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The Maxwell Stress Tensor and Electromagnetic Momentum

Artice Davis[1, *] and Vladimir Onoochin[2]

Abstract—In this paper, we discuss two well-known definitions of electromagnetic momentum, $\rho\mathbf{A}$ and $\epsilon_0[\mathbf{E}\times\mathbf{B}]$. We show that the former is preferable to the latter for several reasons which we will discuss. Primarily, we show in detail—and by example—that the usual manipulations used in deriving the expression $\epsilon_0[\mathbf{E}\times\mathbf{B}]$ have a serious mathematical flaw. We follow this by presenting a succinct derivation for the former expression. We feel that the fundamental definition of electromagnetic momentum should rely upon the interaction of a single particle with the electromagnetic field. Thus, it contrasts with the definition of momentum as $\epsilon_0[\mathbf{E}\times\mathbf{B}]$ which depends upon a (defective) integral over an entire region, usually all space.

1. INTRODUCTION

The correct form of the electromagnetic energy-momentum tensor has been debated for almost a century. A large number of papers have been written [1], some of which concentrate upon a single particle interaction with the electromagnetic field in free space [2, 3]; others discuss the interaction between a field and a dielectric material upon which it impinges [4]. In works which discuss the Minkowski-Abraham controversy, the system considered is usually on the macroscopic level. On this level, singularities caused by point charges cannot be treated. We show that neglecting the effect due the properties of classical charges creates difficulties which cannot be resolved within the framework typically discussed by textbooks on classical electrodynamics [5].

Our present concept of electromagnetic field momentum density has two interpretations:

(i) As $\epsilon_0[\mathbf{E}\times\mathbf{B}]$, the scaled Poynting vector.

(ii) As A, the vector potential.

There have been many papers and books written which advocate one of these interpretations as opposed to the other (see [1, 2, 5, 6]), but no consensus seems to have been reached. This paper was written to expose several fundamental shortcomings with the first interpretation and to offer an important and (we believe) definitive argument in favor of the second.

Let us begin by considering the Lorentz force

$$\mathbf{F_q} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}. \tag{1}$$

We extend this formula to the case of an infinitesimal element of a distributed test charge, assuming that a differential volume has an equivalent charge $dq = \rho d\tau$, where ρ is the charge density and $d\tau$ the differential element of volume, and that it moves rigidly along a trajectory such that its instantaneous velocity is $\mathbf{v}$. Thus, we will assume that the force on a differential element of charge is

$$d\mathbf{F} = \rho d\tau\mathbf{E} + \rho d\tau\mathbf{v}\times\mathbf{B}. \tag{2}$$

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* Corresponding author: Artice Davis (artice.davis@sjsu.edu).

[1] San Jose State University, CA, USA. [2] Sirius, 3A Nikoloyamski Lane, Moscow, Russia.

The force per unit volume of charge is, therefore,

$$\mathbf{f} = \frac{d\mathbf{F}}{d\tau} = \rho\mathbf{E} + \rho\mathbf{v} \times \mathbf{B}. \tag{3}$$

We will now present the "usual" development of the Maxwell stress tensor. Such a development is presented in many textbooks on classical electrodynamics, but we will follow the one by Griffiths [5].

2. DERIVATION OF THE MAXWELL STRESS TENSOR

Consider the Lorentz force Equation (3) and use Maxwell's equations to write

$$\begin{aligned}
\mathbf{f} &= [\epsilon_0 \nabla \cdot \mathbf{E}]\,\mathbf{E} + \left[\frac{1}{\mu_0}\nabla \times \mathbf{B} - \epsilon_0 \partial_t \mathbf{E}\right] \times \mathbf{B} \\
&= [\epsilon_0 \nabla \cdot \mathbf{E}]\,\mathbf{E} - \frac{1}{\mu_0}\mathbf{B} \times [\nabla \times \mathbf{B}] - \epsilon_0 \partial_t\,[\mathbf{E} \times \mathbf{B}] + \epsilon_0 \mathbf{E} \times \partial_t \mathbf{B} \\
&= [\epsilon_0 \nabla \cdot \mathbf{E}]\,\mathbf{E} - \frac{1}{\mu_0}\mathbf{B} \times [\nabla \times \mathbf{B}] - \epsilon_0 \partial_t\,[\mathbf{E} \times \mathbf{B}] \\
&\quad -\epsilon_0 \mathbf{E} \times [\nabla \times \mathbf{E}]\,.
\end{aligned} \tag{4}$$

We now call upon the well-known vector identity

$$\mathbf{g} \times [\nabla \times \mathbf{g}] = \nabla \left[\frac{g^2}{2}\right] - [\mathbf{g} \cdot \nabla]\,\mathbf{g}, \tag{5}$$

and add the zero-valued function $[\nabla \cdot \mathbf{B}]/\mu_0$ to obtain

$$\mathbf{f} = \epsilon_0[\nabla \cdot \mathbf{E}]\mathbf{E} + \frac{1}{\mu_0}[\nabla \cdot \mathbf{B}]\mathbf{B} - \nabla\left[\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\right] + \epsilon_0\,[\mathbf{E} \cdot \nabla]\,\mathbf{E} + \frac{1}{\mu_0}\,[\mathbf{B} \cdot \nabla]\,\mathbf{B} - \partial_t\,[\mathbf{E} \times \mathbf{B}]\,. \tag{6}$$

Consider the ith component:

$$\begin{aligned}
f_i &= \epsilon_0[\partial_j E_j]E_i + \frac{1}{\mu_0}[\partial_j B_j]B_i - \partial_i\left[\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\right] + \epsilon_0[E_j\partial_j]E_i \\
&\quad + \frac{1}{\mu_0}[B_j\partial_j]B_i - \epsilon_0\partial_t[\epsilon_{ijk}EjBk],
\end{aligned} \tag{7}$$

which we can rewrite as

$$\begin{aligned}
f_i &= \partial_j\left[\epsilon_0 E_i E_j + \frac{1}{\mu_0}B_i B_j - \left(\frac{\epsilon_0 E^2}{2} + \frac{E^2}{\mu_0}\right)\delta_{ij}\right] \\
&\quad -\partial_t\left[\epsilon_{ijk}E_j B_k\right] = \partial_j T_{ij} - \partial_t\left[\epsilon_{ijk}E_j B_k\right].
\end{aligned} \tag{8}$$

T_{ij} are the components of a second order tensor — a 3×3 matrix — called the Maxwell stress tensor:

$$T_{ij} = \epsilon_0 E_i E_j + \frac{1}{\mu_0}B_i B_j - \left[\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\right]\delta ij. \tag{9}$$

We see that it is symmetric because $T_{ji} = T_{ij}$ for any choice of the indices i, j. Using dyadic notation, we write this tensor in the form

$$\mathbf{T} = T_{ij}\mathbf{e}_i\mathbf{e}_j \tag{10}$$

and define its divergence by

$$\nabla \cdot \mathbf{T} = \mathbf{e}_k\partial_k \cdot T_{ij}\mathbf{e}_i\mathbf{e}_j = \partial_k T_{ij}\delta_{ki}\mathbf{e}_j = \partial_i T_{ij}\mathbf{e}_j. \tag{11}$$

Since $\mathbf{T}$ is symmetric, we can rewrite this as

$$\nabla \cdot \mathbf{T} = [\partial_i T_{ji}]\,\mathbf{e}_j = [\partial_j T_{ij}]\,\mathbf{e}_i. \tag{12}$$

Therefore, we can rewrite Equation (7) as

$$\mathbf{f} = \nabla \cdot \mathbf{T} - \epsilon_0\partial_t\,[\mathbf{E} \times \mathbf{B}]\,. \tag{13}$$

Equation (13) is the force per unit volume on a distributed charge. Focusing on a differential element having volume $d\tau$, we can write the force on it as

$$d\mathbf{f} = \nabla \cdot \mathbf{T}\, d\tau - \epsilon_0 \partial_t \left[\mathbf{E} \times \mathbf{B}\right] d\tau. \tag{14}$$

Now suppose that we want to find the force on a charge distributed over a region R. We merely integrate the preceding equation over this region, getting

$$\mathbf{F} = \int_R \nabla \cdot \mathbf{T} d\tau - \epsilon_0 \int_R \partial_t \left[\mathbf{E} \times \mathbf{B}\right] d\tau. \tag{15}$$

Note that we are only considering the net force on the charge in the region R. Any self forces are internal to R and serve only to distort the region.

3. CRITIQUE OF THE CONVENTIONAL FIELD MOMENTUM INTERPRETATION

The usual derivation typically proceeds like this. If $\mathbf{T}$ satisfies the conditions for applying the divergence theorem for a region R, we can write the first integral in the form

$$\int_R \nabla \cdot \mathbf{T} d\tau = \oint_{\partial R} d\mathbf{S} \cdot \mathbf{T} = \left[\oint_{\partial R} T_{ij} dS_i\right] \mathbf{e}_j, \tag{16}$$

where ∂R is the boundary of R. Using this result in Equation (15) we find that the net translational force on the region R is

$$\mathbf{F} = \oint_{\partial_R} d\mathbf{S} \cdot \mathbf{T} - \epsilon_0 \int_R \partial_t \left[\mathbf{E} \times \mathbf{B}\right] d\tau. \tag{17}$$

Letting

$$\mathbf{F} = m\mathbf{a} = \frac{d\mathbf{P}_{mech}}{dt} \tag{18}$$

and

$$\epsilon_0 \int_R \partial_t \left[\mathbf{E} \times \mathbf{B}\right] d\tau = \frac{d}{dt}\left[\epsilon_0 \int_R \mathbf{E} \times \mathbf{B} d\tau\right] = \frac{d\mathbf{P}_{em}}{dt}, \tag{19}$$

we can rewrite Equation (17) in the form

$$\frac{d\left(\mathbf{P}_{mech} + \mathbf{P}_{em}\right)}{dt} = \oint_{\partial R} d\mathbf{S} \cdot \mathbf{T}. \tag{20}$$

Equation (19) defines the classical electromagnetic field momentum.

Now, the argument continues, if the fields constituting $\mathbf{T}$ approach zero fast enough at infinity, then we can imagine the region R to expand to be all space, and the surface integral is zero. We are left with

$$\frac{d\left(\mathbf{P}_{mech} + \mathbf{P}_{em}\right)}{dt} = 0. \tag{21}$$

We then see that conservation of the total momentum, namely mechanical momentum plus electromagnetic momentum, holds.

There are a number of things wrong with this typical argument (and the associated definition of field momentum) which we will now discuss.

(i) The conditions for application of the divergence theorem do not apply. If we investigate a typical term in the stress tensor, we see that one component in that term is $\epsilon_0 \partial_j [E_i E_j]$ and $E_i = -\partial_i \phi - \partial_t A_i$. Thus, if we assume for simplicity that $\partial_t \mathbf{A}_i = 0$ we see that we must compute $\partial_j [\partial_i \phi \partial_j \phi]$ (with no sum implied on j. Let us take as an example a single charge q' located at $\mathbf{r}'$ which generates $\mathbf{E}$. Then we have $\phi = q'/(4\pi\epsilon_0 R)$ where, as usual, $R = |\mathbf{r} - \mathbf{r}'|$ and $\mathbf{r}$ is the point at which the field is to be computed. Then

$$\phi = \frac{q'}{4\pi\epsilon_0 R}. \tag{22}$$

Dropping the constant factors, we see that we must compute (with no sum implied on j)

$$\begin{aligned}\partial_j\left[\left(\partial_i R^{-1}\right)\left(\partial_j R^{-1}\right)\right] &= \partial_j\left[\left(-R^{-2}\frac{x_i-x_i'}{R}\right)\left(-R^{-2}\frac{x_j-x_j'}{R}\right)\right]\\ &= \partial_j\left[R^{-6}\left(x_i-x_i'\right)\left(x_j-x_j'\right)\right]\\ &= -6R^{-8}\left(x_i-x_i'\right)\left(x_j-x_j'\right)^2+R^{-6}\delta_{ij}\left(x_j-x_j'\right)\\ &\quad +3R^{-6}\left(x_i-x_i'\right)=-\frac{6\left(x_i-x_i'\right)\left(x_j-x_j'\right)^2}{R^8}+\frac{4\left(x_i-x_i'\right)}{R^6}.\end{aligned}\tag{23}$$

We see, therefore, that the integrand of the surface integral has a singularity of order $1/R^5$ due to the term $\partial_i E_i \sim 1/R^3$ multiplied by the term $E_i \sim 1/R^2$. Thus, we cannot legitimately reason that the surface integral extending to infinity is zero. For this reason the divergence theorem does not apply [7]. But this means that we must interpret the first integral in Eq. (15) as another force, and hence the time derivative of another component of momentum.

(ii) The interpretation of field momentum as the integral over all space is not an adequate depiction of momentum — which one ordinarily thinks of relative to a particle or finite volume of matter in space.

(iii) We note that even if one ignores the aforementioned failure of the divergence theorem to be applicable, there is yet another problem. We cannot simply bring the partial derivative outside the second integral in Eq. (19) because the region is moving; that is, we have $R = R(t)$. The exchange of differentiation and integration becomes a more complex issue, and yet one thinks of the force on a particle or body as being the *total* derivative of momentum with respect to time — not as a partial derivative.

4. A TWO PARTICLE EXAMPLE: A PARADOX.

Our conventional derivation of the stress tensor is presented in many textbooks, but it is too schematic in nature — too abstract — and for this reason the flaws in the derivation which we have remarked on are overlooked. To further illustrate this, let us consider a system of two equal charges of opposite sign with the positive charge fixed and the negative charge rotating around the positive one. If the linear velocity of the rotating charge is small ($v \ll c$) then we are able to use the Darwin approximation [8]. Doing so, we find that the sum of the forces in this system is [9]

$$\sum_l \mathbf{F}_l = \mathbf{F}_+ + \mathbf{F}_- = \int_R \rho_- \mathbf{E}_+ d\tau + \int_R \rho_+ \mathbf{E}_- d\tau = \frac{q^2}{4\pi\varepsilon_0 r^2} - \frac{q^2}{4\pi\varepsilon_0 r^2}\left(1+\frac{v^2}{2c^2}\right) = -\frac{q^2 v^2}{8\pi\varepsilon_0 c^2 r^2}, \tag{24}$$

which is clearly nonzero. The forces in this system are determined only by the electric fields — the magnetic components are zero. This means that the terms involving $\epsilon_0\left[\mathbf{E}\times\mathbf{B}\right]$ are zero. (We are assuming that all fields are bounded, that is they approach zero fast enough for convergence of the integral.) Therefore, the surface integral of the Maxwell tensor is equal to zero if the surface of integration is a sphere of infinitely large radius — despite the fact that there are clearly unbalanced forces in this system.

5. A BETTER DERIVATION OF FIELD MOMENTUM

We start with the classical Lorentz force law in Eq. (1) for a *particle* moving along a trajectory given by $\mathbf{r} = \mathbf{g}(t)$ with corresponding velocity $\mathbf{v}(t) = \dot{\mathbf{g}}(t)$. Thus, we obviate all those assumptions regarding differential elements of charge and subsequent interpretation in terms of fields alone. In invoking the Lorentz force formula, we are assuming that it is only the instantaneous velocity that is important, not the acceleration, etc. We write

$$\mathbf{f}(t) = \frac{d}{dt}[m\mathbf{v}] = q\mathbf{E} + q\mathbf{v}(t)\times\mathbf{B} = -q\nabla\phi - q\partial_t\mathbf{A} + q\mathbf{v}(t)\times\left[\nabla\times\mathbf{A}\right]. \tag{25}$$

Now use the identities

$$\mathbf{v}(t) \times \left[\nabla \times \mathbf{A}\right] = \nabla\left[\mathbf{v} \cdot \mathbf{A}\right] - \left[\mathbf{v}(t) \cdot \nabla\right]\mathbf{A} \tag{26}$$

and

$$\frac{d}{dt}\mathbf{A} = \partial_t \mathbf{A} + \left[\mathbf{v} \cdot \nabla\right]\mathbf{A} \tag{27}$$

to write

$$\mathbf{v}(t) \times \left[\nabla \times \mathbf{A}\right] = \nabla\left[\mathbf{v}(t) \cdot \mathbf{A}\right] + \partial_t \mathbf{A} - \frac{d}{dt}\mathbf{A}. \tag{28}$$

Using this result in our force equation gives

$$\frac{d}{dt}[m\mathbf{v}(t)] = -q\nabla\phi + \nabla[\mathbf{v}(t) \cdot \mathbf{A}] - q\frac{d}{dt}\mathbf{A}. \tag{29}$$

Rearranging, we get

$$\frac{d}{dt}\left[m\mathbf{v} + q\mathbf{A}\right] = -q\nabla\left[\phi - \mathbf{v}(t) \cdot \mathbf{A}\right]. \tag{30}$$

This has the form

$$\frac{d\mathbf{p}}{dt} = -q\nabla\left[\psi\right], \tag{31}$$

where the momentum is

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A} \tag{32}$$

and the velocity dependent potential is

$$\psi = \phi - \mathbf{v} \cdot \mathbf{A}. \tag{33}$$

Thus, if one believes in the original Lorentz force equation then the field momentum *must* be given by $q\mathbf{A}$. We feel that this derivation† is much more logical and rigorous than the conventional one leading to the field momentum in terms of the integral of $\mathbf{E} \times \mathbf{B}$.

The use of this form of the EM momentum gives a simple explanation of the 'paradox' explained above. Since the negative charge rotates in a circular orbit with the constant angular speed ω, its linear velocity is $v = \omega r$, and the EM momentum, limited to the region occupied by the charge, is

$$\mathbf{P}_{\mathrm{EM}} = \int_{\tau_{ch}} \rho \mathbf{A} d\tau = q_+ \mathbf{A} = \frac{1}{4\pi\epsilon_0}\frac{q^2\mathbf{v}}{2c^2 r}$$

where the expression for the vector potential in the Darwin approach

$$\mathbf{A} = q\,\frac{\mathbf{v} + (\mathbf{v} \cdot \hat{\mathbf{n}})\,\hat{\mathbf{n}}}{8\pi\epsilon_0 c^2 r}$$

is used (Equation (65.6) of [10]). Then for the system of two charges

$$\frac{d\mathbf{A}}{dt} = \frac{q}{8\pi\epsilon_0 c^2 r}\frac{d\mathbf{v}}{dt} = \frac{q\hat{\mathbf{n}}}{4\pi\epsilon_0 r^2}\frac{v^2}{2c^2}$$

and with accuracy to $1/c^2$ the law of the total momentum conservation is fulfilled

$$\sum_{l=1}^{2} \mathbf{F}_l + q\frac{d\mathbf{A}}{dt} = 0\,. \tag{34}$$

As one can see, the Maxwellian form of the EM momentum, $\mathbf{P}_{em} = q\mathbf{A}$, allows one to explain the fulfillment of total momentum conservation even in the case of the absence of the magnetic component of the Lorentz force.

Finally, we are able to conclude that the explanations of some paradoxes based on the use of the Poynting-Poincaré form of the EM momentum (see, for example, [9]) should be reconsidered since this form of $\mathbf{P}_{em}$ does not provide the fulfillment of the total momentum conservation.

† Our derivation is similar to that given by Semon and Taylor [3].

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