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Dielectric Wedge Scattering: An Analytic Inroad

Jan A. Grzesik*

Abstract—We provide herein open-form, double series formulae describing the diffraction of electromagnetic waves by a dielectric, dissipative wedge of finite radius a. Our procedure bypasses altogether any attempt to enforce boundary conditions at wedge faces, and relies instead on volume self-consistency for the total electric field, incident plus self-consistently radiated by polarization/ohmic currents distributed throughout the wedge interior. Self-consistency of this sort is formulated as an integral equation over the wedge cross-sectional area, an equation wherein are implicitly subsumed all necessary boundary conditions. The crux of the ensuing solution depends upon a decomposition within the wedge interior of both incoming (here taken as plane) wave field and the underlying Green's (Hankel) function into standard functional buildings blocks individually compliant with the Helmholtz equation as adapted to the reference, exterior medium. With such decomposition in hand, the remainder of the solution follows a more or less routine, Ewald-Oseen route, one eased by function orthogonality, by cancellation across the board of the total field when similarly so decomposed throughout the wedge interior, and an almost rote reading off of interior expansion coefficients against those found on the exterior. The incoming field series decomposition across the wedge interior, it should be noted, avoids the pitfall of a naïve recourse to Fourier series, and invokes instead a root-mean-square minimization. That such a procedure enjoys a measure of validity is confirmed in Appendix C, wherein it is shown that the present analytic apparatus, when permitted to confront a degenerate wedge having its exterior angle γ tending to zero, $\gamma \to 0+$, which is to say, a *bona fide* dielectric cylinder, recovers the classical, boundary-value solution as to its every detail. All in all, while we do hope that the present work will serve to broaden the prevailing viewpoint as to permeable wedge scattering, we nevertheless admit to a measure of regret as to the complexity of the resulting formulae, whose numerical implementation bodes ominously to be a formidable task in its own right. It would seem that we reach here a frontier of diminishing returns as to the applications of classical analysis, a point at which its intellectual allure can honorably surrender to direct, computer-driven point matching methods.

1. INTRODUCTION

The theory of diffraction by perfectly conducting wedges of infinite extent overlooks a vast terrain in the electromagnetic landscape, and has attracted a voluminous, magisterial literature, with its roots anchored to the pioneering efforts of Sommerfeld and Macdonald [1, 2]. References [3–10] [1] give some indication of just how important, how endlessly and urgently popular this theory, now well over a century old, has remained in scientific circles.

While both MacDonald and Sommerfeld base their arguments on field expansions in fractional-index Bessel functions, it was only the latter who was able to recast his theory in terms of canonical

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* Corresponding author: Jan A. Grzesik (jan.grzesik@hotmail.com).
The author is with the Allwave Corporation, 3860 Del Amo Boulevard, Suite 404, Torrance, CA 90503, USA.
[1] To be sure, References [4, 5] abstain from treatment of genuine wedges having their exterior angles γ less than 2π, confining themselves to half-planes with $\gamma = 2\pi$. There is no shortage of literature devoted to this special, $\gamma = 2\pi$ case, be the treatment via the Sommerfeld formalism or else the more modern, Wiener-Hopf approach. One could even be tempted to say that the literature in this niche genre runs legion.

contour integrals of an enduring allure. These integrals exhibit, on the one hand, a periodicity of twice the wedge exterior angle 2γ, and thus invite visualization in terms of image sources placed on nonphysical Riemann sheets, while, on the other, and much, much more cogently so, they exhibit angular symmetry about both wedge faces. This symmetry, emphasized by Pauli [11], is well suited to enforcing both electric and magnetic boundary conditions at wedge faces, and, in our opinion, has a far greater physical impact than any artifice of image sources sitting on submerged Riemann sheets, no matter how attractive such concepts may be from a purely mathematical perspective. Moreover, it becomes self-evident that, once such a two-face symmetry has been duly ascertained, the 2γ periodicity follows as an automatic consequence.

The natural efforts to extend such analyses to permeable, dielectric wedges had to forfeit at once all recourse to *a priori* angular symmetries and thus quickly ground to a halt. There ensued a half century or so of surrogate attempts, proceeding by fits and starts, to mimic penetration into wedge interiors by a variety of surface impedance stratagems, invariably phrased as adaptations of the Sommerfeld formalism, and culminating in the Malyuzhinets program [8].

The work now reported intends to turn this timid approach on its head by precisely such an entry into wedge interiors phrased via a statement of field self-consistency, conveyed by an integral equation wherein wedge polarization/ohmic currents are allowed full rein as radiative sources, for fields both near and far, fields superimposed upon the excitation arriving from the exterior. [2] Once the appropriate integral equation has been set up, here with its Green's function gotten as[3] $(4i)^{-1}\times$ Hankel function $H_0^{(1)}$, its solution evolves along fairly standard lines, save for one major caveat. That caveat springs into view once the need is recognized to produce expansions based on Helmholtz equation solutions within the angular wedge slot not only for the interior field, with its interior radial propagation constant k_2 (*q.v.* Eq. (1)), but also for both the incoming excitation and for the Green's function, the latter two controlled of course by the exterior, reference propagation constant k_1 (Eq. (2)). Once such wedge expansions become available, the orthogonality of their terms under angular quadrature renders virtually routine the subsequent integral equation interior field solution along Ewald-Oseen extinction lines, a solution of the sort recently set out by [12] in a substantially simpler planar slab geometry.

Finally, with the interior wedge field duly in hand, the exterior, scattered field is gotten almost as a quadrature afterthought, the integration over interior sources being carried out now with the Green's function restored to its native, trigonometric/Bessel expansion, unrestricted as to angle, $0 < \varphi \leq 2\pi$, and with Bessel function indices strictly integral. In principle it is all quite simple, save for the fact that, since both integer and non-integer angular components now enter the mix, all simplifying benefits of diagonal outcomes vanish, and one is left to contend with doubly indexed infinite series. And so, it is because of this computational complexity that we choose to defer all numerical implementation/comparison to future work, and to include in our title the prudent, modest qualifier *"inroad"* so as to soften the thrust of our solution claim.

2. PROBLEM SETUP

We consider a dielectric wedge having its transverse extent bounded by radius $r = a$ as shown in Figure 1. It is unlimited along its rim, direction $\hat{\mathbf{e}}_z$, which serves as radial origin $r = 0$ in polar coördinates. The wedge is uniform with dielectric permittivity ϵ_2 and conductivity σ_2, and is embedded in a similarly uniform, nondissipative dielectric medium having permittivity ϵ_1.[4] Exterior angle γ can roam full-range,

[2] For the sake of methodological convenience at this initial level of reporting, we insist in what follows upon exterior excitation fields of an exclusively plane-wave type. The analysis now given shows, *ipso facto*, that restrictions of this sort can be taken in stride by suitable extensions of the steps already taken.

Furthermore, at this groundbreaking level of permeable wedge analysis, we require that wave incidence be perpendicular to the edge, and that it be electrically polarized along it. We relinquish moreover the patent idealization of a wedge having unbounded transverse extent. This infinite idealization is clearly the easiest to abandon, its abandonment providing by way of compensation the freedom to illuminate the wedge from any incident direction $0 < \varphi_0 \leq 2\pi$, and not just from directions bounded by $0 \leq \varphi_0 \leq \gamma$, and it liberates all attendant quadratures from any danger of divergence.

[3] We restrict ourselves to a simple harmonic time dependence $\exp(-i\omega t)$, with a positive angular frequency $\omega > 0$. The requisite analysis for negative frequencies follows under complex conjugation across the board.

[4] The special case $\epsilon_2 = \epsilon_1$ is not excluded, provided only that $\sigma_2 \neq 0$. All quantities, be they geometric or electrical, adhere to SI units, so that $\epsilon_{1,2}$ are to be measured in farads per meter and σ_2 in Siemens per meter. We have in mind of course an essentially nondissipative reference medium 1 with a vanishing conductivity $\sigma_1 \downarrow 0+$ (*cf.* Eq. (2) below).

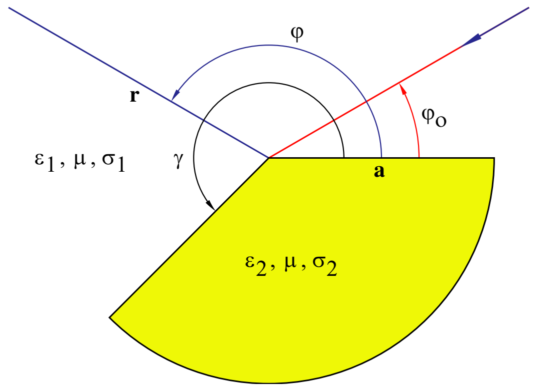

Figure 1. Diffracting wedge geometry: transverse radius a, exterior angle γ, plane-wave arrival along φ_0, reference dielectric medium (ϵ_1, μ, σ_1), dielectric wedge material (ϵ_2, μ, σ_2).

$0 \le \gamma \le 2\pi$, its end points $\gamma = 0$ and $\gamma = 2\pi$ being admissible as geometrically obvious special cases.

A common value is assumed for the magnetic permeability μ, measured in henrys per meter. The parameters thus identified provide

$$\begin{aligned} k_2 &= \omega\sqrt{\mu(\epsilon_2 + i\sigma_2/\omega)} \\ &= \omega\sqrt{\frac{\mu}{2}}\left[\sqrt{\sqrt{\epsilon_2^2 + \sigma_2^2/\omega^2} + \epsilon_2} + i\sqrt{\sqrt{\epsilon_2^2 + \sigma_2^2/\omega^2} - \epsilon_2}\right] \end{aligned} \tag{1}$$

for the propagation constant within the wedge, and

$$\begin{aligned} k_1 &\underset{\sigma_1 \downarrow 0+}{=} \omega\sqrt{\mu(\epsilon_1 + i\sigma_1/\omega)} \\ &\underset{\sigma_1 \downarrow 0+}{=} \omega\sqrt{\epsilon_1\mu}\left(1 + i\,\frac{\sigma_1}{2\epsilon_1\omega}\right) \end{aligned} \tag{2}$$

as its exterior counterpart.

The wedge is illuminated from the exterior by a plane electric wave[5] of unit amplitude, polarized along $\hat{\mathbf{e}}_z$, and incident from direction φ_0, *viz.*, $\mathbf{E}_{\mathrm{inc}}(r,\varphi) = \hat{\mathbf{e}}_z \exp\left(-ik_1 r\cos(\varphi - \varphi_0)\right)$. On physical grounds, the total field, incident plus scattered, is then likewise everywhere endowed with but a single component along $\hat{\mathbf{e}}_z$. Although incidence angle φ_0 is *a priori* unrestricted, $0 \le \varphi_0 < 2\pi$, it is further clear from symmetry that we need consider only one of the half-ranges $\gamma/2 - \pi \le \varphi_0 \le \gamma/2$ and $\gamma/2 \le \varphi_0 \le \gamma/2 + \pi$, a natural 2π periodicity being understood. The ensuing analysis will however offer neither a discernible opportunity for, nor any particular advantage in exploiting such symmetry. Its benefits would only serve to alleviate the labor of some ultimate numerical implementation. And so it is never invoked in the material below.

3. RADIATIVE SELF-CONSISTENCY

Within the wedge one encounters a distribution of ohmic/dielectric polarization currents $\left(\sigma_2 - i\omega\epsilon_2\right)\mathbf{E}(r,\varphi)$, with $\mathbf{E}$ being the total field, incident plus that self-consistently radiated by them. This latter distribution can, however, be regarded as an excess above a similar, reference distribution

[5] A plane wave bespeaks of course nothing more than a useful idealization wherein a somewhat more credible line source recedes to infinity. Moreover, we have the option of taking k_1 to be rigorously real or else, as in Eq. (2), to include a vanishingly small dissipative offset from the real axis. In the latter case we face the standard dilemma of witnessing a steady, physically most welcome attenuation with wave progress, but only at the cost of insisting that the source acquire unbounded strength during its departure to infinity. We postpone to some future date all efforts to account for finite-distance line source illumination.

$-i\omega\epsilon_1\mathbf{E}(r,\varphi)$ pervading all of space. This excess radiates a scattered magnetic field

$$\mathbf{B}_{\text{scatt}}(r,\varphi) = \frac{\mu\left\{\sigma_2 - i\omega(\epsilon_2-\epsilon_1)\right\}}{4\pi}\boldsymbol{\nabla}\times\left[\int_0^a r'dr'\int_\gamma^{2\pi} d\varphi'\,\mathbf{E}(r',\varphi')\left\{\int_{-\infty}^{\infty}\frac{e^{ik_1\sqrt{|\mathbf{r}-\mathbf{r}'|^2+z^2}}dz}{\sqrt{|\mathbf{r}-\mathbf{r}'|^2+z^2}}\right\}\right], \tag{3}$$

or else, since the integral over z is proportional to the Hankel function $H_0^{(1)}(k_1|\mathbf{r}-\mathbf{r}'|)$,

$$\int_{-\infty}^{\infty}\frac{e^{ik_1\sqrt{|\mathbf{r}-\mathbf{r}'|^2+z^2}}dz}{\sqrt{|\mathbf{r}-\mathbf{r}'|^2+z^2}} = i\pi H_0^{(1)}(k_1|\mathbf{r}-\mathbf{r}'|)\,, \tag{4}$$

$$\mathbf{B}_{\text{scatt}}(r,\varphi) = \frac{i\mu\left\{\sigma_2 - i\omega(\epsilon_2-\epsilon_1)\right\}}{4}\boldsymbol{\nabla}\times\left[\int_0^a r'dr'\int_\gamma^{2\pi} H_0^{(1)}(k_1|\mathbf{r}-\mathbf{r}'|)\,\mathbf{E}(r',\varphi')d\varphi'\right]. \tag{5}$$

The total field $\mathbf{B}(r,\varphi)$, gotten by augmenting Eq. (5) with that incident,

$$\begin{aligned}\mathbf{B}(r,\varphi) &= -\frac{i}{\omega}\boldsymbol{\nabla}\times\mathbf{E}_{\text{inc}}(r,\varphi)\\ &\quad+\frac{i\mu\left\{\sigma_2 - i\omega(\epsilon_2-\epsilon_1)\right\}}{4}\boldsymbol{\nabla}\times\left[\int_0^a r'dr'\int_\gamma^{2\pi} H_0^{(1)}(k_1|\mathbf{r}-\mathbf{r}'|)\,\mathbf{E}(r',\varphi')d\varphi'\right],\end{aligned} \tag{6}$$

allows construction of the total, self-consistent field $\mathbf{E}(r,\varphi)$ everywhere, both inside and outside the wedge, the latter indicated in yellow in Figure 1, as

$$\begin{aligned}\epsilon(r,\varphi)\mathbf{E}(r,\varphi) &= \frac{1}{\omega^2\mu}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{E}_{\text{inc}}(r,\varphi)\\ &\quad-\frac{\left\{\sigma_2 - i\omega(\epsilon_2-\epsilon_1)\right\}}{4\omega}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\left[\int_0^a r'dr'\int_\gamma^{2\pi} H_0^{(1)}(k_1|\mathbf{r}-\mathbf{r}'|)\,\mathbf{E}(r',\varphi')d\varphi'\right],\end{aligned} \tag{7}$$

with

$$\epsilon(r,\varphi) = \begin{cases}\epsilon_1; & \{0\le\varphi<\gamma\,,\,0\le r\le a\}\bigcup\{0\le\varphi<2\pi\,,\,r>a\}\\ \epsilon_2+i\sigma_2/\omega; & \{\gamma\le\varphi\le 2\pi\,,\,0\le r\le a\}\,.\end{cases} \tag{8}$$

Simplifications of Eq. (7) accrue on noting, first, that

$$\frac{1}{\omega^2\mu}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{E}_{\text{inc}}(r,\varphi) = \epsilon_1\mathbf{E}_{\text{inc}}(r,\varphi)\,, \tag{9}$$

while, in connection with the second, integral term, the divergence,

$$\begin{aligned}\boldsymbol{\nabla}\cdot\left\{H_0^{(1)}(k_1|\mathbf{r}-\mathbf{r}'|)\,\mathbf{E}(r',\varphi')\right\} &= \left\{\boldsymbol{\nabla}\,H_0^{(1)}(k_1|\mathbf{r}-\mathbf{r}'|)\right\}\cdot\mathbf{E}(r',\varphi')\\ &= 0\,,\end{aligned} \tag{10}$$

vanishes by virtue of the fact that the gradient $\boldsymbol{\nabla}\,H_0^{(1)}(k_1|\mathbf{r}-\mathbf{r}'|)$ is transverse to $\mathbf{E}(r',\varphi')$. Altogether then Eq. (7) becomes

$$\begin{aligned}\epsilon(r,\varphi)\mathbf{E}(r,\varphi) &= \epsilon_1\mathbf{E}_{\text{inc}}(r,\varphi)\\ &\quad+\frac{\left\{\sigma_2 - i\omega(\epsilon_2-\epsilon_1)\right\}}{4\omega}\nabla^2\left[\int_0^a r'dr'\int_\gamma^{2\pi} H_0^{(1)}(k_1|\mathbf{r}-\mathbf{r}'|)\,\mathbf{E}(r',\varphi')d\varphi'\right].\end{aligned} \tag{11}$$

And finally, on invoking the rôle of $(4i)^{-1}H_0^{(1)}(k_1|\mathbf{r}-\mathbf{r}'|)$ as a Green's function for the two-dimensional Helmholtz equation,[6]

$$\left(\nabla^2+k_1^2\right)H_0^{(1)}(k_1|\mathbf{r}-\mathbf{r}'|)=4i\delta\left(\mathbf{r}-\mathbf{r}'\right)\,, \tag{12}$$

Eq. (11) is brought into the form

$$\begin{aligned}\epsilon(r,\varphi)\mathbf{E}(r,\varphi) \;=\;& \epsilon_1\mathbf{E}_{\rm inc}(r,\varphi)\\ &-\frac{\omega\epsilon_1\mu\left\{\sigma_2-i\omega(\epsilon_2-\epsilon_1)\right\}}{4}\int_0^a r'dr'\int_{\gamma-2\pi}^0 H_0^{(1)}(k_1|\mathbf{r}-\mathbf{r}'|)\,\mathbf{E}(r',\varphi')\,d\varphi'\\ &+\frac{i\left\{\sigma_2-i\omega(\epsilon_2-\epsilon_1)\right\}}{\omega}\,\mathbf{E}(r,\varphi)\,,\end{aligned} \tag{13}$$

with the proviso that the third line is present only within the wedge interior, $0\le r\le a$, $\gamma-2\pi\le\varphi\le 0$, the latter statement, in conjunction with the adjusted integration limits in the second line, serving to emphasize the *a priori* 2π angular periodicity.

4. SOLVING THE INTEGRAL EQUATION

Since Eq. (13) confirms the self-consistency of having assumed that both total $\mathbf{E}(r,\varphi)$ and scattered $\mathbf{E}_{\rm scatt}(r,\varphi)=\mathbf{E}(r,\varphi)-\mathbf{E}_{\rm inc}(r,\varphi)$ electric fields share with $\mathbf{E}_{\rm inc}(r,\varphi)$ a common, invariable polarization along edge direction $\hat{\mathbf{e}}_z$, we dispense henceforth with the boldface vector notation and write all relations in terms of complex-valued amplitudes. Thus Eq. (13) regresses into

$$\begin{aligned}\epsilon(r,\varphi)E(r,\varphi) \;=\;& \epsilon_1E_{\rm inc}(r,\varphi)\\ &-\frac{\omega\epsilon_1\mu\left\{\sigma_2-i\omega(\epsilon_2-\epsilon_1)\right\}}{4}\int_0^a r'dr'\int_{\gamma-2\pi}^0 H_0^{(1)}(k_1|\mathbf{r}-\mathbf{r}'|)\,E(r',\varphi')\,d\varphi'\\ &+\frac{i\left\{\sigma_2-i\omega(\epsilon_2-\epsilon_1)\right\}}{\omega}\,E(r,\varphi)\,.\end{aligned} \tag{14}$$

Fixing attention first upon the wedge domain *per se*, with the third line present, we abbreviate its angular extent by setting $2\pi-\gamma=2\pi(1-\gamma/2\pi)=2\pi\tilde{\gamma}$. Throughout this wedge, each member of the (r,φ) function system $\left(J_{n/\tilde{\gamma}}(k_2r)\left\{\cos(n\varphi/\tilde{\gamma}),\,\sin(n\varphi/\tilde{\gamma})\right\}\right)_{n=0}^{\infty}$ obeys the requisite Helmholtz equation appropriate to propagation constant k_2, and all such functions are orthogonal with respect to the inner product

$$(f,g)=\int_0^a rdr\int_{-2\pi\tilde{\gamma}}^0 f(r,\varphi)g^*(r,\varphi)d\varphi\,, \tag{15}$$

the burden of orthogonality being carried strictly by the trigonometric terms.[7] The system is presumably complete, and on the basis of such assumption we write

$$E(r,\varphi)=\sum_{n=0}^{\infty}J_{n/\tilde{\gamma}}(k_2r)\left\{A_n\cos(n\varphi/\tilde{\gamma})+B_n\sin(n\varphi/\tilde{\gamma})\right\} \tag{16}$$

with coefficients A_n, B_n still to be determined.[8]

[6] Since it is commonly accepted that in three dimensions

$$\left(\nabla^2+k_1^2\right)\frac{e^{ik_1\sqrt{|\mathbf{r}-\mathbf{r}'|^2+z^2}}}{\sqrt{|\mathbf{r}-\mathbf{r}'|^2+z^2}}=-4\pi\delta\left(\mathbf{r}-\mathbf{r}'\right)\delta(z)\,,$$

Eq. (12) becomes an easy consequence of (4). It is of course understood, here and elsewhere, that $\mathbf{r}$, $\mathbf{r}'$ represent two-dimensional vectors, as is in any event suggested by Figure 1.

[7] The elementary integrations underlying such orthogonality, and the attendant function normalizations, appear in Appendix A below. For immediate use one may note that both squared sine and cosine integrals evaluate to $\pi\tilde{\gamma}$ when $n\ge 1$.

[8] The prospective reader is placed on high alert not to associate coefficients B_n with any sort of direct magnetic significance. If nothing else, their dimension in the SI system is that of volt/meter, whereas magnetic fields are rendered in tesla.

Now, in order to take advantage of the stated angular orthogonality, we must seek similar developments for both $E_{\rm inc}(r,\varphi)$ and $H_0^{(1)}(k_1|\mathbf{r}-\mathbf{r}'|)$, but of course in a Helmholtz basis

$$\Big(J_{n/\tilde{\gamma}}(k_1 r)\,\big\{\cos(n\varphi/\tilde{\gamma}),\,\sin(n\varphi/\tilde{\gamma})\big\}\Big)_{n=0}^{\infty}$$

adapted to the reference propagation constant k_1. And so we set[9]

$$E_{\rm inc}(r,\varphi)=\sum_{n=0}^{\infty}J_{n/\tilde{\gamma}}(k_1 r)\Big\{C_n\cos(n\varphi/\tilde{\gamma})+D_n\sin(n\varphi/\tilde{\gamma})\Big\} \tag{17}$$

with coefficients C_n, D_n gotten in Appendix A as rather intricate structures under a root-mean-square minimization with respect to the inner product spelled out in (15). The corresponding Hankel function development[10]

$$H_0^{(1)}(k_1|\boldsymbol{r}-\boldsymbol{r'}|)\;=\;\frac{1}{\tilde{\gamma}}\sum_{m=0}^{\infty}\left(2-\delta_0^m\right)J_{m/\tilde{\gamma}}(k_1 r_<)H_{m/\tilde{\gamma}}^{(1)}(k_1 r_>)\cos(m\{\varphi-\varphi'\}/\tilde{\gamma}) \tag{18}$$

follows much more simply by identifying Green's function attributes in a process that emulates a counterpart successfully performed in a full-range, $0\le\varphi<2\pi$ scenario. It goes without saying that representations in Eqs. (17)–(18) are strictly confined to the wedge angular domain $-2\pi\tilde{\gamma}\le\varphi\le 0$. On the wedge exterior one naturally reverts to a full-range embodiment of (18), examined, of course, only throughout $-2\pi<\varphi<-2\pi\tilde{\gamma}$ when $r<a$.

Interplay between Eqs. (16) and (18) gives next[11]

$$\int_0^a r'dr'\int_{-2\pi\tilde{\gamma}}^{0}H_0^{(1)}(k_1|\mathbf{r}-\mathbf{r}'|)\,E(r',\varphi')\,d\varphi'=$$

$$\pi\sum_{n=0}^{\infty}\left(2-\delta_0^n\right)\Big(\Big(1+\delta_0^n\Big)A_n\cos(n\varphi/\tilde{\gamma})+B_n\sin(n\varphi/\tilde{\gamma})\Big)\int_0^a r'J_{n/\tilde{\gamma}}(k_2r')J_{n/\tilde{\gamma}}(k_1r_<)H_{n/\tilde{\gamma}}^{(1)}(k_1r_>)dr'. \tag{19}$$

But now

$$\begin{aligned}\int_0^a r'J_{n/\tilde{\gamma}}(k_2r')J_{n/\tilde{\gamma}}(k_1r_<)H_{n/\tilde{\gamma}}^{(1)}(k_1r_>)dr' \;&=\; H_{n/\tilde{\gamma}}^{(1)}(k_1r)\int_0^r r'J_{n/\tilde{\gamma}}(k_2r')J_{n/\tilde{\gamma}}(k_1r')dr'\\ &\quad+J_{n/\tilde{\gamma}}(k_1r)\int_r^a r'J_{n/\tilde{\gamma}}(k_2r')H_{n/\tilde{\gamma}}^{(1)}(k_1r')dr'\,,\end{aligned} \tag{20}$$

with both integrals on the right well known, and easily evaluated on the basis of the underlying Bessel ODE. Thus

$$\int_0^r r'J_{n/\tilde{\gamma}}(k_2r')J_{n/\tilde{\gamma}}(k_1r')dr'\;=\;\left.\frac{r'\Big(k_1J_{n/\tilde{\gamma}}(k_2r')J'_{n/\tilde{\gamma}}(k_1r')-k_2J_{n/\tilde{\gamma}}(k_1r')J'_{n/\tilde{\gamma}}(k_2r')\Big)}{k_2^2-k_1^2}\right|_{r'=0}^{r'=r} \tag{21}$$

and

$$\int_r^a r'J_{n/\tilde{\gamma}}(k_2r')H_{n/\tilde{\gamma}}^{(1)}(k_1r')dr'\;=\;\left.\frac{r'\Big(k_1J_{n/\tilde{\gamma}}(k_2r')H_{n/\tilde{\gamma}}^{(1)\prime}(k_1r')-k_2H_{n/\tilde{\gamma}}^{(1)}(k_1r')J'_{n/\tilde{\gamma}}(k_2r')\Big)}{k_2^2-k_1^2}\right|_{r'=r}^{r'=a}. \tag{22}$$

[9] One may note in passing that, in (16)–(17) onward, one has available also the option of using, instead of sines and cosines, simple exponentials indexed now from $-\infty$ to ∞, $-\infty<n<\infty$, accompanied by Bessel functions $J_{|n|/\tilde{\gamma}}(k_{1,2}r)$ which avoid the divergence at origin $r=0$ associated with negative, fractional indices. Our seemingly pedestrian, seemingly clumsy reliance on sines and cosines provides an automatic embodiment of such fractional index non-negativity.

[10] In standard notation, $r_<=\min(r,r')$, $r_>=\max(r,r')$. Series development (18) is derived in Appendix B. It is of course understood in Eq. (18) that both φ and φ' are here confined to the wedge angular slot, $-2\pi\tilde{\gamma}\le\varphi,\varphi'\le 0$. All other periodicity slots of width $2\pi\tilde{\gamma}$, artificially adjoined up and down without end, must of course be ignored, and, in any event, do not even enter into the ensuing calculational process.

[11] For the trigonometric integrals, *cf.* Appendix A, Eqs. (39)–(41).

On putting Eq. (20) through Eq. (22) together we thus find

$$\int_0^a r' J_{n/\tilde{\gamma}}(k_2 r') J_{n/\tilde{\gamma}}(k_1 r_<) H^{(1)}_{n/\tilde{\gamma}}(k_1 r_>) dr' =$$
$$J_{n/\tilde{\gamma}}(k_1 r) \left\{ \frac{a\left(k_1 J_{n/\tilde{\gamma}}(k_2 a) H^{(1)'}_{n/\tilde{\gamma}}(k_1 a) - k_2 H^{(1)}_{n/\tilde{\gamma}}(k_1 a) J'_{n/\tilde{\gamma}}(k_2 a)\right)}{k_2^2 - k_1^2} \right\}$$
$$+ J_{n/\tilde{\gamma}}(k_2 r) \left\{ \frac{k_1 r \left(H^{(1)}_{n/\tilde{\gamma}}(k_1 r) J'_{n/\tilde{\gamma}}(k_1 r) - J_{n/\tilde{\gamma}}(k_1 r) H^{(1)'}_{n/\tilde{\gamma}}(k_1 r)\right)}{k_2^2 - k_1^2} \right\} \tag{23}$$

and can then still further condense into

$$\int_0^a r' J_{n/\tilde{\gamma}}(k_2 r') J_{n/\tilde{\gamma}}(k_1 r_<) H^{(1)}_{n/\tilde{\gamma}}(k_1 r_>) dr' =$$
$$J_{n/\tilde{\gamma}}(k_1 r) \left\{ \frac{a\left(k_1 J_{n/\tilde{\gamma}}(k_2 a) H^{(1)'}_{n/\tilde{\gamma}}(k_1 a) - k_2 H^{(1)}_{n/\tilde{\gamma}}(k_1 a) J'_{n/\tilde{\gamma}}(k_2 a)\right)}{k_2^2 - k_1^2} \right\}$$
$$+ \frac{2}{i\pi(k_2^2 - k_1^2)} J_{n/\tilde{\gamma}}(k_2 r) \tag{24}$$

following appeal to the well known Wronskian connection

$$H^{(1)}_{n/\tilde{\gamma}}(k_1 r) J'_{n/\tilde{\gamma}}(k_1 r) - J_{n/\tilde{\gamma}}(k_1 r) H^{(1)'}_{n/\tilde{\gamma}}(k_1 r) = \frac{2}{i\pi k_1 r} . \tag{25}$$

A restrospective glance at Eqs. (1)–(2) shows that $k_2^2 - k_1^2 = i\omega\mu\left(\sigma_2 - i\omega(\epsilon_2 - \epsilon_1)\right)$, whereupon the interior field in Eq. (16) cancels identically in Eq. (14), leaving us with just

$$\sum_{n=0}^{\infty} J_{n/\tilde{\gamma}}(k_1 r) \left(C_n \cos(n\varphi/\tilde{\gamma}) + D_n \sin(n\varphi/\tilde{\gamma})\right) =$$
$$\frac{\pi a}{4i} \sum_{n=0}^{\infty} (2 - \delta_0^n) J_{n/\tilde{\gamma}}(k_1 r) \left((1 + \delta_0^n) A_n \cos(n\varphi/\tilde{\gamma}) + B_n \sin(n\varphi/\tilde{\gamma})\right) \times$$
$$\times \left\{ k_1 J_{n/\tilde{\gamma}}(k_2 a) H^{(1)'}_{n/\tilde{\gamma}}(k_1 a) - k_2 H^{(1)}_{n/\tilde{\gamma}}(k_1 a) J'_{n/\tilde{\gamma}}(k_2 a) \right\} , \tag{26}$$

involving only reference medium propagation and allowing us to simply read off the desired amplitudes A_n and B_n in the form

$$A_n = \frac{2i}{\pi a} C_n \left\{ k_1 J_{n/\tilde{\gamma}}(k_2 a) H^{(1)'}_{n/\tilde{\gamma}}(k_1 a) - k_2 H^{(1)}_{n/\tilde{\gamma}}(k_1 a) J'_{n/\tilde{\gamma}}(k_2 a) \right\}^{-1}$$
$$B_n = \frac{2i}{\pi a} D_n \left\{ k_1 J_{n/\tilde{\gamma}}(k_2 a) H^{(1)'}_{n/\tilde{\gamma}}(k_1 a) - k_2 H^{(1)}_{n/\tilde{\gamma}}(k_1 a) J'_{n/\tilde{\gamma}}(k_2 a) \right\}^{-1} , \tag{27}$$

whereupon the problem is in essence solved once coefficients C_n and D_n have been duly imported from Appendix A below. In connection with coefficients A_n, use has been made of the fact that $(2 - \delta_0^n)\left(1 + \delta_0^n\right) = 2 + \delta_0^n - (\delta_0^n)^2 = 2$ identically for all indices $n \geq 0$, while, in connection with the B_n, the preliminary factor $2 - \delta_0^n$ has been simply set at 2 because B_0 is inoperative, algebraically moot.

5. SCATTERED FIELD ON WEDGE EXTERIOR

When seeking the scattered field on the wedge exterior we naturally revert to the standard, full-range Hankel development [13, p. 374]

$$H_0^{(1)}(k_1|\boldsymbol{r} - \boldsymbol{r'}|) = \sum_{m=0}^{\infty} (2 - \delta_0^m) J_m(k_1 r_<) H^{(1)}_m(k_1 r_>) \cos(m\{\varphi - \varphi'\}) . \tag{28}$$

On dispensing with its final term, Eq. (14) thus gives

$$E_{\text{scatt}}(r,\varphi) = \frac{i(k_2^2-k_1^2)}{4}\sum_{m=0}^{\infty}(2-\delta_0^m)\int_0^a r'dr' \times \\ \times \int_{-2\pi\tilde{\gamma}}^{0} E(r',\varphi')J_m(k_1r_<)H_m^{(1)}(k_1r_>)\cos(m\{\varphi-\varphi'\})d\varphi' \quad (29)$$

which further becomes

$$E_{\text{scatt}}(r,\varphi) = \frac{i(k_2^2-k_1^2)}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(2-\delta_0^m)\int_0^a r'J_m(k_1r_<)H_m^{(1)}(k_1r_>)J_{n/\tilde{\gamma}}(k_2r')dr' \times \\ \times \int_{-2\pi\tilde{\gamma}}^{0}\Big\{A_n\cos(n\varphi'/\tilde{\gamma})+B_n\sin(n\varphi'/\tilde{\gamma})\Big\}\cos(m\{\varphi-\varphi'\})\,d\varphi' \quad (30)$$

once note is taken of Eq. (16). We become obliged thus to dispose first of the elementary integrals

$$\int_{-2\pi\tilde{\gamma}}^{0}\cos(n\varphi'/\tilde{\gamma})\cos(m\{\varphi-\varphi'\})\,d\varphi' = \Re\left\{\int_{-2\pi\tilde{\gamma}}^{0} e^{in\varphi'/\tilde{\gamma}}\cos(m\{\varphi-\varphi'\})\,d\varphi'\right\} \quad (31)$$

and

$$\int_{-2\pi\tilde{\gamma}}^{0}\sin(n\varphi'/\tilde{\gamma})\cos(m\{\varphi-\varphi'\})\,d\varphi' = \Im\left\{\int_{-2\pi\tilde{\gamma}}^{0} e^{in\varphi'/\tilde{\gamma}}\cos(m\{\varphi-\varphi'\})\,d\varphi'\right\}, \quad (32)$$

both of which fall into place on noting that

$$\int_{-2\pi\tilde{\gamma}}^{0} e^{in\varphi'/\tilde{\gamma}}\cos(m\{\varphi-\varphi'\})\,d\varphi' = \frac{\tilde{\gamma}}{2i}\left\{e^{im\varphi}\,\frac{e^{i\{n/\tilde{\gamma}-m\}\varphi'}}{n-m\tilde{\gamma}}\Bigg|_{\varphi'=-2\pi\tilde{\gamma}}^{\varphi'=0} + e^{-im\varphi}\,\frac{e^{i\{n/\tilde{\gamma}+m\}\varphi'}}{n+m\tilde{\gamma}}\Bigg|_{\varphi'=-2\pi\tilde{\gamma}}^{\varphi'=0}\right\} \\ = \frac{\tilde{\gamma}}{2i}\left\{e^{im\varphi}\left(\frac{1-e^{2\pi im\tilde{\gamma}}}{n-m\tilde{\gamma}}\right)+e^{-im\varphi}\left(\frac{1-e^{-2\pi im\tilde{\gamma}}}{n+m\tilde{\gamma}}\right)\right\} \quad (33) \\ = \frac{\tilde{\gamma}}{n^2-m^2\tilde{\gamma}^2}\Bigg\{m\tilde{\gamma}\Big(\sin(m\varphi)-\sin(m\{\varphi+2\pi\tilde{\gamma}\})\Big) - \\ -in\Big(\cos(m\varphi)-\cos(m\{\varphi+2\pi\tilde{\gamma}\})\Big)\Bigg\}$$

whereupon

$$\int_{-2\pi\tilde{\gamma}}^{0}\cos(n\varphi'/\tilde{\gamma})\cos(m\{\varphi-\varphi'\})\,d\varphi' = \frac{m\tilde{\gamma}^2\Big\{\sin(m\varphi)-\sin(m\{\varphi+2\pi\tilde{\gamma}\})\Big\}}{n^2-m^2\tilde{\gamma}^2} \quad (34)$$

whereas

$$\int_{-2\pi\tilde{\gamma}}^{0}\sin(n\varphi'/\tilde{\gamma})\cos(m\{\varphi-\varphi'\})\,d\varphi' = -\frac{n\tilde{\gamma}\Big\{\cos(m\varphi)-\cos(m\{\varphi+2\pi\tilde{\gamma}\})\Big\}}{n^2-m^2\tilde{\gamma}^2}. \quad (35)$$

Standard sine and cosine addition formulae can now be invoked so as to further exhibit Eqs. (34)–(35) as linear combinations of pure $\sin(m\varphi)$, $\cos(m\varphi)$ terms. For *bona fide* wedges, $0<\gamma<2\pi$ and hence $0<\tilde{\gamma}<1$, delicate l'Hôpital limits intrude whenever $\tilde{\gamma}$ becomes rational. But when the wedge degenerates into a complete cylinder, as is later examined in Appendix C, where $\gamma\to 0+$, $\tilde{\gamma}\to 1-$, their limits are best exhibited from the perspective of full-range trigonometric quadratures on the left which give, respectively, $\pi(1+\delta_0^m)\delta_n^m\cos(m\varphi)$ and $\pi(1+\delta_0^m)\delta_n^m\sin(m\varphi)$, factor $(1+\delta_0^m)$ being a *sine qua non* in the case of (34), but a mere formality in that of Eq. (35).

Altogether Eq. (30) now becomes

$$E_{\text{scatt}}(r,\varphi) = \frac{i\tilde{\gamma}(k_2^2-k_1^2)}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{2-\delta_0^m}{n^2-m^2\tilde{\gamma}^2}\right)\times$$
$$\times\left(m\tilde{\gamma}A_n\left\{\sin(m\varphi)-\sin(m\{\varphi+2\pi\tilde{\gamma}\})\right\}-nB_n\left\{\cos(m\varphi)-\cos(m\{\varphi+2\pi\tilde{\gamma}\})\right\}\right)\times$$
$$\times\int_0^a r' J_m(k_1 r_<)H_m^{(1)}(k_1 r_>)J_{n/\tilde{\gamma}}(k_2 r')dr', \tag{36}$$

something which, when specialized to $r > a$, reads

$$E_{\text{scatt}}(r,\varphi) = \frac{i\tilde{\gamma}(k_2^2-k_1^2)}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{2-\delta_0^m}{n^2-m^2\tilde{\gamma}^2}\right)H_m^{(1)}(k_1 r)\int_0^a r' J_m(k_1 r')J_{n/\tilde{\gamma}}(k_2 r')dr'\times$$
$$\times\left(m\tilde{\gamma}A_n\left\{\sin(m\varphi)-\sin(m\{\varphi+2\pi\tilde{\gamma}\})\right\}-nB_n\left\{\cos(m\varphi)-\cos(m\{\varphi+2\pi\tilde{\gamma}\})\right\}\right), \tag{37}$$

while, when instead $0 \le r \le a$, appears in the more complicated form

$$E_{\text{scatt}}(r,\varphi) = \frac{i\tilde{\gamma}(k_2^2-k_1^2)}{4}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{2-\delta_0^m}{n^2-m^2\tilde{\gamma}^2}\right)\left(H_m^{(1)}(k_1 r)\int_0^r r' J_m(k_1 r')J_{n/\tilde{\gamma}}(k_2 r')dr'+\right.$$
$$\left.+J_m(k_1 r)\int_r^a r' H_m^{(1)}(k_1 r')J_{n/\tilde{\gamma}}(k_2 r')dr'\right)\times$$
$$\times\left(m\tilde{\gamma}A_n\left\{\sin(m\varphi)-\sin(m\{\varphi+2\pi\tilde{\gamma}\})\right\}-nB_n\left\{\cos(m\varphi)-\cos(m\{\varphi+2\pi\tilde{\gamma}\})\right\}\right). \tag{38}$$

Angular ranges for φ, even though temporarily declared as $-2\pi < \varphi < 0$ and $-2\pi < \varphi < -2\pi\tilde{\gamma}$ respectively for Eqs. (37) and (38), can naturally revert, by virtue of their inherent 2π periodicity, respectively to $0 < \varphi < 2\pi$ and $0 < \varphi < \gamma$. The nondiagonally indexed quadratures over cylinder function pairs in Eqs. (37)–(38) resist any easy closed-form evaluation, and will require either outright numerical integration, or, at the very least, series multiplication prior to integration term-by-term.

6. PARTING COMMENTS

It had naturally been our fervent hope to witness a full recovery of the Macdonald/Sommerfeld infinitely conducting wedge results obtained from Eq. (38) in the limit as $a \to \infty$ and $\sigma_2 \to \infty$.[12] Such a demonstration, alas, awaits another day. On the other hand, when $\gamma \to 0+$, and thus $\tilde{\gamma} \to 1-$, the wedge closes upon itself to form a complete dielectric cylinder whose known scattering solution is gotten along quite elementary boundary matching lines. This particular limit, by happy contrast, we are able to recover, as is demonstrated in Appendix C. Such limited agreement will have to serve presently by way of validating the analysis herein evolved.

APPENDIX A. INCOMING FIELD SERIES DECOMPOSITION

When seeking to employ the functions

$$\left(J_{n/\tilde{\gamma}}(k_1 r)\left\{\cos(n\varphi/\tilde{\gamma}),\,\sin(n\varphi/\tilde{\gamma})\right\}\right)_{n=0}^{\infty}$$

as an expansion basis, it is useful, albeit not entirely necessary, to be assured of member by member orthogonality with respect to the inner product of Eq. (15). Such orthogonality is most easily established,

[12] In particular, one would have to show that interior field (16), gauging a component parallel to wedge faces, is more and more confined to thin veneers adjacent to wedge faces $\varphi = -2\pi\tilde{\gamma}+$ and $\varphi = 0-$ as $\sigma_2 \to \infty$, its value tending toward zero in such a way that $\sqrt{\sigma_2} \times E$ remains finite [14, Eqs. (31b), (40), (41)].

and rests squarely upon the shoulders of the trigonometric components, as the following calculations are quick to show:

$$\begin{aligned}\int_{-2\pi\tilde{\gamma}}^{0}\cos(m\varphi/\tilde{\gamma})\sin(n\varphi/\tilde{\gamma})d\varphi &= \frac{1}{2}\int_{-2\pi\tilde{\gamma}}^{0}\Big[\sin\big(\{m+n\}\varphi/\tilde{\gamma}\big)-\sin\big(\{m-n\}\varphi/\tilde{\gamma}\big)\Big]d\varphi \\ &= 0\end{aligned} \tag{39}$$

$$\begin{aligned}\int_{-2\pi\tilde{\gamma}}^{0}\cos(m\varphi/\tilde{\gamma})\cos(n\varphi/\tilde{\gamma})d\varphi &= \frac{1}{2}\int_{-2\pi\tilde{\gamma}}^{0}\Big[\cos\big(\{m+n\}\varphi/\tilde{\gamma}\big)+\cos\big(\{m-n\}\varphi/\tilde{\gamma}\big)\Big]d\varphi \\ &= \begin{cases} 0 & \text{if} \quad m \neq n \\ \pi\tilde{\gamma}\Big(1+\delta_0^m\Big) & \text{if} \quad m = n\end{cases}\end{aligned} \tag{40}$$

$$\begin{aligned}\int_{-2\pi\tilde{\gamma}}^{0}\sin(m\varphi/\tilde{\gamma})\sin(n\varphi/\tilde{\gamma})d\varphi &= \frac{1}{2}\int_{-2\pi\tilde{\gamma}}^{0}\Big[\cos\big(\{m-n\}\varphi/\tilde{\gamma}\big)-\cos\big(\{m+n\}\varphi/\tilde{\gamma}\big)\Big]d\varphi \\ &= \begin{cases} 0 & \text{if} \quad m \neq n \\ \pi\tilde{\gamma}\Big(1-\delta_0^m\Big) & \text{if} \quad m = n\end{cases}\end{aligned} \tag{41}$$

So fortified, we set down

$$P = \int_0^a rdr\int_{-2\pi\tilde{\gamma}}^{0}\left|e^{-ik_1r\cos(\varphi-\varphi_0)} - \sum_{n=0}^{\infty} J_{n/\tilde{\gamma}}(k_1r)\Big\{C_n\cos(n\varphi/\tilde{\gamma}) + D_n\sin(n\varphi/\tilde{\gamma})\Big\}\right|^2 d\varphi \tag{42}$$

as a mismatch penalty associated with representation of Eq. (17) and proceed to search for coefficients C_n, D_n which minimize it. We write $C_n = |C_n|\exp i\vartheta_n$, $D_n = |D_n|\exp i\theta_n$, and thus, at each series index $n \geq 1$, have control over four discretionary parameters with respect to which first derivatives of P can be annulled.[13] Now

$$\begin{aligned}P = \pi\tilde{\gamma}a^2 + \pi\tilde{\gamma}\sum_{n=0}^{\infty}\left\{(1+\delta_0^n)\,|C_n|^2 + (1-\delta_0^n)\,|D_n|^2\right\}\left(\int_0^a rJ^2_{n/\tilde{\gamma}}(k_1r)dr\right) - \\ -2\,\Re\sum_{n=0}^{\infty}\Big\{C_n\int_0^a rJ_{n/\tilde{\gamma}}(k_1r)dr\int_{-2\pi\tilde{\gamma}}^{0}e^{ik_1r\cos(\varphi-\varphi_0)}\cos(n\varphi/\tilde{\gamma})d\varphi + \\ +D_n\int_0^a rJ_{n/\tilde{\gamma}}(k_1r)dr\int_{-2\pi\tilde{\gamma}}^{0}e^{ik_1r\cos(\varphi-\varphi_0)}\sin(n\varphi/\tilde{\gamma})d\varphi\Big\}\end{aligned} \tag{43}$$

and so setting the derivative of P with respect to $|C_n|$ equal to zero gives

$$N_n^2\Big(1+\delta_0^n\Big)\,|C_n| = \Re\left\{e^{i\vartheta_n}\int_0^a rJ_{n/\tilde{\gamma}}(k_1r)dr\int_{-2\pi\tilde{\gamma}}^{0}e^{ik_1r\cos(\varphi-\varphi_0)}\cos(n\varphi/\tilde{\gamma})d\varphi\right\}, \tag{44}$$

whereas that with respect to ϑ_n simply reads

$$\Re\left\{ie^{i\vartheta_n}\int_0^a rJ_{n/\tilde{\gamma}}(k_1r)dr\int_{-2\pi\tilde{\gamma}}^{0}e^{ik_1r\cos(\varphi-\varphi_0)}\cos(n\varphi/\tilde{\gamma})d\varphi\right\} = 0\,. \tag{45}$$

With a view to Eq. (43), we have further introduced in Eq. (44) the abbreviation

$$N_n = \sqrt{\pi\tilde{\gamma}\int_0^a rJ^2_{n/\tilde{\gamma}}(k_1r)dr} \tag{46}$$

[13] It is clear *a priori* that D_0 is irrelevant, so that the parameter quartet is reduced to just two when $n = 0$.

for the divisor N_n required to normalize the associated function basis member. Taken together, Eqs. (44)–(45) then imply that

$$C_n = \left(1+\delta_0^n\right)^{-1} N_n^{-2} \int_0^a r J_{n/\tilde{\gamma}}(k_1 r) dr \int_{-2\pi\tilde{\gamma}}^{0} e^{-ik_1 r\cos(\varphi-\varphi_0)} \cos(n\varphi/\tilde{\gamma}) d\varphi \tag{47}$$

whereas identical reasoning, whenever $n > 0$, leads one to

$$D_n = N_n^{-2} \int_0^a r J_{n/\tilde{\gamma}}(k_1 r) dr \int_{-2\pi\tilde{\gamma}}^{0} e^{-ik_1 r\cos(\varphi-\varphi_0)} \sin(n\varphi/\tilde{\gamma}) d\varphi\,. \tag{48}$$

We face now the task of reducing the several quadratures encountered in Eqs. (46)–(48). That in Eq. (46), associated with the name of Lommel, is well known [15] and can be presented in closed form as

$$\begin{aligned} N_n^2 &= \pi\tilde{\gamma} \int_0^a r J_{n/\tilde{\gamma}}^2(k_1 r) dr \\ &= \frac{\pi\tilde{\gamma}}{k_1^2} \int_0^{ka} \zeta J_{n/\tilde{\gamma}}^2(\zeta) d\zeta \\ &= \frac{\pi\tilde{\gamma}}{2k_1^2} \left\{ \left((k_1 a)^2 - (n/\tilde{\gamma})^2\right) J_{n/\tilde{\gamma}}^2(k_1 a) + \left(k_1 a J'_{n/\tilde{\gamma}}(k_1 a)\right)^2 \right\} \\ &= \frac{\pi\tilde{\gamma}a^2}{2} \left\{ J_{n/\tilde{\gamma}}^2(k_1 a) - J_{n/\tilde{\gamma}-1}(k_1 a) J_{n/\tilde{\gamma}+1}(k_1 a) \right\} \\ &= \frac{\pi\tilde{\gamma}a^2}{2} \left\{ J_{n/\tilde{\gamma}-1}^2(k_1 a) - 2(n/\tilde{\gamma}k_1 a) J_{n/\tilde{\gamma}-1}(k_1 a) J_{n/\tilde{\gamma}}(k_1 a) + J_{n/\tilde{\gamma}}^2(k_1 a) \right\}, \end{aligned} \tag{49}$$

the fourth and fifth lines of which follow from the third on the strength of standard Bessel function recurrences.

More cumbersome by far are the integrals on the right in Eqs. (47)–(48), resistant to everything but a piecemeal treatment, treatment which begins by setting

$$\begin{aligned} N_n^2 \left(1+\delta_0^n\right) C_n &= \int_0^a r J_{n/\tilde{\gamma}}(k_1 r) dr \int_{-2\pi\tilde{\gamma}}^{0} e^{-ik_1 r\cos(\varphi-\varphi_0)} \cos(n\varphi/\tilde{\gamma}) d\varphi \\ &= \sum_{m=0}^{\infty} \frac{(-ik_1)^m}{m!} \int_0^a r^{m+1} J_{n/\tilde{\gamma}}(k_1 r) dr \int_{-2\pi\tilde{\gamma}}^{0} \cos^m(\varphi-\varphi_0) \cos(n\varphi/\tilde{\gamma}) d\varphi \end{aligned} \tag{50}$$

and

$$\begin{aligned} N_n^2 D_n &= \int_0^a r J_{n/\tilde{\gamma}}(k_1 r) dr \int_{-2\pi\tilde{\gamma}}^{0} e^{-ik_1 r\cos(\varphi-\varphi_0)} \sin(n\varphi/\tilde{\gamma}) d\varphi \\ &= \sum_{m=0}^{\infty} \frac{(-ik_1)^m}{m!} \int_0^a r^{m+1} J_{n/\tilde{\gamma}}(k_1 r) dr \int_{-2\pi\tilde{\gamma}}^{0} \cos^m(\varphi-\varphi_0) \sin(n\varphi/\tilde{\gamma}) d\varphi\,, \end{aligned} \tag{51}$$

which are usefully united by writing, for want of any better notation,

$$\begin{aligned} W_{m,n} &= \int_{-2\pi\tilde{\gamma}}^{0} \cos^m(\varphi-\varphi_0) e^{in\varphi/\tilde{\gamma}} d\varphi \\ &= \frac{1}{2^m} \sum_{s=0}^{m} \binom{m}{s} e^{-i(m-2s)\varphi_0} \int_{-2\pi\tilde{\gamma}}^{0} e^{i(m-2s+n/\tilde{\gamma})\varphi} d\varphi \\ &= -\frac{i\tilde{\gamma}}{2^m} \sum_{s=0}^{m} \binom{m}{s} e^{-i(m-2s)\varphi_0} \left(\frac{1-e^{-2\pi i\tilde{\gamma}(m-2s)}}{n+\tilde{\gamma}(m-2s)} \right) \end{aligned} \tag{52}$$

so that

$$\int_{-2\pi\tilde{\gamma}}^{0} \cos^m(\varphi-\varphi_0)\cos(n\varphi/\tilde{\gamma})d\varphi = \Re\left\{W_{m,n}\right\} \tag{53}$$
$$= -\frac{\tilde{\gamma}}{2^m}\sum_{s=0}^{m}\binom{m}{s}\left(\frac{\sin(\{m-2s\}\varphi_0)-\sin(\{m-2s\}\{\varphi_0+2\pi\tilde{\gamma}\})}{n+\tilde{\gamma}(m-2s)}\right)$$

and

$$\int_{-2\pi\tilde{\gamma}}^{0} \cos^m(\varphi-\varphi_0)\sin(n\varphi/\tilde{\gamma})d\varphi = \Im\left\{W_{m,n}\right\} \tag{54}$$
$$= -\frac{\tilde{\gamma}}{2^m}\sum_{s=0}^{m}\binom{m}{s}\left(\frac{\cos(\{m-2s\}\varphi_0)-\cos(\{m-2s\}\{\varphi_0+2\pi\tilde{\gamma}\})}{n+\tilde{\gamma}(m-2s)}\right).$$

But, as regards the shared radial integrals in Eqs. (50)–(51), there seems to be no facile option available save for a plodding term-by-term series quadrature. Thus

$$\int_0^a r^{m+1}J_{n/\tilde{\gamma}}(k_1r)dr = \frac{1}{k_1^{m+2}}\int_0^{k_1a}\zeta^{m+1}J_{n/\tilde{\gamma}}(\zeta)d\zeta$$
$$= \frac{2^{m+1}}{k_1^{m+2}}\int_0^{k_1a}\left\{(\zeta/2)^{m+n/\tilde{\gamma}+1}\sum_{s=0}^{\infty}\frac{(-1)^s\left(\zeta/2\right)^{2s}}{s!\,\Gamma\left(n/\tilde{\gamma}+s+1\right)}\right\}d\zeta \tag{55}$$
$$= \left(\frac{2}{k_1}\right)^{m+2}\left(\frac{k_1a}{2}\right)^{m+n/\tilde{\gamma}+2}\sum_{s=0}^{\infty}\frac{(-1)^s}{s!\,(m+n/\tilde{\gamma}+2s+2)\,\Gamma\left(/\tilde{\gamma}+s+1\right)}\left(\frac{k_1a}{2}\right)^{2s}.$$

We must hence be content with a jigsaw, mosaic assembly of coefficients C_n and D_n as

$$C_n = \left(1+\delta_0^n\right)^{-1}N_n^{-2}\sum_{m=0}^{\infty}\frac{(-ik_1)^m}{m!}\Re\left\{W_{m,n}\right\}\int_0^a r^{m+1}J_{n/\tilde{\gamma}}(k_1r)dr \tag{56}$$

and, when $n>0$,

$$D_n = N_n^{-2}\sum_{m=0}^{\infty}\frac{(-ik_1)^m}{m!}\Im\left\{W_{m,n}\right\}\int_0^a r^{m+1}J_{n/\tilde{\gamma}}(k_1r)dr \tag{57}$$

via a path threading itself across Eqs. (53)–(55).

APPENDIX B. GREEN'S FUNCTION SERIES DECOMPOSITION

We seek to imitate the standard, full-range development (28)

$$H_0^{(1)}(k_1|\boldsymbol{r}-\boldsymbol{r'}|) = \sum_{m=0}^{\infty}(2-\delta_0^m)\,J_m(k_1r_<)H_m^{(1)}(k_1r_>)\cos(m\{\varphi-\varphi'\}) \tag{58}$$

by writing, in one fell swoop,

$$H_0^{(1)}(k_1|\boldsymbol{r}-\boldsymbol{r'}|) = \alpha\sum_{m=0}^{\infty}(2-\delta_0^m)\,J_{m/\tilde{\gamma}}(k_1r_<)H_{m/\tilde{\gamma}}^{(1)}(k_1r_>)\cos(m\{\varphi-\varphi'\}/\tilde{\gamma}), \tag{59}$$

with both φ and φ' confined to the wedge slot, $-2\pi\tilde{\gamma}\le\varphi,\varphi'\le 0$, and coefficient α yet to be determined. Such determination is arrived at by noting that the radial derivative with respect to r of series (59) exhibits a jump discontinuity Δ at the transition point $r=r'$ in an amount

$$\Delta=\alpha k_1\sum_{m=0}^{\infty}(2-\delta_0^m)\left\{J_{m/\tilde{\gamma}}(*)H_{m/\tilde{\gamma}}^{(1)'}(*)\Big|_{*=k_1r'+} - J'_{m/\tilde{\gamma}}(*)H_{m/\tilde{\gamma}}^{(1)}(*)\Big|_{*=k_1r'-}\right\}\cos(m\{\varphi-\varphi'\}/\tilde{\gamma}). \tag{60}$$

But now the Wronskian connection in Eq. (25) once more informs us that

$$J_{m/\tilde{\gamma}}(*)H^{(1)'}_{m/\tilde{\gamma}}(*)\Big|_{*=k_1r'+} - J'_{m/\tilde{\gamma}}(*)H^{(1)}_{m/\tilde{\gamma}}(*)\Big|_{*=k_1r'-} = \frac{2i}{\pi k_1 r'} \tag{61}$$

whereas

$$\begin{aligned}\sum_{m=0}^{\infty}(2-\delta_0^m)\cos(m\{\varphi-\varphi'\}/\tilde{\gamma}) &= \sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')/\tilde{\gamma}} \\ &= 2\pi\sum_{m=-\infty}^{\infty}\delta(\{\varphi-\varphi'\}/\tilde{\gamma}-2m\pi) \\ &= 2\pi\tilde{\gamma}\sum_{m=-\infty}^{\infty}\delta(\varphi-\varphi'-2m\pi\tilde{\gamma})\,.\end{aligned} \tag{62}$$

Since, in its application to series (59) of the Helmholtz operator $(\nabla^2+k_1^2)$ one further radial derivative with respect to r is still required, we see that all necessary ingredients are in place to recover Eq. (12) in the form

$$\left(\nabla^2+k_1^2\right)H_0^{(1)}(k_1|\boldsymbol{r}-\boldsymbol{r'}|) = \frac{4i}{r'}\delta(r-r')\delta(\varphi-\varphi') \tag{63}$$

provided that one sets $\alpha = 1/\tilde{\gamma}$. It surely needs no repeating that

$$\frac{1}{r'}\delta(r-r')\delta(\varphi-\varphi') = \delta(\mathbf{r}-\mathbf{r}') \tag{64}$$

since clearly

$$\int r dr \int \frac{1}{r'}\delta(r-r')\delta(\varphi-\varphi')d\varphi = 1 \tag{65}$$

when the quadrature extends over any domain overlapping radius r' at angle φ'. The auxiliary development in Eq. (18) has thus been vindicated. One should perhaps mention in passing that fractional index decompositions of this sort are taken in easy stride in [16], wherein one finds a leisurely treatment of knife edge scattering.

APPENDIX C. RECONCILIATION WITH DIELECTRIC CYLINDER SCATTERING

C.1. Classical Boundary Value Matching

When $\gamma \to 0+$, $\tilde{\gamma} \to 1-$, the wedge defaults into a perfect cylinder which scatters in accordance with elementary formulae based upon interface continuity of tangential electric $E(a,\varphi)$ (axial, directed along $\hat{\mathbf{e}}_z$) and magnetic $B(a,\varphi)$ (azimuthal, directed along $\hat{\mathbf{e}}_\varphi$; there is also a radial magnetic companion, which need not interest us at this point) components. As is well known [13, p. 374],

$$\begin{aligned}E_{\text{inc}}(r,\varphi) &= e^{-ik_1 r\cos(\varphi-\varphi_0)} \\ &= \sum_{n=0}^{\infty}(2-\delta_0^n)i^{-n}J_n(k_1r)\cos(n\{\varphi-\varphi_0\}) \\ &= \sum_{n=0}^{\infty}(2-\delta_0^n)i^{-n}J_n(k_1r)\left\{\cos(n\varphi_0)\cos(n\varphi)+\sin(n\varphi_0)\sin(n\varphi)\right\}\,,\end{aligned} \tag{66}$$

with an azimuthal magnetic partner

$$B_{\text{inc}}(r,\varphi) = \frac{ik_1}{\omega}\sum_{n=0}^{\infty}(2-\delta_0^n)i^{-n}J'_n(k_1r)\left\{\cos(n\varphi_0)\cos(n\varphi)+\sin(n\varphi_0)\sin(n\varphi)\right\}\,. \tag{67}$$

Additional coefficient arrays $\Big\{F_n\Big\}_{n=0}^{\infty}$ and $\Big\{G_n\Big\}_{n=0}^{\infty}$ are required to account for the reradiated or, as one says, scattered electric field

$$E_{\text{scatt}}(r,\varphi) \;=\; \sum_{n=0}^{\infty} H_n^{(1)}(k_1 r)\Big\{F_n\cos(n\varphi)+G_n\sin(n\varphi)\Big\}\;, \tag{68}$$

which must be adjoined to Eq. (66) so as to assemble the total electric field, incident plus scattered,

$$\begin{aligned} E(r,\varphi) \;=\; & \sum_{n=0}^{\infty}(2-\delta_0^n)i^{-n}J_n(k_1 r)\Big\{\cos(n\varphi_0)\cos(n\varphi)+\sin(n\varphi_0)\sin(n\varphi)\Big\}+ \\ & +\sum_{n=0}^{\infty} H_n^{(1)}(k_1 r)\Big\{F_n\cos(n\varphi)+G_n\sin(n\varphi)\Big\}\;, \end{aligned} \tag{69}$$

and a similarly total azimuthal magnetic field

$$\begin{aligned} B(r,\varphi) \;=\; & \frac{ik_1}{\omega}\sum_{n=0}^{\infty}(2-\delta_0^n)i^{-n}J_n'(k_1 r)\Big\{\cos(n\varphi_0)\cos(n\varphi)+\sin(n\varphi_0)\sin(n\varphi)\Big\}+ \\ & +\frac{ik_1}{\omega}\sum_{n=0}^{\infty} H_n^{(1)'}(k_1 r)\Big\{F_n\cos(n\varphi)+G_n\sin(n\varphi)\Big\} \end{aligned} \tag{70}$$

on the cylinder exterior $r > a$.

Interior to the cylinder, $0 \le r \le a$, we likewise have, from Eq. (16),

$$E(r,\varphi) \;=\; \sum_{n=0}^{\infty} J_n(k_2 r)\Big\{A_n\cos(n\varphi)+B_n\sin(n\varphi)\Big\} \tag{71}$$

and

$$B(r,\varphi) \;=\; \frac{ik_2}{\omega}\sum_{n=0}^{\infty} J_n'(k_2 r)\Big\{A_n\cos(n\varphi)+B_n\sin(n\varphi)\Big\}\;. \tag{72}$$

On taking into account the natural orthogonality of the trigonometric functions $\cos(m\varphi)$, $\sin(m\varphi)$ on the full angular interval $0 \le \varphi < 2\pi$, the tangential field continuity at cylinder boundary $r = a$ emerges as four linear equations

$$\left.\begin{aligned} A_n J_n(k_2 a) \;&=\; (2-\delta_0^n)i^{-n}J_n(k_1 a)\cos(n\varphi_0)+F_n H_n^{(1)}(k_1 a) \\ B_n J_n(k_2 a) \;&=\; (2-\delta_0^n)i^{-n}J_n(k_1 a)\sin(n\varphi_0)+G_n H_n^{(1)}(k_1 a) \\ k_2 A_n J_n'(k_2 a) \;&=\; k_1(2-\delta_0^n)i^{-n}J_n'(k_1 a)\cos(n\varphi_0)+k_1 F_n H_n^{(1)'}(k_1 a) \\ k_2 B_n J_n'(k_2 a) \;&=\; k_1(2-\delta_0^n)i^{-n}J_n'(k_1 a)\sin(n\varphi_0)+k_1 G_n H_n^{(1)'}(k_1 a) \end{aligned}\right\} \tag{73}$$

mode by mode. This quartet clearly decouples into two 2×2 linear systems respectively for A_n, F_n and B_n, G_n having

$$A_n \;=\; k_1(2-\delta_0^n)i^{-n}\cos(n\varphi_0)\left(\frac{J_n(k_1 a)H_n^{(1)'}(k_1 a)-J_n'(k_1 a)H_n^{(1)}(k_1 a)}{k_1 J_n(k_2 a)H_n^{(1)'}(k_1 a)-k_2 J_n'(k_2 a)H_n^{(1)}(k_1 a)}\right) \tag{74}$$

$$F_n \;=\; (2-\delta_0^n)i^{-n}\cos(n\varphi_0)\left(\frac{k_2 J_n'(k_2 a)J_n(k_1 a)-k_1 J_n(k_2 a)J_n'(k_1 a)}{k_1 J_n(k_2 a)H_n^{(1)'}(k_1 a)-k_2 J_n'(k_2 a)H_n^{(1)}(k_1 a)}\right) \tag{75}$$

$$B_n = k_1(2-\delta_0^n)i^{-n}\sin(n\varphi_0)\left(\frac{J_n(k_1a)H_n^{(1)'}(k_1a)-J_n'(k_1a)H_n^{(1)}(k_1a)}{k_1J_n(k_2a)H_n^{(1)'}(k_1a)-k_2J_n'(k_2a)H_n^{(1)}(k_1a)}\right) \tag{76}$$

$$G_n = (2-\delta_0^n)i^{-n}\sin(n\varphi_0)\left(\frac{k_2J_n'(k_2a)J_n(k_1a)-k_1J_n(k_2a)J_n'(k_1a)}{k_1J_n(k_2a)H_n^{(1)'}(k_1a)-k_2J_n'(k_2a)H_n^{(1)}(k_1a)}\right) \tag{77}$$

as their composite solution. Equations (74) and (76) are then still further condensed into

$$A_n = -\frac{2(-i)^{n+1}(2-\delta_0^n)\cos(n\varphi_0)}{\pi a\left(k_1J_n(k_2a)H_n^{(1)'}(k_1a)-k_2J_n'(k_2a)H_n^{(1)}(k_1a)\right)} \tag{78}$$

and

$$B_n = -\frac{2(-i)^{n+1}(2-\delta_0^n)\sin(n\varphi_0)}{\pi a\left(k_1J_n(k_2a)H_n^{(1)'}(k_1a)-k_2J_n'(k_2a)H_n^{(1)}(k_1a)\right)} \tag{79}$$

on appeal to Wronskian in Eq. (61).

C.2. Self-consistent Reradiated Field Solution

In the limit indicated, our corresponding analysis from Appendix A simplifies enormously. Fixing attention on Eqs. (47)–(48), we note that

$$\begin{aligned}\int_{-2\pi}^{0} e^{-ik_1r\cos(\varphi-\varphi_0)}\cos(n\varphi)d\varphi &= \int_0^{2\pi} e^{-ik_1r\cos(\varphi)}\Big\{\cos(n\varphi_0)\cos(n\varphi)-\sin(n\varphi_0)\sin(n\varphi)\Big\}d\varphi \\ &= \cos(n\varphi_0)\int_0^{2\pi} e^{-ik_1r\cos(\varphi)}\cos(n\varphi)d\varphi \\ &= 2\pi(-i)^n\cos(n\varphi_0)J_n(k_1r)\end{aligned} \tag{80}$$

by virtue of a well known Bessel function identity [17, p. 360]. In similar fashion,

$$\begin{aligned}\int_{-2\pi}^{0} e^{-ik_1r\cos(\varphi-\varphi_0)}\sin(n\varphi)d\varphi &= \int_0^{2\pi} e^{-ik_1r\cos(\varphi)}\Big\{\sin(n\varphi_0)\cos(n\varphi)+\cos(n\varphi_0)\sin(n\varphi)\Big\}d\varphi \\ &= \sin(n\varphi_0)\int_0^{2\pi} e^{-ik_1r\cos(\varphi)}\cos(n\varphi)d\varphi \\ &= 2\pi(-i)^n\sin(n\varphi_0)J_n(k_1r)\,.\end{aligned} \tag{81}$$

Equations (50)–(51), on the basis of their first lines alone, thus become

$$N_n^2C_n = 2(-i)^n\left(1+\delta_0^n\right)^{-1}N_n^2\cos(n\varphi_0) \tag{82}$$

and

$$N_n^2D_n = 2(-i)^nN_n^2\sin(n\varphi_0)\,, \tag{83}$$

with normalizer N_n cancelling across the board, Eq. (83) clearly destined to be utilized only when $n > 0$,[14] and since, as already remarked, $(2-\delta_0^n)(1+\delta_0^n) = 2$, Eq. (82) more usefully written as

$$C_n = (-i)^n\left(2-\delta_0^n\right)\cos(n\varphi_0)\,. \tag{84}$$

Inserting Eq. (84) into the first line of Eq. (27) reproduces Eq. (78) exactly; inserting Eq. (83) into its second similarly recovers Eq. (79), such recovery being of course purely formal when $n = 0$.

[14] Since coefficients B_n, D_n, and G_n accompany a sine having an argument proportional to n, it is self-evident that they are operative only when $n > 0$. Although repeating this over and over again has no doubt belabored the point, so be it.

On turning finally to the scattered field (37)–(38), we note that Eq. (38) is moot, whereas Eq. (37) undergoes index diagonalization and a common mode scaling in accordance with the remarks following Eqs. (34)–(35). Thus we get, as the counterpart of Eq. (37),

$$\begin{aligned} E_{\text{scatt}}(r,\varphi) \;=\; & \frac{i\pi(k_2^2-k_1^2)}{4}\sum_{n=0}^{\infty}(2-\delta_0^n)\,(1+\delta_0^n)\,H_n^{(1)}(k_1r)\int_0^a r'J_n(k_2r')J_n(k_1r')dr' \,\times \\ & \times\left(A_n\cos(n\varphi)+B_n\sin(n\varphi)\right)\,. \end{aligned} \tag{85}$$

After the pattern of Eq. (21) we further find

$$\int_0^a r'J_n(k_2r')J_n(k_1r')dr' \;=\; -a\left(\frac{k_2J_n'(k_2a)J_n(k_1a)-k_1J_n(k_2a)J_n'(k_1a)}{k_2^2-k_1^2}\right) \tag{86}$$

at which point an appeal to Eqs. (78)–(79), their compatibility with both solution methods having now been acknowledged, gives the highly symmetric form

$$\begin{aligned} E_{\text{scatt}}(r,\varphi) \;=\; & \sum_{n=0}^{\infty} i^{-n}\left(2-\delta_0^n\right)\left(\frac{k_2J_n'(k_2a)J_n(k_1a)-k_1J_n(k_2a)J_n'(k_1a)}{k_1J_n(k_2a)H_n^{(1)'}(k_1a)-k_2J_n'(k_2a)H_n^{(1)}(k_1a)}\right)\times \\ & \times H_n^{(1)}(k_1r)\left(\cos(n\varphi_0)\cos(n\varphi)+\sin(n\varphi_0)\sin(n\varphi)\right) \end{aligned} \tag{87}$$

fully compliant with Eqs. (68), (75), and (77). Our self-consistent formalism has thus been shown to fly by with full colors unfurled, at least in this special, cylindrical case. Form Eq. (87) conforms moreover to the obvious demand that the scattering abate entirely as the cylinder blends into the background medium, $k_2 \to k_1$.

The opposite, vacuous limit, $\gamma \to 2\pi-$, $\tilde{\gamma} \to 0+$, wherein the wedge simply evaporates, generates throughout integrals that tend toward zero by virtue of a vanishing range $-2\pi\tilde{\gamma} \le \varphi \le 0$, and are then still further weighed down by the circumstance that Bessel functions $J_{n/\tilde{\gamma}}(k_1r)$, at least for real arguments k_1r, exhibit, as indices $n/\tilde{\gamma} \to \infty$, ever longer *"carpets"* of vanishingly small values preceding the onset of their first, nonneglible peaks. An apropos asymptotic estimate is found in [17, p. 365].

APPENDIX D. AN ALTERNATIVE, UNSUITABLE SERIES DEVELOPMENT

It should be stressed that series developments in Eqs. (16)–(18), while they may appear to be Fourier series, such they are not, and this despite the fact of their manifest periodicity over angular slots of width $2\pi\tilde{\gamma}$, none of which, save for the wedge interval *per se*, are of any relevance to our work. Their members were all chosen to vanish under application of the Helmholtz operator appropriate to the medium at hand, and their coefficients C_n and D_n were fixed on the basis of a global minimization of the mismatch penalty P from Eqs. (42)–(43), only obliquely attentive to a demand for interior point convergence to the incoming field as specified, and a concurrent convergence to average value/average slope of this field at wedge boundaries.

We can suggest this divergence in viewpoint by setting down

$$\sum_{n=0}^{\infty}\left\{\tilde{C}_n(k_1r)\cos(n\varphi/\tilde{\gamma})+\tilde{D}_n(k_1r)\sin(n\varphi/\tilde{\gamma})\right\} \;=\; e^{-ik_1r\cos(\varphi-\varphi_0)} \tag{88}$$

as a Fourier series at each radius $0 \le r \le a$, so that

$$\tilde{C}_n(k_1r) \;=\; \left\{\pi\tilde{\gamma}\left(1+\delta_0^n\right)\right\}^{-1}\int_{-2\pi\tilde{\gamma}}^{0} e^{-ik_1r\cos(\varphi-\varphi_0)}\cos(n\varphi/\tilde{\gamma})d\varphi \tag{89}$$

and, when $n \ge 1$,

$$\tilde{D}_n(k_1r) \;=\; \left\{\pi\tilde{\gamma}\right\}^{-1}\int_{-2\pi\tilde{\gamma}}^{0} e^{-ik_1r\cos(\varphi-\varphi_0)}\sin(n\varphi/\tilde{\gamma})d\varphi\,. \tag{90}$$

Now, as before in Eqs. (50)–(51), we could seek open forms for each one of $\tilde{C}_n(k_1 r)$ and $\tilde{D}_n(k_1 r)$, but it is presently more illuminating to subject them to the Bessel operator

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + k_1^2 - \frac{n^2}{\tilde{\gamma}^2 r^2}\right).$$

And so, via a painstaking process involving a multitude of routine integrations by parts, we duly arrive at

$$\begin{aligned}&\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + k_1^2 - \frac{n^2}{\tilde{\gamma}^2 r^2}\right)\tilde{C}_n(k_1 r)\\ &= \left(2-\delta_0^n\right)\frac{ik_1}{2\pi\tilde{\gamma}r}\left(\sin(\varphi_0)e^{-ik_1 r\cos(\varphi_0)} + \sin(\gamma-\varphi_0)e^{-ik_1 r\cos(\gamma-\varphi_0)}\right)\end{aligned} \quad (91)$$

and

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + k_1^2 - \frac{n^2}{\tilde{\gamma}^2 r^2}\right)\tilde{D}_n(k_1 r) = \frac{n}{\pi\tilde{\gamma}^2 r^2}\left(e^{-ik_1 r\cos(\varphi_0)} - e^{-ik_1 r\cos(\gamma-\varphi_0)}\right), \quad (92)$$

the right-hand source terms in Eqs. (91)–(92) being proportional to slot boundary mismatch differences respectively in angular derivative and function value. Solutions to Eqs. (91)–(92), composed of both homogeneous and source-driven portions as dictated by a standard variation-of-parameters process, include of course not only an appearance of the desired Bessel function $J_{n/\tilde{\gamma}}(k_1 r)$, but also of its partner $Y_{n/\tilde{\gamma}}(k_1 r)$ and something more built around interlaced function-times-integral of function products, all of it adjusted to assure finiteness at radial origin $r = 0$.

There is of course no point in completing these analytical details here. What is at issue instead is the present demonstration that *ansatz* Eq. (17) and its analogues are not to be viewed through the canonical prism of Fourier series, that their Bessel function multipliers $J_{n/\tilde{\gamma}}(k_1 r)$ and so on, by themselves, are inadequate to provide the necessary amplitudes for a radial continuum of Fourier series, all of them laid down across a common, $-2\pi\tilde{\gamma} \leq \varphi \leq 0$ angular slot. Such Fourier series continua would, conversely, spoil the otherwise automatic compliance of vanishing under application of the relevant Helmholtz operators.

On the other hand, since the right-hand sides of both Eqs. (91) and (92) vanish when $\gamma \to 0+$, functions $\tilde{C}_n(k_1 r)$ and $\tilde{D}_n(k_1 r)$ individually revert to being simply proportional to Bessel's $J_n(k_1 r)$, and so representation Eq. (17), as also Eq. (16), do become genuine Fourier series. It ceases thus to be surprising that our formalism should similarly degenerate so gracefully, that our global fitting should capture so well the scattering features otherwise determined by boundary matching in Appendix C.

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