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Generalized Current Green's Function Formalism for Electromagnetic Radiation by Composite Systems

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Abstract—We provide an explicit geometric generalisation of the antenna current Green's function (ACGF) formalism from the perfect electric conducting (PEC) to generic interacting N-body systems composed of arbitrarily shaped coupled PEC and dielectric objects, with the main emphasis on the mathematical foundations and the rigorous construction of the Green's function using distributional limits. Starting from mainly reciprocity, surface equivalence theorems, and other typical regularity conditions, we carefully construct the current Green's function by employing a combination of methods including Riemannian geometry, distribution theory, and functional analysis. The formalism outlined here for composite domains turns out to be more complicated than the PEC-only formulation due to the former's need to explicitly account for the coupling interaction between the magnetic and electric degrees of freedom. The approach is developed for extremely general systems, and use is made of Riemannian geometry to avoid working with specific or concrete configurations, hence retaining high generality in our final conclusions. While the ACGF tensor's matrix representations depend on the coordinate system on the manifolds supporting the electromagnetic boundary conditions, we focus here on providing coordinate-independent integral expressions for the induced current. With the ACGF it is possible to theoretically treat arbitrary N-body coupled PEC-dielectric configurations as space-frequency linear systems with an exact and rigorous response function being the current Green's function itself. While the derivation is very general, it still leaves open questions regarding whether the ACGF can be constructed for nonreciprocal systems or using volume integral equations.

1. INTRODUCTION

The current Green's function of a material electromagnetic system is defined as the surface current distribution transfer function dictating how an induced current is generated in response to an external electromagnetic field [1]. The concept itself is rooted in earlier circuit representations of antennas as "admittance transfer matrices" (a discrete approximation) [2, 3] and is formally identical to the nonlocal response of low-dimensional materials to external electromagnetic fields, for example, light-matter interactions in nanotubes, graphene, and other novel nano-surfaces [4–6]. In conventional classical electrodynamics, radiation by mico- or macro-sources is often described by the dyadic Green's function of free space, where a given radiating current $\mathbf{J}(\mathbf{r})$ is taken as an input while the electromagnetic fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ produced by this source are the output (the "radiating electromagnetic system.") [7–12]. Following a proposal originally stated by Schelkunoff within the framework of transmission line circuit theory [13] but not rigoursly implemented within exact electromagnetic theory till very recently, it was shown that radiation by electromagnetic systems is much better described as the cascade of two distinct linear electromagnetic processes, Modes A and B, where in Mode A we first induce a current $\mathbf{J}(\mathbf{r})$ by an external *field* excitation $\mathbf{E}^{\text{ex}}(\mathbf{r})$, e.g., the antenna source or the illumination field

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in scattering problems, while in Mode B this induced current is allowed to radiate according to the classical dyadic Green's function. The former process, Mode A, is described by a *new* dyadic Green's function, often denoted by $\overline{\mathbf{F}}(\mathbf{r},\mathbf{r}')$ [14–16]. If the traditional radiation dyadic Green's function is given by $\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')$, then the two dyadic Green's functions $\overline{\mathbf{F}}(\mathbf{r},\mathbf{r}')$ and $\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')$ together constitute a complete space-frequency representation of the electromagnetic radiation problem describing how a macroscopic material structure responds to an external field by producing another field. This formulation is more natural because it allows a uniform treatment of a wider class of electromagnetic problems as input-output systems with the same signal type present at *both* the excitation and response "terminals": i.e., electromagnetic fields [1].

Applications of the ACGF method in engineering spans topics like mutual coupling analysis [17], mutual coupling compensation [18–20], fast EM computation of large system [21–23], alternative methods to compute the near-field response [24], and wireless communications [25–27] (more applications are discussed in [1].) Most past researches have been so far highlighting applications, with considerably less emphasis on the theoretical, physical, and mathematical foundations, especially for extremely generic radiating systems. Consequently, *the present work will be mainly focused on the fundamental theoretical aspects of the generalized ACGF formalism*. Readers interested in numerical and experimental results connected with the applications of ACGF method to PEC or composite antenna systems may wish to consult the references quoted above.[†]

From the viewpoint of mathematical physics, the current Green's functions represent a new species of Green's function fundamentally different from other types like those associated with the heat kennel or the wave equation, though significant similarities also exist in several other aspects. The most important technical difference is that the current Green's function is the Green's function of the inverse of an *integro*-differential — hence *global* — operator, while the traditional Green's functions of mathematical physics are those associated with the inversion of a *differential* — hence *local* — operators. This makes the analysis inherently more complicated and less concrete since no exact analytical expressions in the case of the current Green's function are available and one must resort to a creative — rather nontrivial — combination of analytical and numerical methods.

However, the ultimate mathematical foundations of the antenna current Green's function (ACGF) formalism itself were made available by means of a mixture of functional analysis, distribution theory, and Riemannian geometry [1, 15]. Careful analysis has also revealed recently that because of the global nature of the inverse operator associated with the ACGF, considerations related to causality enters into the picture of spacetime electromagnetic systems [27]. Some additional differences include that the ACGF is essentially a Green's function on a curved Riemannian manifold, while the Green's functions of mathematical physics are typically constructed on flat Euclidean spaces. While an extensive mathematical theory of operators on manifolds and the partial differential equations living on them already exists in the literature, e.g., see [28, 29], we note that most of these researches focus on *differential* operators on manifolds. The case of nonlocal operators globally defined on the entire manifold is more complex since it requires careful analysis of the topology of the underlying manifold. Indeed, the global geometry of the radiating surface enters into the behaviour of the response function at every point. It has been already suggested in [15, 16] that this deep connection between the topology and geometry of the radiating surface and the electromagnetic response is one of the most fundamental applications of the ACGF formalism to the physics of electromagnetic systems.

Even though the ACGF formalism has been utilized in the analysis and design of various antenna systems [1, 14–16, 20, 21, 24, 27, 30], including mixed PEC-dielectric configurations [18, 22, 23], to the best of our knowledge there currently exists no published account of how the formalism can be derived and formulated for the general case of multiple PEC and dielectric objects existing simultaneously in the system. Moreover, even the single-dielectric type case, i.e., one all-dielectric inhomogeneous domain in PEC-free system, has never been treated in print anywhere. This is in direct contrast to the extreme importance of mixed PEC-dielectric radiating (and receiving) systems, most prominently of course printed antenna systems such as patches and planner dipoles, and volumetric-type dielectric resonator antennas. In addition, high-power radiators like horn antennas are sometimes loaded with dielectric media in order to control their radiation characteristics, making them another example of antenna

[†] Also see Sec. 3 for some additional remarks on the applications to composite domains treated in literature so far and the references quoted therein.

types that need to be handled using an expanded or generalized ACGF formalism going beyond the now standard PEC-based ACGF theory.

This paper focuses exclusively on the purely mathematical aspects of the generalized ACGF formalism, in addition to extended discussion of some physical and conceptual issues related to the mathematical relations derived herein. No detailed discussion of the computational and experimental applications will be given here since these have been treated at length elsewhere. However, we provide toward the end of this paper (Sec. 3) a brief discussion of the literature on the computational implementation of the ACGF method and how it is related to some applications. Readers more interested in applications may consult Part 4 in the book [1], in addition to some of the more recent literature, such as [19, 25, 27]. Our mathematical treatment will be schematic and the rigid formal theorem-proof format will not be followed here to increase the accessibility of the work to wider audience.

2. EXACT DERIVATION OF THE GENERALIZED ACGF TENSOR ARRAY IN GENERIC PEC-DIELECTRIC SYSTEMS

2.1. Background Preparation: The Operator Formalism of Minimal Coupled PEC-Dielectric Systems

Throughout this paper, we assume a time-harmonic excitation of the form $\exp(\mathrm{i}\omega t)$. That is, we develop here a frequency-domain (space-frequency) current Green's function formalism, while the generalization to spacetime is left for future work. Let us first consider the minimal system composed of two material types, one PEC, and the other electric-magnetic type described by electric permitivity tensors $\overline{\overline{\varepsilon}}(\omega)$ and permeability $\overline{\overline{\mu}}(\omega)$, where both are allowed to be functions of the operating frequency ω.[‡]

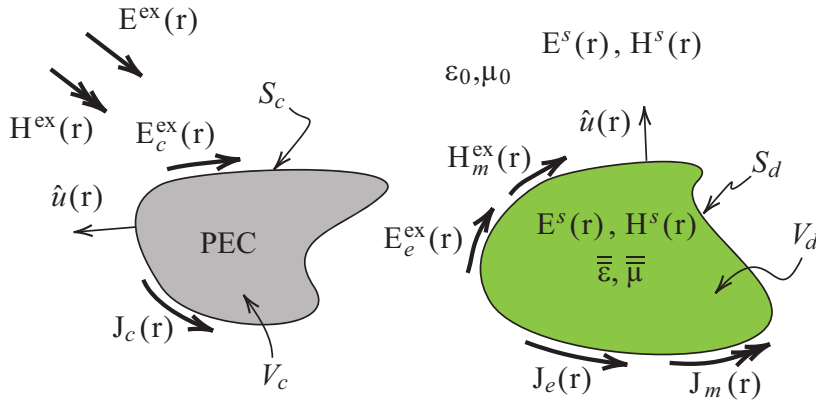

Figure 1. Minimal coupled PEC-dielectric system composed of exactly one PEC subdomain V_c enclosed by surface $S_c = \partial V_c$, and exactly one dielectric subdomain V_d with boundary $S_d = \partial V_d$. The dielectric subdomain electric and magnetic properties are described by permittivity and permeability tensors $\overline{\overline{\varepsilon}}(\omega), \overline{\overline{\mu}}(\omega)$, respectively, and these are assumed to be reciprocal. The surfaces S_c and S_d are modeled as orientable differential manifold with the Riemannian metric inherited from the embedding ambient Euclidean space $\mathbb{R}^3$.

The system is depicted in Fig. 1, where the externally applied fields are $\mathbf{E}^{\mathrm{ex}}(\mathbf{r})$ and $\mathbf{H}^{\mathrm{ex}}(\mathbf{r})$, while $\mathbf{E}^s(\mathbf{r})$ and $\mathbf{H}^s(\mathbf{r})$ are the secondary (scattered) fields. The total fields are given by

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{\mathrm{ex}}(\mathbf{r}) + \mathbf{E}^s(\mathbf{r}), \;\; \mathbf{H}(\mathbf{r}) = \mathbf{H}^{\mathrm{ex}}(\mathbf{r}) + \mathbf{H}^s(\mathbf{r}). \tag{1}$$

For simplicity, throughout this paper each dielectric domain is assumed to be *homogeneous*, so we drop out any possible dependence of the material profiles on position within each sub-region.[§] The region

[‡] Here, with the risk of abusing the common terminology, we include magnetic properties in those material domains we called 'dielectric'.

[§] That does not constitute a serious loss of generality since we will later generalize the formalism to deal with arbitrary number of

supporting the dielectric medium is denoted by V_d with enclosing surface S_d, while the PEC region has the corresponding V_c and S_c, respectively. All enclosing surfaces are assumed to enjoy enough regularity to allow the use of standard vector calculus and function analysis, e.g., see [31, 32]. Associated with each closed 2-manifold S_d and S_c is a smooth normal unit vector field $\hat{u}(\mathbf{r})$, with $\mathbf{r}$ in either S_c or S_d, which is always clear from the context. All regions will be embedded into an infinite free space exterior domain $V_{\text{ext}} := \mathbb{R}^3 - (V_d \cup V_c)$, with permittivity and permeability ε_0 and μ_0. We will assume the applicability of the radiation condition at infinity, and hence the surface S_∞ is not mentioned here. In what follows, the relevant integral representation theorems [33] will be used in conjunction with this radiation condition since we are interested in working with scattering/radiation problems.

The electromagnetic boundary conditions are [10, 34]

$$\hat{u}(\mathbf{r}) \times \mathbf{E}(\mathbf{r}) = 0, \quad \mathbf{r} \in S_c, \tag{2}$$

for the PEC domain, and the following continuity relations of the tangential electric and magnetic fields, respectively, for the dielectric domains:

$$\hat{u}(\mathbf{r}) \times \mathbf{E}\left(\mathbf{r}^+\right) = \hat{u}(\mathbf{r}) \times \mathbf{E}\left(\mathbf{r}^-\right), \hat{u}(\mathbf{r}) \times \mathbf{H}\left(\mathbf{r}^+\right) = \hat{u}(\mathbf{r}) \times \mathbf{H}\left(\mathbf{r}^-\right), \ \mathbf{r} \in S_d. \tag{3}$$

From these boundary conditions and using the standard surface equivalence theorem [10, 35], we introduce three types of currents in this system: conducting currents $\mathbf{J}_c(\mathbf{r})$, electric currents $\mathbf{J}_e(\mathbf{r})$, and magnetic currents $\mathbf{J}_m(\mathbf{r})$ given by [35–37]

$$\mathbf{J}_c(\mathbf{r}) := \begin{cases} \hat{u}(\mathbf{r}) \times \mathbf{H}(\mathbf{r}), & \mathbf{r} \in S_c, \\ 0, & \text{otherwise.} \end{cases}, \tag{4}$$

$$\mathbf{J}_e(\mathbf{r}) := \begin{cases} \hat{u}(\mathbf{r}) \times \mathbf{H}(\mathbf{r}), & \mathbf{r} \in S_d, \\ 0, & \text{otherwise.} \end{cases}, \quad \mathbf{J}_m(\mathbf{r}) := \begin{cases} -\hat{u}(\mathbf{r}) \times \mathbf{E}(\mathbf{r}), & \mathbf{r} \in S_d, \\ 0, & \text{otherwise.} \end{cases}. \tag{5}$$

Each of these currents will radiate both electric and magnetic field components into the interior and exterior domains of the problem. In other words, we are using the special form of Schelkunoff's surface equivalence theorem [37] implied in Eqs. (4) and (5), which gives correct fields only in the *exterior* region, while the internal fields are zero (Love's Principle [36].) However, for this to happen, the *entire* space of the problem, the unbounded domain $\mathbb{R}^3$ must be treated as free space ε_0, μ_0.[‖] This is the *exterior domain formulation* depicted in Fig. 2 where the presence of the primary external (excitation) fields is included in the exterior domain (but they can also be present in the interior domain with obvious modifications.)

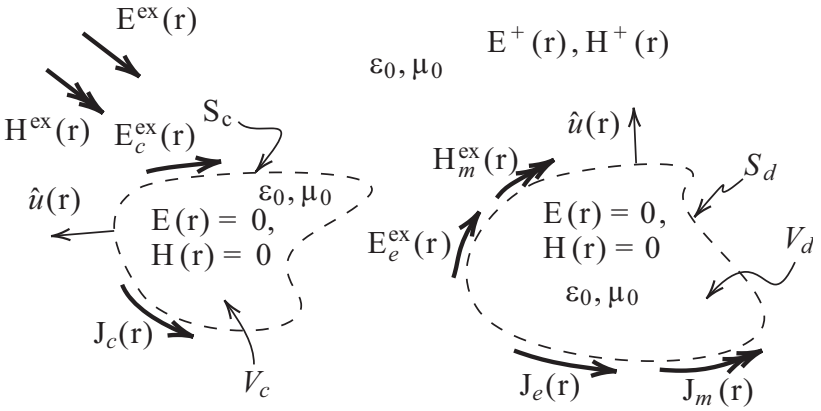

Figure 2. Exterior (radiation) problem reformulation using the surface equivalence theorem (Love-Schelkunoff Principle [37]).

sub-domains. Since inhomogeneous regions can be approximated (at least in first attempts) by a finite number of smaller homogeneous microdomains, the generic N-domain formalism of this paper should be sufficient for a large number of practical applications.

‖ Or the same as the properties of the exterior domain in the original problem if the radiation is not in free space

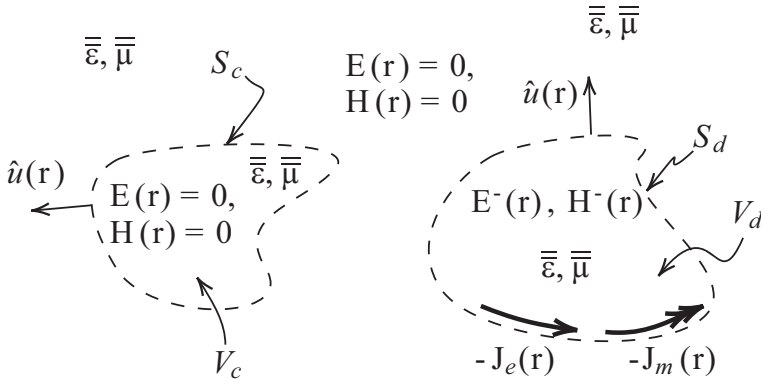

Figure 3. Interior (radiation) problem reformulation using the surface equivalence theorem (Love-Schelkunoff [37] Principle).

On the other hand, the surface equivalent currents $-\mathbf{J}_e(\mathbf{r})$ and$-\mathbf{J}_m(\mathbf{r})$ will radiate *null* fields in the exterior domain but produce the correct fields in the interior region V_d provided the entire unbounded domain is filled with the medium having $\overline{\overline{\varepsilon}}, \overline{\overline{\mu}}$ as depicted in Fig. 3. In this formulation, the exterior domain fields $\mathbf{E}^+(\mathbf{r}), \mathbf{H}^+(\mathbf{r})$ in Fig. 2 are identical to the correct original fields $\mathbf{E}(\mathbf{r}), \mathbf{H}(\mathbf{r})$, the latter due to the system Fig. 1 and defined by Eq. (1), but only for $\mathbf{r} \in \mathbb{R}^3 - V_d$, while the interior domain fields $\mathbf{E}^-(\mathbf{r}), \mathbf{H}^-(\mathbf{r})$ in Fig. 3 are equal to the original fields (1) for $\mathbf{r} \in V_d$.

Finally, using the principle of linear superposition, the total electric and magnetic fields, denoted by $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$, respectively, are given by

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{\text{ex}}(\mathbf{r}) + \underbrace{\mathbf{E}_e(\mathbf{r}) + \mathbf{E}_m(\mathbf{r}) + \mathbf{E}_c(\mathbf{r})}_{\mathbf{E}^s(\mathbf{r})}, \mathbf{H}(\mathbf{r}) = \mathbf{H}^{\text{ex}}(\mathbf{r}) + \underbrace{\mathbf{H}_e(\mathbf{r}) + \mathbf{H}_m(\mathbf{r}) + \mathbf{H}_c(\mathbf{r})}_{\mathbf{H}^s(\mathbf{r})}, \tag{6}$$

Here, $\mathbf{E}^{\text{ex}}(\mathbf{r})$ and $\mathbf{H}^{\text{ex}}(\mathbf{r})$ are the same as the externally supplied excitation in Fig. 1, which are assumed to exist in the exterior region only. In the remaining field symbols, each subscript indicates which *source current* type contributed to the production of the field component under consideration. For example, $\mathbf{E}_m(\mathbf{r})$ is the electric field produced by the magnetic current $\mathbf{J}_m(\mathbf{r})$, $\mathbf{H}_c(\mathbf{r})$ is the magnetic field produced by the conducting current $\mathbf{J}_c(\mathbf{r})$, etc.. Using this notation, the total exterior and interior fields may be expressed as

$$\begin{aligned} &\mathbf{E}^+ = \mathbf{E}^{\text{ex}} + \mathbf{E}_e^+ + \mathbf{E}_m^+ + \mathbf{E}_c^+, \quad \mathbf{H}^+ = \mathbf{H}^{\text{ex}} + \mathbf{H}_e^+ + \mathbf{H}_m^+ + \mathbf{H}_c^+, \\ &\mathbf{E}^- = \mathbf{E}_e^- + \mathbf{E}_m^-, \quad \mathbf{H}^- = \mathbf{H}_e^- + \mathbf{H}_m^-, \end{aligned} \tag{7}$$

respectively, where the convention that $\mathbf{E}^{\text{ex}}(\mathbf{r}) = \mathbf{H}^{\text{ex}}(\mathbf{r}) = 0$ for $\mathbf{r} \in V_d$ was adopted here. Also note that $\mathbf{E}_c = \mathbf{H}_c = 0$ since $\mathbf{J}_c = 0$ for the interior problem Fig. 3.

Under such conditions of the surface equivalence theorem, the field integral representation theorems can be used to express the total interior and exterior electric and magnetic via the operator relations [9, 10, 32–35, 38, 39]

$$\begin{aligned} &\mathbf{E}_e^{\pm}(\mathbf{r}) = \mathcal{G}_{ee}^{\pm}(\mathbf{r}, \mathbf{r}')\mathbf{J}_e(\mathbf{r}'), \ \mathbf{H}_m^{\pm}(\mathbf{r}) = \mathcal{G}_{mm}^{\pm}(\mathbf{r}, \mathbf{r}')\mathbf{J}_m(\mathbf{r}'), \\ &\mathbf{E}_c^{\pm}(\mathbf{r}) = \mathcal{G}_{ec}^{\pm}(\mathbf{r}, \mathbf{r}')\mathbf{J}_c(\mathbf{r}'), \ \mathbf{H}_e^{\pm}(\mathbf{r}) = \mathcal{G}_{me}^{\pm}(\mathbf{r}, \mathbf{r}')\mathbf{J}_e(\mathbf{r}'), \\ &\mathbf{E}_m^{\pm}(\mathbf{r}) = \mathcal{G}_{em}^{\pm}(\mathbf{r}, \mathbf{r}')\mathbf{J}_m(\mathbf{r}'), \ \mathbf{H}_c^{\pm}(\mathbf{r}) = \mathcal{G}_{mc}^{\pm}(\mathbf{r}, \mathbf{r}')\mathbf{J}_c(\mathbf{r}'), \end{aligned}$$

Here, the operator $\mathcal{G}_{ab}^{\pm}(\mathbf{r}, \mathbf{r}'), a \in \{e, m\}, b \in \{e, m, c\}$, is defined by the integral relation (Green's function operator)

$$\mathcal{G}_{ab}^{\pm}(\mathbf{r}, \mathbf{r}')\mathbf{J}_b(\mathbf{r}') := \int_{S(b)} \mathrm{d}s' \, \overline{\mathbf{G}}_{ab}^{\pm}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}_b(\mathbf{r}'), \tag{8}$$

where

$$S(b) := \begin{cases} S_c, & b = c, \\ S_d, & b = e, m. \end{cases} \tag{9}$$

The quantities $\overline{\mathbf{G}}^{+}_{ab}(\mathbf{r},\mathbf{r}')$ and $\overline{\mathbf{G}}^{-}_{ab}(\mathbf{r},\mathbf{r}')$ are the dyadic Green's functions for the infinite and unbounded domains filled with materials having $(\varepsilon_0,\mu_0),(\overline{\overline{\varepsilon}},\overline{\overline{\mu}})$, respectively. They connect a current excitation of the type $\mathbf{J}_b$ with either the electric field on S_c or electric/magnetic field on S_d. Not all of them are independent; e.g., it is obvious that $\overline{\mathbf{G}}^{\pm}_{ac}(\mathbf{r},\mathbf{r}')=\overline{\mathbf{G}}^{\pm}_{ae}(\mathbf{r},\mathbf{r}')$ for $a=e,m$. Moreover, other dyadic Green's functions are strongly related to each due to the symmetry (duality) relations. Since we will not need the specific forms of these functions in what follows, such additional relations will not be listed here since they are readily available in the literature.¶

It is significant to notice that the field produced by each current source is obtained via *nonlocal* relation, i.e., Green's functions operators (8), which are essentially integro-differential operators that can be defined on some convenient Banach or Sobolev space, see [32, 33]. We denoted a generic such operator by the symbol $\mathcal{G}^{\pm}_{ab}$, where $a\in\{e,m\}, b\in\{e,m,c\}$, which can be physically interpreted in the following manner: An entity expressed by $\mathcal{G}^{+}_{em}$, for example, denotes the radiation operator responsible of the production of the exterior electric field component $\mathbf{E}_e$ due to the magnetic current $\mathbf{J}_m$ when radiating into an infinite domain filled with ε_0,μ_0, etc.+ For the case of generic *anisotropic* media, the Green's functions are less well understood, and their computation is considerably more complicated. In general, no closed-form expressions exist for $\overline{\mathbf{G}}^{-}_{ab}(\mathbf{r},\mathbf{r}')$ in the anisotropic medium scenario, in contrast to the case with $\overline{\mathbf{G}}^{+}_{ab}(\mathbf{r},\mathbf{r}')$ where the latter are very well understood since they possess a relatively simple analytical form.♯

Let us now define the *surface excitation fields*, i.e., the pertinent illumination fields interacting with the system, which originate from an *external source*. These are constructed in terms of which surface they interact with. As will become apparent immediately, in the generalized ACGF formalism, it is absolutely essential that fields interacting with different boundary types are distinguished from each other. Most conspicuous is the case of electric fields, where whether they interact with PEC surface or material interface makes a fundamental difference in the mathematical formulation. The detailed definitions are

$$\mathbf{E}^{\text{ex}}_{e}(\mathbf{r}):=\mathbf{E}^{\text{ex}}(\mathbf{r})\psi(\mathbf{r};S_d),\quad \mathbf{H}^{\text{ex}}_{m}(\mathbf{r}):=\mathbf{H}^{\text{ex}}(\mathbf{r})\psi(\mathbf{r};S_d),\quad \mathbf{E}^{\text{ex}}_{c}(\mathbf{r}):=\mathbf{E}^{\text{ex}}(\mathbf{r})\psi(\mathbf{r};S_c), \tag{10}$$

where the surface membership function $\psi: S\to\mathbb{R}$, denoted by $\psi(\mathbf{r};S)$, is given by

$$\psi(\mathbf{r};S):=\begin{cases}1, & \mathbf{r}\in S,\\ 0, & \text{otherwise.}\end{cases} \tag{11}$$

Remember that the magnetic field does not contribute to the ACGF for PEC objects [1] (because the electric-field integral equations was chosen there to formulate the problem.) We may now put together all these notations and general theorems and combine them with the boundary conditions (2) and (3) in conjunction with the two surface equivalence problems depicted in Figs. 2 and 3. Using Eq. (2.1) in the electromagnetic boundary conditions (2) and (3), we find

$$\begin{aligned}
\hat{u}(\mathbf{r})\times\left[\mathbf{E}^{\text{ex}}(\mathbf{r})+\mathbf{E}^{+}_{e}(\mathbf{r})+\mathbf{E}^{+}_{m}(\mathbf{r})+\mathbf{E}^{+}_{c}(\mathbf{r})\right]&=0,\quad \mathbf{r}\in S_c,\\
\hat{u}(\mathbf{r})\times\left[\mathbf{E}^{\text{ex}}(\mathbf{r})+\mathbf{E}^{+}_{e}(\mathbf{r})+\mathbf{E}^{+}_{m}(\mathbf{r})+\mathbf{E}^{+}_{c}(\mathbf{r})\right]_{+}&=\hat{u}(\mathbf{r})\times\left[\mathbf{E}^{-}_{e}(\mathbf{r})+\mathbf{E}^{-}_{m}(\mathbf{r})\right]_{-},\ \mathbf{r}\in S_d,\\
\hat{u}(\mathbf{r})\times\left[\mathbf{H}^{\text{ex}}(\mathbf{r})+\mathbf{H}^{+}_{e}(\mathbf{r})+\mathbf{H}^{+}_{m}(\mathbf{r})+\mathbf{H}^{+}_{c}(\mathbf{r})\right]_{+}&=\hat{u}(\mathbf{r})\times\left[\mathbf{E}^{-}_{e}(\mathbf{r})+\mathbf{H}^{-}_{m}(\mathbf{r})\right]_{-},\ \mathbf{r}\in S_d.
\end{aligned} \tag{12}$$

Here, the subscripts '+' and '−' indices attached to the brackets in the second and third equations indicate that we will approach the surface S_d from the outside and inside, respectively. In general, *jump conditions*, which depend on the Green's functions $\overline{\mathbf{G}}^{\pm}_{ab}(\mathbf{r},\mathbf{r}')$ being used, must be employed to further simplify the system of Equation (12). Moreover, several combinations of electric and magnetic type boundary conditions might be used, leading to electric field integral equations (EFIE), magnetic field integral equation (MFIE), and combined field integral equation (CFIE) [38]. The well-known PMCHWT

¶ We also further note that since in the interior problem of Fig. 3, $\mathbf{J}_c=0$, the operators $\mathcal{G}^{-}_{ac}$ will not be actually used in what follows.

+ Most advanced classical electromagnetic theory books present derivations of these nonlocal operators, which we will not give here, but see [9, 10, 32, 33, 35, 38–41] for extensive further details and discussion regarding the mathematical theory and the computational aspects of these radiation operators.

♯ It is also worth mentioning that in the engineering community, the dyadic Green's functions of inhomogeneous media, especially those of stratified domains, have received very extensive attention and are often formulated and computed efficiently, e.g., see [9, 12, 38].

(after Poggio-Miller-Chang-Harrington-Wu-Tsai) has been used to solve EM scattering from dielectric objects and PEC/dielectric systems [38].

All of the above cases (in addition to others) can be put in the following extremely general system of operator equations [10, 35, 38, 40, 42–44]

$$\begin{aligned}
\mathcal{L}_{ee}(\mathbf{r},\mathbf{r}')\mathbf{J}_e(\mathbf{r}') + \mathcal{L}_{em}(\mathbf{r},\mathbf{r}')\mathbf{J}_m(\mathbf{r}') + \mathcal{L}_{ec}(\mathbf{r},\mathbf{r}')\mathbf{J}_c(\mathbf{r}') &= \hat{u}(\mathbf{r}) \times \mathbf{E}_e^{\text{ex}}(\mathbf{r}), \qquad \mathbf{r} \in S_d, \\
\mathcal{L}_{me}(\mathbf{r},\mathbf{r}')\mathbf{J}_e(\mathbf{r}') + \mathcal{L}_{mm}(\mathbf{r},\mathbf{r}')\mathbf{J}_m(\mathbf{r}') + \mathcal{L}_{mc}(\mathbf{r},\mathbf{r}')\mathbf{J}_c(\mathbf{r}') &= \hat{u}(\mathbf{r}) \times \mathbf{H}_m^{\text{ex}}(\mathbf{r}), \quad \mathbf{r} \in S_d, \\
\mathcal{L}_{ce}(\mathbf{r},\mathbf{r}')\mathbf{J}_e(\mathbf{r}') + \mathcal{L}_{cm}(\mathbf{r},\mathbf{r}')\mathbf{J}_m(\mathbf{r}') + \mathcal{L}_{cc}(\mathbf{r},\mathbf{r}')\mathbf{J}_c(\mathbf{r}') &= \hat{u}(\mathbf{r}) \times \mathbf{E}_c^{\text{ex}}(\mathbf{r}), \qquad \mathbf{r} \in S_c.
\end{aligned} \tag{13}$$

In this system, the unknowns are the three current distributions $\mathbf{J}_e(\mathbf{r}')$, $\mathbf{J}_m(\mathbf{r}')$, and $\mathbf{J}_c(\mathbf{r})$, while the input excitation (forcing terms) are the fields $\mathbf{E}_e^{\text{ex}}(\mathbf{r}')$, $\mathbf{H}_m^{\text{ex}}(\mathbf{r}')$, and $\mathbf{E}_c^{\text{ex}}(\mathbf{r}')$ defined by Eq. (10). The exact form of the operators $\mathcal{L}_{ab}, a, b \in \{e, m, c\}$, depends on the jump conditions and which surface integral equation was used. For the fundamental constructive proof of the generalized ACGF formalism, as long as the system is solvable the exact details of these sub-operators are not needed, hence the powerful generality of the method.

Through the special manner by which we defined the unit normal vector field $\hat{u}(\mathbf{r})$, which has a support only on the manifolds S_c and S_d, the RHS of Eq. (13) is ensured to be nonzero only on either a PEC surface or a material interface. In other words, from the perspective of linear system theory, the Equations (13) express the antenna/scattering problem as a linear response problem where input excitation fields, $\mathbf{E}^{\text{ex}}$ and $\mathbf{H}^{\text{ex}}$ interact only with 2-manifolds, here S_c and S_d, in spite of the fact that the interacting fields are 3-dimensional spacetime processes. This "dimensional change" is one of the main reasons why the current Green's function formalism requires extra care when formulating its main equations and results as was originally noted in [15].

The three linear operator equations in Eq. (13) must be solved for the unknown currents simultaneously, very much like in ordinary linear algebraic equations [38, 40]. This amounts to inverting the *global* operator of the system, denoted by $\mathcal{L}(\mathbf{r},\mathbf{r}')$, which is defined by the following operator relation

$$\mathcal{L}(\mathbf{r},\mathbf{r}') \cdot [\mathbf{J}(\mathbf{r}')] = [\mathbf{V}^{\text{ex}}(\mathbf{r})], \tag{14}$$

where

$$[\mathbf{J}(\mathbf{r})] := \begin{pmatrix} \mathbf{J}_e(\mathbf{r}) \\ \mathbf{J}_m(\mathbf{r}) \\ \mathbf{J}_c(\mathbf{r}) \end{pmatrix} \tag{15}$$

is a compact form of the three unknown vector current distributions, while

$$[\mathbf{V}^{\text{ex}}(\mathbf{r})] := \hat{u}(\mathbf{r}) \times \begin{pmatrix} \mathbf{E}_e^{\text{ex}}(\mathbf{r}) \\ \mathbf{H}_m^{\text{ex}}(\mathbf{r}) \\ \mathbf{E}_c^{\text{ex}}(\mathbf{r}) \end{pmatrix} \tag{16}$$

is the corresponding form of the excitation fields. In component form, the global integro-differential operator of the PEC-dielectric system can be expressed as

$$\mathcal{L}(\mathbf{r},\mathbf{r}';\omega,S,\overline{\overline{\varepsilon}},\overline{\overline{\mu}}) := \begin{pmatrix} \mathcal{L}_{ee}(\mathbf{r},\mathbf{r}';\omega,S,\overline{\overline{\varepsilon}},\overline{\overline{\mu}}) & \mathcal{L}_{em}(\mathbf{r},\mathbf{r}';\omega,S,\overline{\overline{\varepsilon}},\overline{\overline{\mu}}) & \mathcal{L}_{ec}(\mathbf{r},\mathbf{r}';\omega,S,\overline{\overline{\varepsilon}},\overline{\overline{\mu}}) \\ \mathcal{L}_{me}(\mathbf{r},\mathbf{r}';\omega,S,\overline{\overline{\varepsilon}},\overline{\overline{\mu}}) & \mathcal{L}_{mm}(\mathbf{r},\mathbf{r}';\omega,S,\overline{\overline{\varepsilon}},\overline{\overline{\mu}}) & \mathcal{L}_{mc}(\mathbf{r},\mathbf{r}';\omega,S,\overline{\overline{\varepsilon}},\overline{\overline{\mu}}) \\ \mathcal{L}_{ce}(\mathbf{r},\mathbf{r}';\omega,S,\overline{\overline{\varepsilon}},\overline{\overline{\mu}}) & \mathcal{L}_{cm}(\mathbf{r},\mathbf{r}';\omega,S,\overline{\overline{\varepsilon}},\overline{\overline{\mu}}) & \mathcal{L}_{cc}(\mathbf{r},\mathbf{r}';\omega,S,\overline{\overline{\varepsilon}},\overline{\overline{\mu}}) \end{pmatrix}, \tag{17}$$

where the frequency dependence was deliberately emphasized though it will be suppressed most of the time hereafter. The sub-operators appearing in Eq. (17) are all functions of the material composition tensors $\overline{\overline{\varepsilon}}$ and $\overline{\overline{\mu}}$ as well as the geometry of the interface manifolds S, but mention of this dependence will be also dropped out to simplify the notation (though it remains implied.) On the other hand, the $\mathbf{r}$ and $\mathbf{r}'$ variables appearing here are essentially formal and are used to emphasize the *operator* character of $\mathcal{L}$ as transformations mapping functions of $\mathbf{r}'$ into functions of $\mathbf{r}$.

2.2. Exact Derivation of the Generalized ACGF Tensor Array in Minimal PEC-Dielectric Systems

We are ready now to construct the current Green's function of the mixed PEC-dielectric system using the techniques proposed in [15]. It will be seen that similar to the PEC case, the ACGF of the PEC-

dielectric configuration is not an ordinary function, but a *distribution*, i.e., a generalized function [45–47]. A Green's function is a response function representing how a *linear* system reacts to influences or external inputs [47]. However, not every linear system can automatically possess a Green's function, but in each case the Green's function must be explicitly shown to exist [47, 48]. In mathematical physics, since most physical processes of interest are described by differential equations (partial or ordinary), an extensive literature on the subject of Green's functions for linear partial differential equations has been developed, especially for the most important cases of first- and second-order systems [49]. In our case, the linear operator system in Eq. (13) is quite different since the operators appearing in Eq. (2.1) composing the global operator in Eq. (17) are *integro*-differential operators, hence not strictly speaking local. This implies that the existence of the current Green's function in this case is not obvious and requires explicit construction. To do so, we follow the same path proposed earlier in [1] where a specialized *surface* delta function is first defined then later deployed to prove the existence of the ACGF by actually constructing one.

Generalized functions like the Green's function or the Dirac delta function are defined with respect (or found to be equivalent) to equivalence classes of sequences of ordinary functions satisfying certain properties [45–47, 49–51]. A surface delta function $\delta_S(\mathbf{r},\mathbf{r}')$ is a delta function living only on the smooth manifold S. It is defined as the generalized function [15]

$$\delta_S(\mathbf{r},\mathbf{r}') \overset{\text{g.f.}}{:=} (\ d_1(\mathbf{r},\mathbf{r}';S)\quad d_2(\mathbf{r},\mathbf{r}';S)\quad \dots\quad d_n(\mathbf{r},\mathbf{r}';S)\quad \dots\ ), \tag{18}$$

where the equality is not point-like as in ordinary function, but to be understood in the sense of equivalence class-type of equality [45, 46, 50]. The functions d_n, however, are all ordinary smooth functions by design. Explicit expressions for the function sequence $d_n(\mathbf{r},\mathbf{r}'), n = 1, 2, \dots$, were given in [1, 15], where the Riemannian metric tensor field $g(\mathbf{r})$, describing the inner product on two tangent vectors at $\mathbf{r}$, enters into the definition of δ_S. Moreover, due to the inhomogeneous nature of the space described by the smooth manifold S, the delta function is not shift-invariant and hence cannot be written as function of $\mathbf{r}-\mathbf{r}'$ as in the case with the more familiar Dirac delta functions on Euclidean spaces $\mathbb{R}^n$ [46]. It can be shown that [1, 15],

$$V(\mathbf{r}) = \lim_{n\to\infty}\int_S \mathrm{d}s' d_n(\mathbf{r},\mathbf{r}';S)V(\mathbf{r}') \tag{19}$$

is valid for any smooth scalar field $V(\mathbf{r})$ on S. In fact, one may consider Eq. (19) *the* defining relation of the surface delta function δ_S of which the form Eq. (18) is merely its rigorous rendering via distribution theory. Indeed, both Eqs. (18) and (19) are combined into

$$V(\mathbf{r}) = \int_S \mathrm{d}s'\delta_S(\mathbf{r},\mathbf{r}')V(\mathbf{r}'), \tag{20}$$

which is the formal relation usually invoked in textbooks as the definition of the Dirac delta function (the sifting property.)

Next, we need to "vectorize" the surface delta function in order to fulfill the specific needs expected for completing the mixed PEC-dielectric ACGF construction under consideration. To do so, let us introduce the vector array

$$[\delta_{\mathrm{S}}^{\mathrm{ex}}(\mathbf{r})] := \begin{pmatrix} \hat{\alpha}_e\ \delta_{S_d}(\mathbf{r},\mathbf{r}') \\ \hat{\alpha}_m\ \delta_{S_d}(\mathbf{r},\mathbf{r}') \\ \hat{\alpha}_c\ \delta_{S_c}(\mathbf{r},\mathbf{r}') \end{pmatrix}. \tag{21}$$

It captures in a compact form the three *independent* excitation fields pertinent to the system's current Green's function. Here, the three vectors $\hat{\alpha}_e$, $\hat{\alpha}_m$, and $\hat{\alpha}_c$ are independent unit vectors encoding the source polarization data, while the impulsive nature of these excitation inputs is captured by the surface delta function defined on the *proper* 2-manifold corresponding to each source. Using Eq. (18), the delta source array in Eq. (21) can be explicated as

$$[\delta_{\mathrm{S}}^{\mathrm{ex}}(\mathbf{r})] \overset{\text{g.f}}{:=} (\ [\mathbf{f}_1^{\mathrm{ex}}(\mathbf{r};S)]\quad [\mathbf{f}_2^{\mathrm{ex}}(\mathbf{r};S)]\quad \dots\quad [\mathbf{f}_n^{\mathrm{ex}}(\mathbf{r};S)]\quad \dots\ ), \tag{22}$$

where the regular source (ordinary) fields are

$$[\mathbf{f}_n^{\mathrm{ex}}(\mathbf{r};S)] := \begin{pmatrix} \hat{\alpha}_e d_n(\mathbf{r},\mathbf{r}';S_d) \\ \hat{\alpha}_m d_n(\mathbf{r},\mathbf{r}';S_d) \\ \hat{\alpha}_c d_n(\mathbf{r},\mathbf{r}';S_c) \end{pmatrix}. \tag{23}$$

The surface vector field function in Eq. (23) presents a manageable ordinary excitation function on which one may apply the normal rules of calculus and functional analysis, in particular the traditional boundary-value problem and its functional analytic formulation as will be seen shortly.

From classical electromagnetic theory, it can be shown that the general operator relation in Eq. (14) is invertible under very reasonable restrictions. The details can be found in [32, 33, 38]. Indeed, the mixed PEC-dielectric system under illumination by smooth fields, with regularity conditions imposed on the geometry — already ensured by assuming that S possesses a differential (smooth) manifold structure — is a well-posed boundary-value problem with the proper existence and uniqueness theorems applicable to it. Under these circumstances, we write

$$[\mathbf{J}(\mathbf{r})] = \mathcal{L}^{-1}(\mathbf{r},\mathbf{r}') \cdot [\mathbf{V}(\mathbf{r}')], \tag{24}$$

where $\mathcal{L}^{-1}$ is the inverse $\mathcal{L}^{-1} := (\mathcal{L})^{-1}$ of the global operator (17) given by

$$\mathcal{L}^{-1}(\mathbf{r},\mathbf{r}') = \begin{pmatrix} \mathcal{L}_{ee}^{-1}(\mathbf{r},\mathbf{r}') & \mathcal{L}_{em}^{-1}(\mathbf{r},\mathbf{r}') & \mathcal{L}_{ec}^{-1}(\mathbf{r},\mathbf{r}') \\ \mathcal{L}_{me}^{-1}(\mathbf{r},\mathbf{r}') & \mathcal{L}_{mm}^{-1}(\mathbf{r},\mathbf{r}') & \mathcal{L}_{mc}^{-1}(\mathbf{r},\mathbf{r}') \\ \mathcal{L}_{ce}^{-1}(\mathbf{r},\mathbf{r}') & \mathcal{L}_{cm}^{-1}(\mathbf{r},\mathbf{r}') & \mathcal{L}_{cc}^{-1}(\mathbf{r},\mathbf{r}') \end{pmatrix}. \tag{25}$$

Therefore, like the forward global operator, the inverse operator *must* be an array of nine sub-operators, here denoted by $\mathcal{L}_{ab}^{-1}(\mathbf{r},\mathbf{r}'), a, b \in \{e, m, c\}$. It should be immediately noted that

$$\mathcal{L}_{ab}^{-1}(\mathbf{r},\mathbf{r}') \neq \left(\mathcal{L}_{ab}(\mathbf{r},\mathbf{r}')\right)^{-1}, \quad \forall a, b \in \{e, m, c\}. \tag{26}$$

In other words, it is not possible to obtain the elements of the individual sub-operators composing $\mathcal{L}^{-1}$ by simply inverting the corresponding original *forward* operators $\mathcal{L}_{ab}(\mathbf{r},\mathbf{r}')$ appearing in Eq. (17). This observation is important because it can be easily overlooked. The confusion may arise from the fact that the various forward sub-operators $\mathcal{L}_{ab}(\mathbf{r},\mathbf{r}')$ possess expressions independent (formally) from each other, and hence one may indeed mathematically invert the individual operators. However, since the three Equations (13) must be satisfied *simultaneously* to ensure the correctness of the proper electromagnetic boundary conditions of the system under consideration, it is necessary to invert the entire operator *array* in Eq. (17).

Following [1, 14, 15], we define the current Green's function (ACGF) of the PEC-dielectric system by the relation

$$[\overline{\mathbf{F}}(\mathbf{r},\mathbf{r}')] := \mathcal{L}^{-1}(\mathbf{r},\mathbf{r}') \cdot \left[\delta_{\mathrm{S}}^{\mathrm{ex}}(\mathbf{r}')\right]. \tag{27}$$

Note that this is a formal procedure for constructing Green's function in electromagnetics in which one starts with a vector (or array of vectors like $[\delta_{\mathrm{S}}^{\mathrm{ex}}(\mathbf{r}')]$) and ends up with a *tensor* (or array of tensors), i.e., a higher-order object (the increase in order is due to the polarization structure of the electromagnetic field). Our notation reflects this in the sense that whenever an inverse operator is applied to a delta function a tensor is produced [1]. Therefore, Equation (27) is interpreted differently from Eq. (24). In the latter, the LHS is an array of vectors, while in Eq. (27) the LHS is a tensor array. The difference comes from the special nature of the input in the RHS: in Eq. (24) is an ordinary vector field (current), while in Eq. (27) is a specialized surface Dirac delta current source. The advantages of the notation in Eq. (27) should become obvious now: it avoids writing too many equations whenever a Green's function is introduced.

Furthermore, it should be noted that the definition in Eq. (27) must be ultimately understood in the sense of *generalized functions*. Hence, by invoking the distributional definition of the excitation field in Eq. (22), we introduce

$$[\overline{\mathbf{F}}_n(\mathbf{r},\mathbf{r}')] := \mathcal{L}^{-1}(\mathbf{r},\mathbf{r}') \cdot \left[\mathbf{f}_n^{\mathrm{ex}}(\mathbf{r}')\right]. \tag{28}$$

which is defined as the nth level distributional approximation of the ACGF.* As can be readily verified, the elements of the function sequence $[\overline{\mathbf{F}}_n(\mathbf{r},\mathbf{r}')], n = 1, 2, \ldots, \infty$, are:

(i) ordinary smooth functions, and

(ii) tensorial fields defined on the 2-dimensional Riemannian manifold S.

* Again, and similar to Eq. (27), a tensor is produced by the application of $\mathcal{L}^{-1}$ to each delta-function approximating level $\mathbf{f}_n^{\mathrm{ex}}$. While the latter is an ordinary vector field for every level n, we still would like to avoid writing too many equations for constructing the ordinary tensor approximation $[\overline{\mathbf{F}}_n(\mathbf{r},\mathbf{r}')]$, so the convention is that whenever an inverse operator $\mathcal{L}^{-1}$ is applied to a delta function or an approximation level of a delta function, a tensor (dyad) is produced.

Fact (i) implies that all rules of normal vector analysis apply when working with the distributional approximation functions $[\overline{\mathbf{F}}_n(\mathbf{r},\mathbf{r}')], n=1,2,\ldots,\infty$. In particular, this ensures that *ordinary full-wave numerical methods in computational electromagnetics (CEM) can be deployed to calculate these level functions themselves*. This observation was heavily exploited in [1] in order to approximate the ACGF of various PEC systems. Moreover, fact (ii) says that the approximating functions $[\overline{\mathbf{F}}_n(\mathbf{r},\mathbf{r}')]$, and hence the ACGF itself, have acquired an additional structure, that of a *tensor on 2-manifold*. Therefore, $[\overline{\mathbf{F}}(\mathbf{r},\mathbf{r}')]$ represents a Green's tensor living in a curved Riemannian manifold S. We will come again to the explicit form of this Green's tensor shortly. Putting in all these information together, we may now construct a formal distributional definition of the ACGF through

$$[\overline{\mathbf{F}}(\mathbf{r},\mathbf{r}')] \overset{\text{g.f.}}{:=} \begin{pmatrix} [\overline{\mathbf{F}}_1(\mathbf{r},\mathbf{r}')] & [\overline{\mathbf{F}}_2(\mathbf{r},\mathbf{r}')] & [\overline{\mathbf{F}}_n(\mathbf{r},\mathbf{r}')] & \ldots \end{pmatrix}. \tag{29}$$

This is the fundamental distributional *constructive* definition of the generalized ACGF. Everything to follow is based on it.

With the help of Eq. (25), the definition in Eq. (27) implies that the generalized ACGF should be expanded into an array of nine sub-ACGFs, i.e., $[\overline{\mathbf{F}}(\mathbf{r},\mathbf{r}')]$ possesses the form

$$[\overline{\mathbf{F}}(\mathbf{r},\mathbf{r}')] = \begin{pmatrix} \overline{\mathbf{F}}_{ee}(\mathbf{r},\mathbf{r}') & \overline{\mathbf{F}}_{em}(\mathbf{r},\mathbf{r}') & \overline{\mathbf{F}}_{ec}(\mathbf{r},\mathbf{r}') \\ \overline{\mathbf{F}}_{me}(\mathbf{r},\mathbf{r}') & \overline{\mathbf{F}}_{mm}(\mathbf{r},\mathbf{r}') & \overline{\mathbf{F}}_{mc}(\mathbf{r},\mathbf{r}') \\ \overline{\mathbf{F}}_{ce}(\mathbf{r},\mathbf{r}') & \overline{\mathbf{F}}_{cm}(\mathbf{r},\mathbf{r}') & \overline{\mathbf{F}}_{cc}(\mathbf{r},\mathbf{r}') \end{pmatrix}. \tag{30}$$

In details, the individual ACGFs composing this generalized ACGF array are given by the following expression

$$[\overline{\mathbf{F}}(\mathbf{r},\mathbf{r}')] = \begin{pmatrix} \underbrace{\mathcal{L}^{-1}_{ee}(\mathbf{r},\mathbf{r}')\hat{\alpha}_e\delta_{S_d}(\mathbf{r},\mathbf{r}')}_{\overline{\mathbf{F}}_{ee}(\mathbf{r},\mathbf{r}')} & \underbrace{\mathcal{L}^{-1}_{em}(\mathbf{r},\mathbf{r}')\hat{\alpha}_m\delta_{S_d}(\mathbf{r},\mathbf{r}')}_{\overline{\mathbf{F}}_{em}(\mathbf{r},\mathbf{r}')} & \underbrace{\mathcal{L}^{-1}_{ec}(\mathbf{r},\mathbf{r}')\hat{\alpha}_c\delta_{S_c}(\mathbf{r},\mathbf{r}')}_{\overline{\mathbf{F}}_{ec}(\mathbf{r},\mathbf{r}')} \\ \underbrace{\mathcal{L}^{-1}_{me}(\mathbf{r},\mathbf{r}')\hat{\alpha}_e\delta_{S_d}(\mathbf{r},\mathbf{r}')}_{\overline{\mathbf{F}}_{me}(\mathbf{r},\mathbf{r}')} & \underbrace{\mathcal{L}^{-1}_{mm}(\mathbf{r},\mathbf{r}')\hat{\alpha}_m\delta_{S_d}(\mathbf{r},\mathbf{r}')}_{\overline{\mathbf{F}}_{mm}(\mathbf{r},\mathbf{r}')} & \underbrace{\mathcal{L}^{-1}_{mc}(\mathbf{r},\mathbf{r}')\hat{\alpha}_c\delta_{S_c}(\mathbf{r},\mathbf{r}')}_{\overline{\mathbf{F}}_{mc}(\mathbf{r},\mathbf{r}')} \\ \underbrace{\mathcal{L}^{-1}_{ce}(\mathbf{r},\mathbf{r}')\hat{\alpha}_e\delta_{S_d}(\mathbf{r},\mathbf{r}')}_{\overline{\mathbf{F}}_{ce}(\mathbf{r},\mathbf{r}')} & \underbrace{\mathcal{L}^{-1}_{cm}(\mathbf{r},\mathbf{r}')\hat{\alpha}_m\delta_{S_d}(\mathbf{r},\mathbf{r}')}_{\overline{\mathbf{F}}_{cm}(\mathbf{r},\mathbf{r}')} & \underbrace{\mathcal{L}^{-1}_{cc}(\mathbf{r},\mathbf{r}')\hat{\alpha}_c\delta_{S_c}(\mathbf{r},\mathbf{r}')}_{\overline{\mathbf{F}}_{cc}(\mathbf{r},\mathbf{r}')} \end{pmatrix}, \tag{31}$$

Each ACGF is in itself a tensor on its associated 2-dimensional Riemannian manifold S.[&] In order to effectively describe the structure of this tensor. i.e., in a coordinate-independent manner, we need to work with the differential atlas associated with the manifold S on which the ACGF tensor is defined.

Let $(U_i,\varphi_i), i\in I$ be such an atlas, where

$$\varphi_i : U_i \to \mathbb{R}^2 \tag{32}$$

is the ith coordinate system associated with the open set $U_i \subset S$, while I is the index set. The coordinates themselves are given by $x^n = \varphi^n_i(\mathbf{r})$, with $n=1,2$ to account for the two degrees of freedom on a 2-dimensional manifold. The manifold itself is the union

$$S = \cup_{i\in I} U_i \tag{33}$$

with possible overlap between individual U_i. The tangent vector at each point $\mathbf{r}\in U_i\subset S$ can be expressed in terms of the coordinates of that patch through the relation [49, 52]

$$\hat{\alpha}(\mathbf{r}) = \alpha_1(\mathbf{r})\frac{\partial}{\partial x^1} + \alpha_2(\mathbf{r})\frac{\partial}{\partial x^2}, \tag{34}$$

where α_1 and α_2 are smooth real functions on S. This is the expression of the tangent vector $\hat{\alpha}$ at $\mathbf{r}$ in terms of local (intrinsic) coordinate systems. But we can also expand the same vector in terms of external global Cartesian coordinate system associated with the ambient space $\mathbb{R}^3$ into which S is embedded, so we write $\hat{\alpha}_1$ and $\hat{\alpha}_2$ as the Cartesian form for the vectors $\partial/\partial x_1$ and $\partial/\partial x_2$, respectively.

[&] The same Green's tensor, being essentially a geometric object in our construction, can be expressed in several forms, e.g., 2D tensor, 3D tensor, or the forms Eq. (31), depending on the context. Regardless to the specific tensorial representation used, the physically measured quantities, in our case induced currents, are the same. See further discussion of this in Sec. 3.1 and also [1].

The two vectors are clearly linearly independent but they are not generally required to be orthogonal. For convenience, in this paper we assume that they are further *orthonormal*, i.e., we have

$$\hat{\alpha}_i(\mathbf{r}) \cdot \hat{\alpha}_j(\mathbf{r}) = \delta_{ij}, \ \mathbf{r} \in S, i, j = 1, 2, \tag{35}$$

where δ_{ij} is the Kronecker delta function. Based on this geometric formulation, the ACGF can be readily expanded into dyadic basis in the form $\hat{\alpha}_i(\mathbf{r}) \otimes \hat{\alpha}_j(\mathbf{r}')$. If we drop the tensor product sign $\otimes$ for simplicity, we obtain

$$\overline{\mathbf{F}}_{ab}(\mathbf{r}, \mathbf{r}') = \sum_{i,j=1}^{2} \hat{\alpha}_i(\mathbf{r})\hat{\alpha}_j(\mathbf{r}')F_{ab}^{ij}(\mathbf{r}, \mathbf{r}') = \begin{pmatrix} \hat{\alpha}_1(\mathbf{r})\hat{\alpha}_1(\mathbf{r}')F_{ab}^{11}(\mathbf{r}, \mathbf{r}') & \hat{\alpha}_1(\mathbf{r})\hat{\alpha}_2(\mathbf{r}')F_{ab}^{12}(\mathbf{r}, \mathbf{r}') \\ \hat{\alpha}_2(\mathbf{r})\hat{\alpha}_1(\mathbf{r}')F_{ab}^{21}(\mathbf{r}, \mathbf{r}') & \hat{\alpha}_2(\mathbf{r})\hat{\alpha}_2(\mathbf{r}')F_{ab}^{22}(\mathbf{r}, \mathbf{r}') \end{pmatrix}, \tag{36}$$

where $a, b \in \{e, m, c\}$. In Appendix B, all of the concrete generalized ACGF array component expressions are listed in order to spell out which manifolds (S_d or S_c) are relevant to each ACGF.

The final — and most crucial — step in the derivation of the generalized ACGF formalism is demonstrating an analog to the sifting property in Eq. (20) valid for the current distribution array response $[\mathbf{J}(\mathbf{r})]$. Here, as in [15], it turns out that the proof of the main relation must use reciprocity (consequent on the self-adjointness of the relevant operators). Let us start by considering the n-level distributional approximation of the surface current

$$[\mathbf{J}_n(\mathbf{r})] = \int_S \mathrm{d}s' \ [\overline{\mathbf{F}}_n(\mathbf{r}, \mathbf{r}')] \cdot [\mathbf{V}(\mathbf{r}')]. \tag{37}$$

Here, the n-level ACGF approximation $[\overline{\mathbf{F}}_n(\mathbf{r}, \mathbf{r}')]$ can be computed with the help of

$$[\overline{\mathbf{F}}_n(\mathbf{r}, \mathbf{r}')] = \begin{pmatrix} \mathcal{L}_{ee}^{-1}(\mathbf{r}, \mathbf{r}')\hat{\alpha}_e d_n(\mathbf{r}, \mathbf{r}'; S_d) & \mathcal{L}_{em}^{-1}(\mathbf{r}, \mathbf{r}')\hat{\alpha}_m d_n(\mathbf{r}, \mathbf{r}'; S_d) & \mathcal{L}_{ec}^{-1}(\mathbf{r}, \mathbf{r}')\hat{\alpha}_c d_n(\mathbf{r}, \mathbf{r}'; S_c) \\ \mathcal{L}_{me}^{-1}(\mathbf{r}, \mathbf{r}')\hat{\alpha}_e d_n(\mathbf{r}, \mathbf{r}'; S_d) & \mathcal{L}_{mm}^{-1}(\mathbf{r}, \mathbf{r}')\hat{\alpha}_m d_n(\mathbf{r}, \mathbf{r}'; S_d) & \mathcal{L}_{mc}^{-1}(\mathbf{r}, \mathbf{r}')\hat{\alpha}_c d_n(\mathbf{r}, \mathbf{r}'; S_c) \\ \mathcal{L}_{ce}^{-1}(\mathbf{r}, \mathbf{r}')\hat{\alpha}_e d_n(\mathbf{r}, \mathbf{r}'; S_d) & \mathcal{L}_{cm}^{-1}(\mathbf{r}, \mathbf{r}')\hat{\alpha}_m d_n(\mathbf{r}, \mathbf{r}'; S_d) & \mathcal{L}_{cc}^{-1}(\mathbf{r}, \mathbf{r}')\hat{\alpha}_c d(\mathbf{r}, \mathbf{r}'; S_c) \end{pmatrix}, \tag{38}$$

where Eqs. (18), (22), (23), (31) were used. The integral (37) is in fact a global integral defined over the entire surface S. It is originally written in terms of the local coordinates φ_i on the patch U_i. Afterwards, the partition of unity technique [52] is used to aggregate all the local patches in order to compute the net integral.

The details of this construction are straightforward but rather technical so a summary of the background is relegated to Appendix A. Using that material, the expression (37) can be expanded as

$$[\mathbf{J}_n(\mathbf{r})] = \sum_{j \in J} \int_{\varphi_j(U_j)} \left(\psi_j(\mathbf{r}') \sqrt{|g(\mathbf{r}')|} [\overline{\mathbf{F}}_n(\mathbf{r}, \mathbf{r}')] \cdot [\mathbf{V}(\mathbf{r}')] \right) \circ \varphi_j^{-1} \mathrm{d}^2 x'. \tag{39}$$

Here, $g(\mathbf{r}')$ is the Riemannian metric tensor field on the particular 2-manifold being integrated over, i.e., the surface S_c or S_d. Each of the individual integrals in the RHS of Eq. (39) is performed over a local *Euclidean* patch of the entire surface, namely $\varphi(U_i) \subset \mathbb{R}^2$, i.e., the *coordinate* patch corresponding to the geometric piece U_i over which the integration will be numerically preformed. Note that the infinitesimal surface differential element $\mathrm{d}^2 x'$ is mapped by the function φ_j^{-1} to the global language (intrinsic geometric) representation of points as $\mathbf{r}, \mathbf{r}' \in S$. In what follows, all global integrals must be understood in the *geometric* sense given above, i.e., *the values obtained are always independent of which coordinate system was used*. However, for the sake of brevity, we will not spell out fine-grained expressions like Eq. (19) every time an integral over an entire manifold S_d or S_c is written.

Because all of the media in our system are assumed to be reciprocal, the global operator $\mathcal{L}$ in Eq. (17) is itself reciprocal and hence Lorenz reciprocity theorem is valid for our case [7, 9, 10, 35]. Using a technical argument identical to the one developed in details in [1, 15], it can be shown that the actual current response array $[\mathbf{J}(\mathbf{r})]$ is the limit of the n-level approximation $[\mathbf{J}_n(\mathbf{r})]$, i.e., we have

$$[\mathbf{J}(\mathbf{r})] = \lim_{n \to \infty} [\mathbf{J}_n(\mathbf{r})] = \lim_{n \to \infty} \int_S \mathrm{d}s' \ [\overline{\mathbf{F}}_n(\mathbf{r}, \mathbf{r}')] \cdot [\mathbf{V}(\mathbf{r}')]. \tag{40}$$

Since the limit in the RHS can be further pushed inside the integral (the exchange of limit and operators is proved based on reciprocity [15]), it can be concluded that

$$[\overline{\mathbf{F}}(\mathbf{r}, \mathbf{r}')] = \lim_{n \to \infty} [\overline{\mathbf{F}}_n(\mathbf{r}, \mathbf{r}')], \tag{41}$$

which is paramount to saying that the generalized ACGF of the system is the distributional limit of the sequence of n-level ordinary tensorial functions $[\overline{\mathbf{F}}_n(\mathbf{r},\mathbf{r}')]$.[^] The details of the proof of the above mentioned limit-operator order exchange, already implicit in the statement of distributional approximation of Eq. (41), are quite lengthy and identical to the PEC case treated in [1] and so will not be given here.[~]

It now immediately follows from Eqs. (40) and (41) that

$$\boxed{[\mathbf{J}(\mathbf{r})] = \int_S \mathrm{d}s'\ [\overline{\mathbf{F}}(\mathbf{r},\mathbf{r}')]\cdot[\mathbf{V}(\mathbf{r}')].} \tag{42}$$

This is the main relation in the generalized ACGF formalism. If we expand it in details, we find

$$\mathbf{J}_e(\mathbf{r}) = \int_{S_d} \mathrm{d}s'\ \overline{\mathbf{F}}_{ee}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}^{\mathrm{ex}}_e(\mathbf{r}') + \int_{S_d} \mathrm{d}s'\ \overline{\mathbf{F}}_{em}(\mathbf{r},\mathbf{r}')\cdot\mathbf{H}^{\mathrm{ex}}_m(\mathbf{r}') + \int_{S_c} \mathrm{d}s'\ \overline{\mathbf{F}}_{ec}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}^{\mathrm{ex}}_c(\mathbf{r}'), \tag{43}$$

$$\mathbf{J}_m(\mathbf{r}) = \int_{S_d} \mathrm{d}s'\ \overline{\mathbf{F}}_{me}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}^{\mathrm{ex}}_e(\mathbf{r}') + \int_{S_d} \mathrm{d}s'\ \overline{\mathbf{F}}_{mm}(\mathbf{r},\mathbf{r}')\cdot\mathbf{H}^{\mathrm{ex}}_m(\mathbf{r}') + \int_{S_c} \mathrm{d}s'\ \overline{\mathbf{F}}_{mc}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}^{\mathrm{ex}}_c(\mathbf{r}'), \tag{44}$$

$$\mathbf{J}_c(\mathbf{r}) = \int_{S_d} \mathrm{d}s'\ \overline{\mathbf{F}}_{ce}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}^{\mathrm{ex}}_e(\mathbf{r}') + \int_{S_d} \mathrm{d}s'\ \overline{\mathbf{F}}_{cm}(\mathbf{r},\mathbf{r}')\cdot\mathbf{H}^{\mathrm{ex}}_m(\mathbf{r}') + \int_{S_c} \mathrm{d}s'\ \overline{\mathbf{F}}_{cc}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}^{\mathrm{ex}}_c(\mathbf{r}'). \tag{45}$$

The relations in Eqs. (42), (43), (44), and (45) provide a complete space-frequency representation of a composite PEC-dielectric system response to external field illuminations $\mathbf{E}^{\mathrm{ex}}(\mathbf{r})$ and $\mathbf{H}^{\mathrm{ex}}(\mathbf{r})$. They are valid for arbitrary inhomogeneous configuration composed of reciprocal domains in which the interfaces can be modeled by orientable smooth 2-manifolds. Further discussion of these final results will be briefly outlined in Sec. 3.

2.3. Generalization to Generic PEC-Dielectric Systems

We end this section by the generalization to the case of arbitrary number of PEC and dielectric domains. The main derivation is identical to the minimal case of one PEC and one dielectric domain developed in details above. We merely expand each surface type into the set-theoretic union of the corresponding sub-surfaces as follows:

$$S = S_c \cup S_d, \quad S_c = \bigcup_{i=1}^{N_c} S^i_c, \quad S_d = \bigcup_{i=1}^{N_d} S^i_d, \tag{46}$$

where N_c and N_d are the numbers of conducting and dielectric subdomains, respectively. Since each of these sub-surfaces constitutes by itself a smooth Riemannian manifold, the set $S = S_c \cup S_d$ is also a Riemannian manifold, as well as S_c and S_d taken individually. Each PEC/dielectric subdomain is enclosed by S_c, S_d, respectively. Furthermore, it is clear that Eqs. (15) and (16) can be recast into larger arrays with structures

$$[\mathbf{J}(\mathbf{r})]_{(2N_e+N_c)\times 1} := \begin{pmatrix} [\mathbf{J}_e(\mathbf{r})]_{N_e\times 1} \\ [\mathbf{J}_m(\mathbf{r})]_{N_e\times 1} \\ [\mathbf{J}_c(\mathbf{r})]_{N_c\times 1} \end{pmatrix}, [\mathbf{V}^{\mathrm{ex}}(\mathbf{r})]_{(2N_e+N_c)\times 1} := \hat{u}(\mathbf{r}) \times \begin{pmatrix} [\mathbf{E}^{\mathrm{ex}}_e(\mathbf{r})]_{N_e\times 1} \\ [\mathbf{H}^{\mathrm{ex}}_m(\mathbf{r})]_{N_e\times 1} \\ [\mathbf{E}^{\mathrm{ex}}_c(\mathbf{r})]_{N_c\times 1} \end{pmatrix}. \tag{47}$$

Here, the components of these augmented arrays are

$$[\mathbf{J}_e(\mathbf{r})]_{N_e\times 1} = \left[\mathbf{J}^i_e(\mathbf{r})\right]^{N_e}_{i=1}, [\mathbf{J}_m(\mathbf{r})]_{N_e\times 1} = \left[\mathbf{J}^i_m(\mathbf{r})\right]^{N_e}_{i=1}, [\mathbf{J}_c(\mathbf{r})]_{N_c\times 1} = \left[\mathbf{J}^i_c(\mathbf{r})\right]^{N_c}_{i=1}, \tag{48}$$

where the individual currents with superscript i labels the currents on the surfaces S^i_c or S^i_d. Similarly, we have

$$\left[\mathbf{E}^{\mathrm{ex},i}_e(\mathbf{r})\right]_{N_e\times 1} = \left[\mathbf{E}^{\mathrm{ex},i}_e(\mathbf{r})\right]^{N_e}_{i=1}, \left[\mathbf{H}^{\mathrm{ex},i}_m(\mathbf{r})\right]_{N_e\times 1} = \left[\mathbf{H}^{\mathrm{ex},i}_m(\mathbf{r})\right]^{N_e}_{i=1}, \left[\mathbf{E}^{\mathrm{ex},i}_c(\mathbf{r})\right]_{N_c\times 1} = \left[\mathbf{E}^{\mathrm{ex},i}_c(\mathbf{r})\right]^{N_c}_{i=1}. \tag{49}$$

[^] An equivalent version of Eq. (41) has been anticipated in Eq. (29).

[~] However, the reader should notice that it is not currently known whether the ACGF can be constructed in nonreciprocal systems.

The pattern is now clear. We will not give the analog to all intermediate relations, but mention the generalized ACGF augmented array, which in the most generic scenario acquires the form

$$[\overline{\mathbf{F}}(\mathbf{r},\mathbf{r}')]_{(2N_e+N_c)\times(2N_e+N_c)} = \begin{pmatrix} [\overline{\mathbf{F}}_{ee}(\mathbf{r},\mathbf{r}')]_{N_e\times N_e} & [\overline{\mathbf{F}}_{em}(\mathbf{r},\mathbf{r}')]_{N_e\times N_e} & [\overline{\mathbf{F}}_{ec}(\mathbf{r},\mathbf{r}')]_{N_e\times N_c} \\ [\overline{\mathbf{F}}_{me}(\mathbf{r},\mathbf{r}')]_{N_e\times N_e} & [\overline{\mathbf{F}}_{mm}(\mathbf{r},\mathbf{r}')]_{N_e\times N_e} & [\overline{\mathbf{F}}_{mc}(\mathbf{r},\mathbf{r}')]_{N_e\times N_c} \\ [\overline{\mathbf{F}}_{ce}(\mathbf{r},\mathbf{r}')]_{N_c\times N_e} & [\overline{\mathbf{F}}_{cm}(\mathbf{r},\mathbf{r}')]_{N_c\times N_e} & [\overline{\mathbf{F}}_{cc}(\mathbf{r},\mathbf{r}')]_{N_c\times N_c} \end{pmatrix}. \tag{50}$$

Here, each element of the augmented generalized ACGF array $[\overline{\mathbf{F}}(\mathbf{r},\mathbf{r}')]$ is written as $\overline{\mathbf{F}}_{ab}^{ij}$, itself a tensor on the appropriate 2-manifold, with data structure conforming to

$$[\overline{\mathbf{F}}(\mathbf{r},\mathbf{r}')]_{(2N_e+N_c)\times(2N_e+N_c)} = \left[\overline{\mathbf{F}}_{ab}^{ij}(\mathbf{r},\mathbf{r}')\right]_{a,b\in\{e,m,c\};i,j\in\{1,\ldots,\max(N_c,N_e)\}}. \tag{51}$$

The detailed forms for all these sub-components are listed in Appendix C, which must be consulted in order to correctly implement the generalized ACGF tensor array $[\overline{\mathbf{F}}(\mathbf{r},\mathbf{r}')]$.

It is to be further observed that each tensor sub-component $\overline{\mathbf{F}}_{ab}^{ij}$ is itself expressible in terms of one of the formulas listed in Appendix B. Consequently, combining the expressions in both the latter Appendix with those found in Appendix C, the global generalized ACGF tensor array can be ultimately traced back to individual scalar components that might be implemented numerically or measured in the lab. Based on this notation, the final generalized ACGF relation becomes

$$\boxed{[\mathbf{J}(\mathbf{r})]_{(2N_e+N_c)\times 1} = \int_S \mathrm{d}s'\ \left[\overline{\mathbf{F}}\left(\mathbf{r},\mathbf{r}'\right)\right]_{(2N_e+N_c)\times(2N_e+N_c)} \cdot \left[\mathbf{V}\left(\mathbf{r}'\right)\right]_{(2N_e+N_c)\times 1}.} \tag{52}$$

In details, the various currents flowing on the sub-surfaces S_c^i and S_d^i an be expressed in terms of all other excitation field through the relations

$$\begin{aligned} \mathbf{J}_e^i(\mathbf{r}) \ =\ & \sum_{j=1}^{N_e}\int_{S_d^j} \mathrm{d}s'\ \overline{\mathbf{F}}_{ee}^{ij}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}_e^{\mathrm{ex},j}(\mathbf{r}') + \sum_{j=1}^{N_e}\int_{S_d^j} \mathrm{d}s'\ \overline{\mathbf{F}}_{em}^{ij}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}_m^{\mathrm{ex},j}(\mathbf{r}') \\ & + \sum_{j=1}^{N_c}\int_{S_c^j} \mathrm{d}s'\ \overline{\mathbf{F}}_{ec}^{ij}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}_c^{\mathrm{ex},j}(\mathbf{r}'), \end{aligned} \tag{53}$$

$$\begin{aligned} \mathbf{J}_m^i(\mathbf{r}) \ =\ & \sum_{j=1}^{N_e}\int_{S_d^j} \mathrm{d}s'\ \overline{\mathbf{F}}_{me}^{ij}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}_e^{\mathrm{ex},j}(\mathbf{r}') + \sum_{j=1}^{N_e}\int_{S_d^j} \mathrm{d}s'\ \overline{\mathbf{F}}_{mm}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}_m^{\mathrm{ex},j}(\mathbf{r}') \\ & + \sum_{j=1}^{N_c}\int_{S_c^j} \mathrm{d}s'\ \overline{\mathbf{F}}_{mc}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}_c^{\mathrm{ex},j}(\mathbf{r}'), \end{aligned} \tag{54}$$

$$\begin{aligned} \mathbf{J}_c^i(\mathbf{r}) \ =\ & \sum_{j=1}^{N_e}\int_{S_d^j} \mathrm{d}s'\ \overline{\mathbf{F}}_{ce}^{ij}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}_e^{\mathrm{ex},j}(\mathbf{r}') + \sum_{j=1}^{N_e}\int_{S_d^j} \mathrm{d}s'\ \overline{\mathbf{F}}_{cm}^{ij}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}_m^{\mathrm{ex},j}(\mathbf{r}') \\ & + \sum_{j=1}^{N_c}\int_{S_c^j} \mathrm{d}s'\ \overline{\mathbf{F}}_{cc}^{ij}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}_c^{\mathrm{ex},j}(\mathbf{r}'). \end{aligned} \tag{55}$$

The three relations in Eqs. (53)–(55) represent the most general form possible of the current Green's function method in generic composite PEC-dielectric systems and is stated and derived here for the first time but under the assumption that the media composing the system are reciprocal.

3. DISCUSSION OF THE RESULTS

3.1. General Remarks

We provide in this last section a set of general observations and remarks with the objective of situating the fundamental mathematical formulation presented above within the general context of both theory and applications.

First, we would like to compare the coupled PEC-dielectric formalism with the previously published PEC-only ACGF method [1]. It is immediately noticed that while there are several similarities, for example the formal structure of the superposition integral in Eq. (52), there are also fundamental differences. The most important, in our view, is the considerable increase in complexity due to the need to take into account the magnetic field interaction. Indeed, the magnetic field will contribute at every point in the surface S_d, and even the conduction current $\mathbf{J}_c^i(\mathbf{r})$, where $\mathbf{r} \in S_c^i$, will receive contribution from not only the electric fields at all other surfaces s_d^j, but also the magnetic fields there. This curious situation shows that mixed PEC-dielectric systems are inherently more complex. Mutual coupling exists between every two domains, and this coupling is taken into account through coupling Green's functions of the form $\overline{\mathbf{F}}_{ab}^{ij}(\mathbf{r}, \mathbf{r}')$.

Next, we highlight not only the formal complexity of the generalized ACGF method, but also anticipate a considerable increase in the required computational burden placed on the expanded method. This is already clear in the generic formulas treated in Sec. 2.3 where each tensor was replaced by a tensor *array*. In fact, the generalized ACGF of a generic composite PEC-dielectric appearing in Eq. (50) is an $(2N_e + N_c) \times (2N_e + N_c)$ square array, where N_e and N_c are the total numbers of dielectric and PEC sub-domains, numbers that can easily increase with complicated antenna types like multi-layered microstrip antennas [53] or multiple-scattering systems [9, 10].

Now every "element" of this array, i.e., the generic term $\overline{\mathbf{F}}_{ab}^{ij}(\mathbf{r}, \mathbf{r}')$, is itself a tensor, or simply a dyad, with representation in a chosen (local) coordinate system on the 2-manifold on which this dyad is defined, giving rise to a 2×2 *matrix* as per the detailed expressions in Appendices B and C. Therefore, the generalized ACGF array is effectively a rather complicated and involved matrix of $(4N_e + 2N_c) \times (4N_e + 2N_c)$ scalar functions. This matrix is coordinate dependent in the sense that its entries' exact numerical values do depend on the coordinate system under consideration. However, since we use *local* charts (intrinsic manifold language), the final results, the current induced on the surface S after interaction with incident excitation fields, as given by the integral in Eq. (52), is independent of the coordinate system. In fact, a concrete calculus for transforming ACGF tensors was derived in [1] and can be also developed here with minor changes. Since the idea is the same but the details are lengthy, this method will not be given here but the reader need to be aware that the ACGF as constructed here is like all tensors a geometric quantity, i.e., not attached to a specific coordinate system.[▷]

A few remarks on the range of validity of the derived formalism. It is important to note that technically the existence of the generalized ACGF depends crucially on the assumption that the system is reciprocal. This, however, is a characteristic of our particular construction. It is currently an open question whether a different approach can be applied to construct the ACGF for nonreciprocal systems. It is in general believed that Green's functions and reciprocity are intimately related to each other, but since our constructive method is not unique, we prefer to leave the question of whether ACGFs exist in nonreciprocal radiating systems to future investigations.

Another restriction that must be noted is that our use of the surface equivalence theorem to construct the generalized ACGF formalism formulates the problem exclusively in terms of surface currents. It is currently an open question whether the ACGF can also be constructed using *volume* integral equations, leading to volume current response. That will lead to substantial modification (if the formulation indeed is possible) since deploying Riemannian geometry will become less warranted than in our formulation above where surface integral equations have been chosen as the starting point from which we build the foundations of the ACGF method.

[▷] In practical applications to complex antennas that involve closed surfaces this is important since full-wave numerical simulators tend to use local (geometric) coordinate systems, e.g., the curvilinear coordinates of the old Gaussian theory of 2-surfaces, e.g., see [39, 40].

3.2. On the Numerical Implementation of the ACGF Method

We finally say a few words about the numerical implementation of the ACGF formalism. Strictly speaking, the ACGF technique was originally introduced essentially as an alternative numerical method in electromagnetics, with applications to the analysis of wire antennas [30]. After developing some of the theoretical foundations in [14, 15], a direct computational technique using the Method of Moment (MoM) was proposed in [16], which is similar to [30]. The basic idea is to use MoM with specialized mesh based on discretizing in Eq. (13) to model an increasing number of n-level distributional approximations of the ACGF, namely the $\overline{\mathbf{F}}_n$ appearing in Eq. (28). The computation themselves are enacted by means of the MoM version of the operator Equation (14), which now becomes a matrix equation. This idea was investigated numerically in [24, 54] and experimentally in [18] for mixed PEC-dielectric type antennas (strongly coupled microstrip antennas backed by dielectric substrate and ground plane), without, however, explicit mention or derivation of the integral formulas (52). Afterwards, this MoM-based approach has been further systematized and integrated with more applications, e.g., see [1].

However, it is very important to note that the MoM-based approach to the ACGF method is not the only option available as we will discuss shortly. In fact, the ACGF, being more fundamental than the discrete mesh approximation upon which all numerical methods like the MoM are based, can be even measured directly in the lab, though to the best of our knowledge this has not been accomplished yet. The problem with MoM-based estimation is that it can be very expensive and may lead to convergence problems. Indeed, each n-level approximation $\overline{\mathbf{F}}_n(\mathbf{r}, \mathbf{r}')$ requires a dense, most often nonuniform, MoM discretization mesh in order to properly model the physics of the progressively narrowing excitation $d_n(\mathbf{r}, \mathbf{r}')$, which in turn approximate the surface delta function as per Eq. (18). Since the MoM is well known to suffer from convergence problems when the matrix to be inverted becomes denser [55], this direct approach via the distributional approximation relation in Eq. (41) may become prohibitively expensive or even simply not convergent if the original (exact) limit is replaced by a MoM-based operator approximation.[◇]

It is interesting to note that in recent years two alternative techniques have been proposed to estimate the ACGF. One technique is in fact as old as the ACGF method itself, and was already used in the earliest publications [30]. This is the proposal to use the *spatial singularity expansion method* (S-SEM) to approximate the ACGF as a finite series of complex exponentials, in a way that resembles the well known eigenfunction expansion method or singularity expansion method in the time domain [56]. While the spatial SEM itself was developed systematically only very recently in [57], it had been already applied to estimate the ACGF of printed antennas in [58–60]. This method rely on measuring some near- or far- field and then using machine learning to perform inverse source modeling to determine the unknown current on the radiator using the S-SEM expansion as learning model to be trained with available data [61, 62].

The second new method alternative to MoM-based approximation is *the infinitesimal dipole model* (IDM) approach, which is referred to here as the ACGF-IDM framework. The IDM method is a very general strategy in computational electromagnetics aiming at replacing a complex radiating current by an equivalent small number of concentrated Dirac-like sources [63]. The method intersects a large number of applications in areas as diverse as imaging, geo-sensing, radars, wireless communications, tomography, and others.[††] It was shown recently that a combination of the ACGF and IDM methods can effectively model complex radiating systems using measured far-field data [23]. This approach was applied there with experimental and numerical results to conformal microstrip antenna arrays, which also involve composite PEC-dielectric objects.

Finally, we mention also some *hybrid methods* involving a combination of standard ACGF and physical optics approach as in [21, 22]. To summarize, aside from direct lab measurement of the ACGF itself (not realized yet), so far four different numerical methods have been used to implement the ACGF technique:

(i) The direct (dense) full-wave method approximation (most often using the MoM).

(ii) The ACGF-SEM method (based on measured field data).

[◇] Unfortunately, not much has been published so far on the pure numerical issue of convergence rate and speed in the MoM-based approach to the ACGF method.

[††] See the literature quoted in [64] for some recent survey on the IDM method.

(iii) The ACGF-IDM method (based on measured field data).

(iv) Hybrid methods combining the full-wave numerical approximation approach to the ACGF with other approximations such as ray tracing, physical optics, etc..

The state of research on the computational efficiency of the proposed ACGF formalism as an alternative method in computational electromagnetics (CEM) is still evolving and we hope more work on this will be published in the new decade that just started.

4. CONCLUSION

Starting from surface integral equations, we provided an extensive rigorous mathematical investigation of the foundations and derivation of the main tenets of the antenna current Green's function (ACGF) formalism for coupled PEC-dielectric systems, filling an existing gap in the literature where most investigations using the ACGF method have focused so far on numerical and experimental aspects. In fact, previously only the perfect-electric conducting (PEC) case was investigated mathematically at the fundamental level. It was found that in contrast to the PEC-only ACGF formalism, including dielectric domains with possible magnetic properties brings both electric and magnetic fields into the picture, i.e., not only the electric field as in the PEC case. Also, both electric and magnetic currents are introduced at dielectric interfaces and the needed general surface integral relations in terms of the ACGF (Green's kernel) were derived. The construction of the mixed PEC-dielectric ACGF gave rise to a higher-order tensor array, considerably more complicated than the PEC-only's tensor. Our derivation depends crucially on assuming reciprocity in additional to linearity and it is currently unknown if the ACGF can be constructed for nonreciprocal systems or whether the ACGF can be constructed based on volume integral equations.

APPENDIX A. COORDINATE-INDEPENDENT EXTENSION OF INTEGRALS FROM LOCAL TO GLOBAL MANIFOLD DOMAINS

Let (M, g) be an n-dimensional Riemannian manifold with topological manifold M and metric tensor field g. An atlas on M is given by $U_i, \varphi_i, i \in I$. We say that $[U_j, \varphi_j(\mathbf{r}), \psi_j(\mathbf{r})], j \in J, \mathbf{r} \in M$, is a partition of unity subordinate to $U_i, \varphi_i, i \in I$, when:

(i) The collection of functions $\psi_j : M \to \mathbb{R}$ on M constitutes a smooth partition of unity subordinate to the covering $U_i, i \in I$.

(ii) $(U_j, \varphi_j), j \in J$, in itself constitutes an atlas on M.

(iii) For any $j \in J$, the support of of $\psi_j(\mathbf{r})$ is contained within U_j, i.e., $\text{supp}(\psi_j) \subset U_j$.

It can be shown that for any atlas there exists a partition of unity subordinate to it in the sense given above [52].

Since $x = \varphi_j(\mathbf{r})$ is in $\mathbb{R}^n$, then we can immediately invoke the classic Lebesgue measure on Euclidean spaces to write integrals with respect to x, which will be in the form $\int \mathrm{d}^n x$. We then define Riemannian measure integral on a given function $f : M \to \mathbb{R}$ by the formula

$$\int_M \mathrm{d}vf := \sum_{j\in J} \int_{\varphi_j(U_j)} \left(\psi_j \sqrt{|g|} f\right) \circ \varphi_j^{-1} \mathrm{d}^n x, \tag{A1}$$

where $|g|$ is the determinant of the metric tensor g when its components are expressed in terms of the local coordinate system of the jth patch. In this way, the integral defined *globally* over the entire manifold is expanded into aggregates of *smaller*, i.e., *local*, integrals over the local patches U_j which when pieced up together give back the total manifold M.

For the definition to work, we assume that the integrated function has a compact support, a condition automatically satisfied in our case because all manifolds dealt with in this paper, namely S_c and S_d are compact. Finally, it can be shown that the construction (A1) is independent of which coordinate system was used [52]. In our case, the surface manifold definition corresponding to $n = 2$ is the only one needed. Additional information regarding differential manifolds and how to perform integration on them can be found in [28, 49, 65, 66].

APPENDIX B. TENSORIAL EXPANSIONS FOR ALL GLOBAL GENERALIZED ACGF COMPONENTS

$$\overline{\mathbf{F}}_{ee}(\mathbf{r},\mathbf{r}') = \begin{cases} \sum_{i,j=1}^{2} \hat{\alpha}_i(\mathbf{r})\hat{\alpha}_j(\mathbf{r}')F_{ee}^{ij}(\mathbf{r},\mathbf{r}'), & \mathbf{r}' \in S_d, \mathbf{r} \in S_d, \\ 0, & \text{elsewhere.} \end{cases} . \tag{B1}$$

$$\overline{\mathbf{F}}_{em}(\mathbf{r},\mathbf{r}') = \begin{cases} \sum_{i,j=1}^{2} \hat{\alpha}_i(\mathbf{r})\hat{\alpha}_j(\mathbf{r}')F_{em}^{ij}(\mathbf{r},\mathbf{r}'), & \mathbf{r}' \in S_d, \mathbf{r} \in S_d, \\ 0, & \text{elsewhere.} \end{cases} . \tag{B2}$$

$$\overline{\mathbf{F}}_{ec}(\mathbf{r},\mathbf{r}') = \begin{cases} \sum_{i,j=1}^{2} \hat{\alpha}_i(\mathbf{r})\hat{\alpha}_j(\mathbf{r}')F_{ec}^{ij}(\mathbf{r},\mathbf{r}'), & \mathbf{r}' \in S_c, \mathbf{r} \in S_d, \\ 0, & \text{elsewhere.} \end{cases} . \tag{B3}$$

$$\overline{\mathbf{F}}_{me}(\mathbf{r},\mathbf{r}') = \begin{cases} \sum_{i,j=1}^{2} \hat{\alpha}_i(\mathbf{r})\hat{\alpha}_j(\mathbf{r}')F_{me}^{ij}(\mathbf{r},\mathbf{r}'), & \mathbf{r}' \in S_d, \mathbf{r} \in S_d, \\ 0, & \text{elsewhere.} \end{cases} . \tag{B4}$$

$$\overline{\mathbf{F}}_{mm}(\mathbf{r},\mathbf{r}') = \begin{cases} \sum_{i,j=1}^{2} \hat{\alpha}_i(\mathbf{r})\hat{\alpha}_j(\mathbf{r}')F_{mm}^{ij}(\mathbf{r},\mathbf{r}'), & \mathbf{r}' \in S_d, \mathbf{r} \in S_d, \\ 0, & \text{elsewhere.} \end{cases} . \tag{B5}$$

$$\overline{\mathbf{F}}_{mc}(\mathbf{r},\mathbf{r}') = \begin{cases} \sum_{i,j=1}^{2} \hat{\alpha}_i(\mathbf{r})\hat{\alpha}_j(\mathbf{r}')F_{mc}^{ij}(\mathbf{r},\mathbf{r}'), & \mathbf{r}' \in S_c, \mathbf{r} \in S_d, \\ 0, & \text{elsewhere.} \end{cases} . \tag{B6}$$

$$\overline{\mathbf{F}}_{ce}(\mathbf{r},\mathbf{r}') = \begin{cases} \sum_{i,j=1}^{2} \hat{\alpha}_i(\mathbf{r})\hat{\alpha}_j(\mathbf{r}')F_{ce}^{ij}(\mathbf{r},\mathbf{r}'), & \mathbf{r}' \in S_d, \mathbf{r} \in S_c, \\ 0, & \text{elsewhere.} \end{cases} . \tag{B7}$$

$$\overline{\mathbf{F}}_{cm}(\mathbf{r},\mathbf{r}') = \begin{cases} \sum_{i,j=1}^{2} \hat{\alpha}_i(\mathbf{r})\hat{\alpha}_j(\mathbf{r}')F_{cm}^{ij}(\mathbf{r},\mathbf{r}'), & \mathbf{r}' \in S_d, \mathbf{r} \in S_c, \\ 0, & \text{elsewhere.} \end{cases} . \tag{B8}$$

$$\overline{\mathbf{F}}_{cc}(\mathbf{r},\mathbf{r}') = \begin{cases} \sum_{i,j=1}^{2} \hat{\alpha}_i(\mathbf{r})\hat{\alpha}_j(\mathbf{r}')F_{cc}^{ij}(\mathbf{r},\mathbf{r}'), & \mathbf{r}' \in S_c, \mathbf{r} \in S_c, \\ 0, & \text{elsewhere.} \end{cases} . \tag{B9}$$

APPENDIX C. STRUCTURE OF THE GENERALIZED ACGF ARRAY COMPONENTS FOR MULTIPLE COUPLED PEC-DIELECTRIC SYSTEMS

$$[\overline{\mathbf{F}}_{ee}(\mathbf{r},\mathbf{r}')]_{N_e\times N_e} = \left[\overline{\mathbf{F}}_{ee}^{ij}(\mathbf{r},\mathbf{r}')\right]_{i,j=1,\ldots,N_e}, \tag{C1}$$

$$[\overline{\mathbf{F}}_{me}(\mathbf{r},\mathbf{r}')]_{N_e\times N_e} = \left[\overline{\mathbf{F}}_{me}^{ij}(\mathbf{r},\mathbf{r}')\right]_{i,j=1,\ldots,N_e}, \tag{C2}$$

$$[\overline{\mathbf{F}}_{em}(\mathbf{r},\mathbf{r}')]_{N_e\times N_e} = \left[\overline{\mathbf{F}}_{em}^{ij}(\mathbf{r},\mathbf{r}')\right]_{i,j=1,\ldots,N_e}, \tag{C3}$$

$$[\overline{\mathbf{F}}_{mm}(\mathbf{r},\mathbf{r}')]_{N_e\times N_e} = \left[\overline{\mathbf{F}}^{ij}_{mm}(\mathbf{r},\mathbf{r}')\right]_{i,j=1,\ldots,N_e}, \tag{C4}$$

$$[\overline{\mathbf{F}}_{ec}(\mathbf{r},\mathbf{r}')]_{N_e\times N_c} = \left[\overline{\mathbf{F}}^{ij}_{ec}(\mathbf{r},\mathbf{r}')\right]_{i=1,\ldots,N_e;j=1,\ldots,N_c}, \tag{C5}$$

$$[\overline{\mathbf{F}}_{ce}(\mathbf{r},\mathbf{r}')]_{N_c\times N_e} = \left[\overline{\mathbf{F}}^{ij}_{ce}(\mathbf{r},\mathbf{r}')\right]_{i=1,\ldots,N_c;j=1,\ldots,N_e}, \tag{C6}$$

$$[\overline{\mathbf{F}}_{cm}(\mathbf{r},\mathbf{r}')]_{N_c\times N_e} = \left[\overline{\mathbf{F}}^{ij}_{cm}(\mathbf{r},\mathbf{r}')\right]_{i=1,\ldots,N_c;j=1,\ldots,N_e}, \tag{C7}$$

$$[\overline{\mathbf{F}}_{mc}(\mathbf{r},\mathbf{r}')]_{N_e\times N_c} = \left[\overline{\mathbf{F}}^{ij}_{mc}(\mathbf{r},\mathbf{r}')\right]_{i=1,\ldots,N_e;j=1,\ldots,N_c}, \tag{C8}$$

$$[\overline{\mathbf{F}}_{cc}(\mathbf{r},\mathbf{r}')]_{N_c\times N_c} = \left[\overline{\mathbf{F}}^{ij}_{cc}(\mathbf{r},\mathbf{r}')\right]_{i,j=1,\ldots,N_c}. \tag{C9}$$

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