z, \tag{116}$$

at which point (61) devolves into $\mathbf{Q}^+ = ik_1\hat{\mathbf{n}}$ and thus obliterates the mixed dot/cross product in the first line of Eq. (79), leaving us with just

$$\mathbf{E}^{+}_{\text{half}-\text{low}} = -\zeta_+^{-1}e^{-\zeta_- a}\hat{\mathbf{n}} \times \left\{\hat{\mathbf{n}} \times \sum_{\pm} \gamma_\pm e^{\gamma_\mp a}\mathbf{E}^{\pm}_{\text{slab}}\right\}. \tag{117}$$

At the same time $\zeta_+ = 2ik_1\hat{n}_z$ whereas $\zeta_- = 0$, whence Eq. (117) becomes

$$\mathbf{E}_{\text{half}-\text{low}} = \frac{i}{2k_1\hat{n}_z}\hat{\mathbf{n}} \times \left\{\hat{\mathbf{n}} \times \sum_{\pm} \gamma_\pm e^{\gamma_\mp a}\mathbf{E}^{\pm}_{\text{slab}}\right\}, \tag{118}$$

which replicates Eq. (115) down to its last iota. Needless to say, at this juncture, $\mathbf{E}^{\pm}_{\text{slab}}$ are to be assigned their limiting values as determined by Eqs. (22)–(23) alone, note being taken of the fact that, since $\zeta_- = 0$, (76) had duly blended by now into Eq. (23) so as to pose no countervailing claim, whereas Eq. (75), in its metamorphosis as (79), had been fully utilized in its alternative buildup of $\mathbf{E}_{\text{half}-\text{low}}$ in Eq. (118).

Already mentioned under the footnote immediately preceding is the disconcerting fact that linear system in Eq. (83) becomes singular when the enclosing half spaces regress toward identity. It is simply the case that determinant $\Gamma\Theta - \Lambda\Psi$ lapses into zero. If this limit is to have then any chance of remaining graceful, we must further require that

$$\gamma_+ e^{-\gamma_- a}\left(\mathbf{P}^+ \cdot \hat{\mathbf{n}}\right)\left(\mathbf{P}^+ \cdot \mathbf{E}^{-}_{\text{slab}}\right) - \gamma_- e^{-\gamma_+ a}\left(\mathbf{P}^- \cdot \hat{\mathbf{n}}\right)\left(\mathbf{P}^- \cdot \mathbf{E}^{+}_{\text{slab}}\right) = 0 \tag{119}$$

in both entries, high and low, of the source vector on the extreme right of Eq. (84). But, when the half spaces become identical, Eq. (31) prevails, and gives

$$\begin{aligned}\mathbf{P}^+ \cdot \mathbf{E}^{-}_{\text{slab}} &= \alpha\gamma_- e^{\gamma_- a}\left(\mathbf{P}^- \cdot \hat{\mathbf{n}}\right)\\ \mathbf{P}^- \cdot \mathbf{E}^{+}_{\text{slab}} &= \alpha\gamma_+ e^{\gamma_+ a}\left(\mathbf{P}^+ \cdot \hat{\mathbf{n}}\right),\end{aligned} \tag{120}$$

whence the left-hand side of Eq. (119) in its adjusted form

$$\alpha\gamma_+\gamma_-\left\{\left(\mathbf{P}^+ \cdot \hat{\mathbf{n}}\right)\left(\mathbf{P}^- \cdot \hat{\mathbf{n}}\right) - \left(\mathbf{P}^- \cdot \hat{\mathbf{n}}\right)\left(\mathbf{P}^+ \cdot \hat{\mathbf{n}}\right)\right\} = 0 \tag{121}$$

properly collapses into zero. Critical use has of course been made of identities in Eq. (4) and those which attend Eq. (24).

In this same limit, with $\mathbf{Q}^+ = ik_1\hat{\mathbf{n}}$, we find that $\hat{\mathbf{n}} \cdot \mathbf{E}^{\pm}_{\text{slab}}$ as required on the extreme left of Eq. (84) can be gotten directly from an interplay between Eqs. (31) and (37) as

$$\hat{\mathbf{n}} \cdot \mathbf{E}^{-}_{\text{slab}} = \gamma_- e^{\gamma_- a}\left\{\alpha\left(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}'\right) - \gamma_+^{-1}e^{-\gamma_+ a}\left(\hat{\mathbf{n}} \cdot \mathbf{E}^{+}_{\text{slab}}\right)\right\} \tag{122}$$

and

$$\hat{\mathbf{n}} \cdot \mathbf{E}^{+}_{\text{slab}} = \left(\mathbf{P}^+ \cdot \boldsymbol{\Omega}\right) / \left(\mathbf{P}^+ \cdot \hat{\mathbf{n}}\right), \tag{123}$$

the latter simplified by the observation that, on the basis of Eq. (33), $\hat{\mathbf{n}} \cdot \boldsymbol{\Omega} = 0$. We should, in principle, be able to extract the very same values so determined as a suitable l'Hôpital limit on the right of Eq. (84). But an analytic feat of this sort lies not only well beyond the reach of our own dexterity, but doubtless also beyond the interest horizon of any prospective reader.

♯ Further evidence of an abrupt methodological transition arrives from the observation that linear system (83), having the matrix entries Γ, Λ, Ψ, and Θ as given by Eqs. (85)–(88), becomes singular, and must be abandoned. Matrix singularity occurs as a consequence of the value $\zeta_- = 0$ which, as previously noted, (60) assigns in this limit.

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