

The Upper Bound of the Speed of Propagation of Waves along a Transmission Line

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Abstract—According to theory, once certain conditions are fulfilled, current and voltage pulses propagate along ideal transmission lines with the speed of light. One can reach such a conclusion only when the conductors are assumed to be perfectly conducting, which cannot be realized in practice. A wave can only propagate along a transmission line with the speed of light if no energy has to be spent in establishing the current in the conductor. However, in establishing a current in a transmission line, energy has to be supplied to the electrons to set them in motion since they have a mass. The energy transfer to the electrons manifests itself in the form of an inductance which is called the kinetic inductance. The effect of the kinetic inductance has to be taken into account in signal propagation along high carrier mobility conductors including super conductors. In the case of transmission lines, the kinetic inductance leads to a change in the characteristic impedance and a reduction in the speed of propagation of waves along the transmission line. The goal of this paper is to show that the kinetic inductance will set an upper bound to the speed of propagation of waves along transmission lines, which is smaller than the speed of light.

1. INTRODUCTION

Transmission lines are utilized to transfer electrical energy from one location to another. The mode of action of the transmission line is to transfer voltage and current pulses over long distances. The principles pertinent to transmission lines are also used to describe the action of transmitting antennas. The propagation characteristics of voltage and current waves moving along the transmission lines can be described by two equations known as telegrapher's equations [1–3]. From these equations, one can predict the speed of propagation of the current and voltage pulses along the transmission line. Under ideal (lossless) conditions, the predicted speed of propagation is equal to the speed of light in the medium where the transmission line is embedded. The two goals of this paper are as follows. First, to show that due to the finite mass of the electrons that carry currents in transmission lines, the speed of propagation of pulses can never be rigorously equal to the speed of light. Second, to find the upper bound for the speed of propagation.

The paper is organized as follows. First, we will give the standard equations that describe the propagation of voltages and currents along a transmission line and the conditions under which these equations are applicable. After that, we will demonstrate that the speed of propagation of voltage or current waves along transmission lines can never reach the speed of light due to physical properties of the conductors along which the waves are guided. The demonstration is followed by an estimation of an upper bound for the speed of propagation of waves along transmission lines. The paper ends with a discussion and general conclusions.

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2. TEM (TRANSVERSE ELECTROMAGNETIC) PROPAGATION ALONG A TRANSMISSION LINE

Let us consider the simplest configuration of a transmission line. It is represented by a cylindrical conductor of radius a located above a conducting ground plane at a height h . The line is located along the x -axis of the Cartesian coordinate system. Let us denote the current and voltage of the line by I and V . The two telegrapher's equations that describe the voltage and current for a uniform and lossless transmission line are given by [1, 2]

$$\frac{\partial I}{\partial x} + C' \frac{\partial V}{\partial t} = 0 \quad (1)$$

$$\frac{\partial V}{\partial x} + L' \frac{\partial I}{\partial t} = 0 \quad (2)$$

In the above equations, C' and L' are, respectively, the per-unit-length capacitance and inductance of the transmission line. These quantities are given by $C' = 2\pi\epsilon_0/\ln(2h/a)$ and $L' = \mu_0 \ln(2h/a)$. The assumptions that lead to these equations are as follows [3, 4]: 1) The current propagates along the line axis. 2) The sum of the line currents at any cross section of the line is zero. 3) The electromagnetic fields produced by electrical charges and currents along the line are confined in the transverse plane and perpendicular to the line axis (transverse electromagnetic — TEM). 4) The wavelengths associated with the signals propagating along the transmission line have to be much larger than the cross sectional dimension of the conductors and the separation between conductors. If the cross sectional dimension of the conductor is electrically small, then the first assumption is satisfied. The second assumption is satisfied if the ground plane is perfectly conducting so that the response of the ground can be represented by an image current. The third assumption can be satisfied only up to a certain threshold frequency, above which other modes of propagation appear. Under these assumptions, the voltage and current of the transmission line can be described by the above set of equations. The speed of propagation of the waves along the line, v , is then given by $v = 1/\sqrt{L'C'}$. For a transmission line in free space, this speed is equal to the speed of light. In other words, once all the conditions mentioned above are satisfied, the theory predicts that the speed of propagation of the current (or voltage) pulses is equal to the speed of light. This is exactly true if one assumes that the wires of the transmission lines are perfectly conducting. When the finite conductivity of the actual material of which the conductor is made is taken into account, one has to consider the effect of the resistance and the impedance created by the skin effect. In the presence of a finite conductivity conductor and/or ground, the propagation constant γ of the wave propagating along the transmission line is given by [1, 2]

$$\gamma = \sqrt{R'G' + (R'C' + L'G')j\omega + L'C'(j\omega)^2} \quad (3)$$

In the above equation, G' is the per-unit-length conductance of the medium in which the transmission line is embedded, R' the per-unit-length resistance, and ω the angular frequency of the wave propagating along the transmission line. In the case of a transmission line in air, one can make $G' = 0$ and the propagation constant reduces to

$$\gamma = \sqrt{R'C'j\omega + L'C'(j\omega)^2} \quad (4)$$

The resistance per unit length is controlled both by the conductivity of the material of the conductors and the penetration of the waves into the conductor as determined by the skin depth. Note that, in what follows, without loss of generality, we will consider only the losses in the conductor and assume the ground is perfectly conducting. The skin depth δ is given by [1, 2]

$$\delta = \sqrt{\frac{2}{\mu_0\omega\sigma}} \quad (5)$$

in which ω is the angular frequency, σ the conductivity of the material of the conductor, and μ_0 the magnetic permeability of free space.

In the presence of the skin effect, the resistance per unit length, R' , has to be replaced by the impedance Z' . The classical exact formula for the internal impedance per-unit-length of solid cylindrical

conductors, which takes the skin effect into account but ignores the proximity effect of other conductors, is given by [3, 5, 6]

$$Z' = \frac{k}{\sigma 2\pi a} \frac{J_0(ka)}{J_1(ka)} \quad (6)$$

In the above equation, a is the radius of the conductor, and J_0 and J_1 are the Bessel functions of the first kind with order zero and one, respectively. The parameter k in Equation (6) is given by

$$k = \frac{1-j}{\delta} \quad (7)$$

Thus, in the presence of the skin effect, the propagation constant of Equation (4) becomes

$$\gamma = \sqrt{Z'C'j\omega + L'C'(j\omega)^2} \quad (8)$$

If $\delta \ll a$, the expression for the impedance can be simplified by assuming that the current flows uniformly through a layer of thickness δ based on the DC resistivity of the material. In this case, the impedance reduces to

$$Z' \approx R' = \frac{1}{2\pi a \delta \sigma} \quad (9)$$

If $\delta \gg a$, the impedance reduces to the DC resistance per unit length as given by

$$R' = \frac{1}{2\pi a^2 \sigma} \quad (10)$$

These equations show that the speed of propagation of the waves along the transmission line is controlled in a complex manner by the material properties of the conductor. Now, we will consider another material property, namely the finite mass of the charge carriers or electrons, that has never been taken into account, to the best of the authors' knowledge, in the analysis of the propagation of waves along transmission lines.

3. THE KINETIC INDUCTANCE

Consider an electromagnetic wave propagating along a transmission line. The transmission line could be a single wire above a perfectly conducting ground plane or two identical parallel and cylindrical wires separated by a certain distance. Let us denote the area of the cross section of the wires of the transmission line by A . The electromagnetic wave propagating along this transmission line is associated with a current and a voltage wave in the transmission line. The current, I , in the conductor, can be expressed as [7]

$$I = n_e e A v_d \quad (11)$$

In the above expression, n_e is the number of free electrons per unit volume in the conductor, A the cross section of the conductor where current is flowing, e the electronic charge, and v_d the drift velocity of the electrons. The average kinetic energy density of the charge carriers per unit volume, in our case electrons, transporting the current is given by

$$E_k = \frac{1}{2} m_e n_e v_d^2 \quad (12)$$

In the above equation m_e is the mass of the electrons and v_d is the drift velocity. Note that this energy becomes zero if the mass of the electron is zero. The energy supplied to electrons per unit length of the conductor is given by

$$E_{k,ul} = \frac{1}{2} A m_e n_e v_d^2 \quad (13)$$

Combining Equations (11) and (13), the total kinetic energy per-unit-length transferred to the electrons can be expressed in terms of the transport current I and an equivalent per-unit-length inductance L_k as follows:

$$E_{k,ul} = \frac{m_e I^2}{2 n_e A e^2} = \frac{1}{2} L_k I^2 \quad (14)$$

This gives

$$L_k = \frac{m_e}{n_e A e^2} \quad (15)$$

It is important to point out that the parameter A is the effective cross section of the region available for current propagation. As mentioned in the previous section, this area depends on the frequency of the wave propagating along the transmission line, in connection with the skin effect. For frequencies where the skin depth is much larger than the radius of the conductor, it is equal to πr^2 . For pulses where the effective skin depth is much smaller than the radius, it is given by $2\pi r\delta$. In the literature, L_k is known as the kinetic inductance [8].

4. THE EFFECT OF KINETIC INDUCTANCE ON THE SPEED OF PROPAGATION

Equation (8) given in Section 2 is an expression for the propagation constant of the waves when one disregards the kinetic inductance. The speed of propagation of the wave, v , can be extracted from this expression and the result is

$$v = \frac{1}{\sqrt{L'C'}} \frac{1}{\text{Re} \left[\sqrt{\frac{Z'}{j\omega L'} + 1} \right]} \quad (16)$$

In the above expression 'Re' represents the real part of the expression inside the brackets. When the kinetic inductance is taken into account, the speed becomes

$$v = \frac{1}{\sqrt{(L' + L_k)C'}} \frac{1}{\text{Re} \left[\sqrt{\frac{Z'}{j\omega(L' + L_k)} + 1} \right]} \quad (17)$$

Observe that the effect of the kinetic inductance is to reduce the speed of propagation of the waves along the transmission line. Note also that, with the kinetic inductance included in the expression, the speed will never reach the speed of light even if the ideal condition $Z' = 0$ is satisfied. Thus, the ultimate speed that a wave propagating along a transmission line can reach is given by

$$v = \frac{1}{\sqrt{(L' + L_k)C'}} \quad (18)$$

For a transmission line in air, this ultimate speed is given by

$$v = c \frac{1}{\sqrt{1 + \frac{m_e}{\mu_0 n_e e^2 A \ln(2h/a)}}} \quad (19)$$

The speed becomes equal to the speed of light only when either the mass of the charge carriers becomes zero or their density becomes infinity. This will never be the case even under superconducting conditions.

5. DISCUSSION

In transmission line theory, the conductivity of the conductors of an ideal transmission line is considered to be infinite [1]. The only way to realize a perfect conductor is to make the electron density infinite. In this case, Equations (18) and (19) predict that the speed of propagation of pulses along the transmission line is equal to the speed of light, provided of course that other conditions necessary for TEM propagation are satisfied. However, the density of free electrons in a material is controlled by the quantum nature of atoms and it is always limited by the physical laws. Even in a superconductor, the electron density is not much different to that of a normal conductor and energy has to be spent in moving the electrons from their stationary state to the state where they are moving with drift speed. Thus, the kinetic inductance is applicable even in the case of superconductors [9].

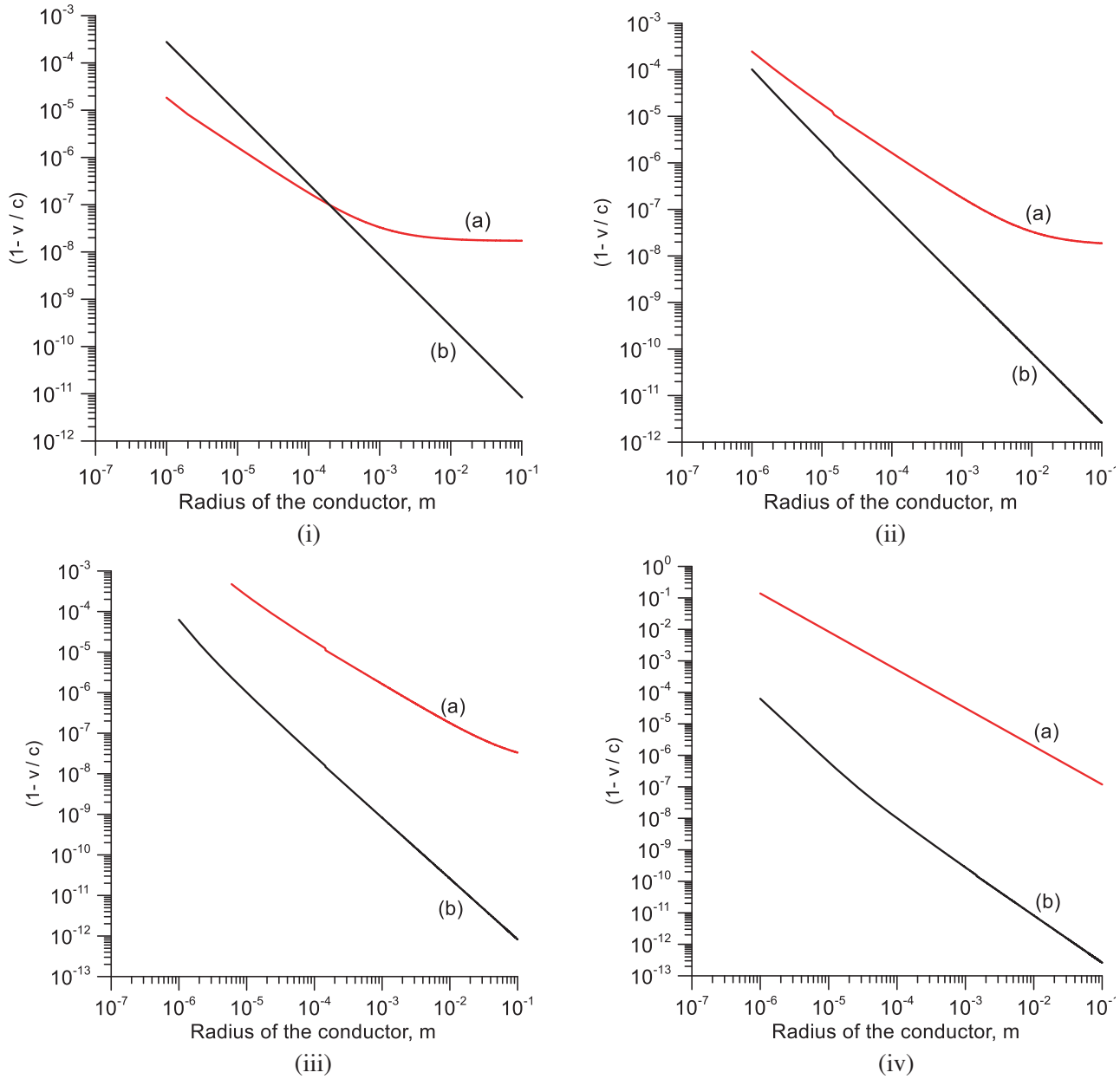


Figure 1. Wave speed as determined by Equation (16) (Curve (a), lossy conductor without kinetic inductance) and Equation (18) (Curve (b), lossless conductor with kinetic inductance). The maximum possible speed of propagation for a given configuration is given by curve (b). In the diagrams, the height of the conductor above the ground plane is equal to $10a$ and the wavelength of the wave is given by (i) $10^2 a$, (ii) $10^3 a$ (iii) $10^4 a$ and (iv) $10^5 a$.

One can utilize Equation (19) to study how the ultimate speed of propagation of a current pulse varies for different parameters of transmission lines. Consider a single conductor transmission line with a conductor of radius a located at a height h above a perfectly conducting ground plane. Let σ be the conductivity of the material of the conductor. In order to satisfy the condition that $h \gg a$, we assume that $h = 10a$. We consider a copper conductor for which $\sigma = 58 \times 10^6$ S/m. The free electron density of copper, n_e in our equations, is equal to $8.49 \times 10^{28}/\text{m}^3$. Figure 1 shows the speeds as predicted by Equations (16) and (18) for different values of conductor radius and for different frequencies. For each conductor height h , the wavelength is selected to be either $10h$, $100h$, $1000h$ or $10000h$. The

results are shown in plots (i), (ii), (iii) and (iv), respectively, in Figure 1. In each plot, the speed is plotted as $1 - v/c$. The curve (a) in each plot corresponds to Equation (16), and the curve (b) refers to Equation (18), respectively corresponding to a lossy and a lossless conductor.

One can see from the plots given in Figure 1, for example plot (i), that for some configurations the speed determined by Equation (16) lies much closer to the speed of light than the upper bound as predicted by Equation (18). In this case, the actual speed of propagation is equal to the upper bound as determined by Equation (18). In other cases, for example plots (ii), (iii) and (iv), the speed determined by Equation (16) lies above the upper bound as determined by Equation (18). In these cases the actual speed of propagation takes place at a value less than the upper bound given by Equation (18). In either case, the actual speed of propagation is either less than or equal to the upper bound.

In the analysis presented here, we have considered TEM propagation along transmission lines. Even under conditions where the TEM mode of propagation is not valid, waves can propagate in other modes along transmission lines. Even in these cases, currents are set up in the conductors and, as in the TEM mode of propagation considered here, some energy has to be diverted from the wave to the kinetic energy of the electrons. This will again cause these waves to propagate at speeds less than the speed of light.

In this paper, we have considered the effect of the kinetic inductance only on the speed of propagation of waves on transmission lines. However, note that the presence of the kinetic inductance not only affects the speed of propagation but also the impedance of the transmission line. This can affect both the shielding effectiveness of the cables and the microstrip interconnected signal propagation [10, 11]. These effects can be studied using numerical procedures such as FDTD or FETD. Such studies are under investigation.

6. CONCLUSIONS

Standard transmission line theory suggests that under appropriate conditions, the propagation speed of current pulses along conductors is equal to the speed of light in free space. The results presented in this paper show that this is true only if the mass of the electrons that transport current in conductors is assumed equal to zero. The results show that when the mass of the electrons is taken into account, the speed of propagation of the waves along transmission lines can never reach the speed of light unless one makes the unphysical assumption that the electron density in the conductor is infinite.

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