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Novel Computational Technique for Time-Dependent Heat Transfer Analysis Using Fast Inverse Laplace Transform

Seiya Kishimoto*, Shohei Nishino, and Shinichiro Ohnuki

Abstract—A novel computational technique is proposed for heat conduction analysis. The heat transfer equation is expanded in the complex frequency domain and solved using the finite difference method (FDM). The results in the complex frequency domain are transformed into the time domain via fast inverse Laplace transform. In the proposed approach, the instantaneous temperature at a specific time can be easily obtained. Moreover, the computation time for the conventional explicit FDM is reduced by employing the time-division parallel computing method.

1. INTRODUCTION

Heat conduction analysis is crucial for designing electrical devices. Thus, various methods have been applied in the development stage of electrical devices and have been implemented in consumer software [1–7]. The conventional explicit time-domain finite difference method (TD-FDM) is easy to implement and suitable for the treatment of multiple media [8, 9]. The time response of the temperature can be analyzed by performing a sequential calculation for updating the observation time. However, based on the step size of the space discretization, the stable condition constrains time increment. Conversely, implicit methods do not constrain the stable condition. Moreover, the computational accuracy of the implicit methods depends on the time step size.

In this paper, a novel computational method relying on the finite difference method (FDM) and fast inverse Laplace transform (FILT) is proposed for heat conduction analysis in the time domain [10]. FILT has been applied to the time analysis of electromagnetic problems [11, 12]. In heat conduction problems, the heat transfer equation is solved by expanding the FDM in the complex frequency domain. The solution is transformed into the time domain using FILT. In the proposed method, the instantaneous temperature at a specific time can be obtained without any sequential calculation. The time step size of our method is not restricted by the requirements of stable conditions. The computation time required for time response analysis can be reduced by selecting a large time step. Moreover, the computation time for the conventional TD-FDM can be reduced by employing the time-division parallel computing method proposed in [12]. To perform parallel computing, the initial temperature distributions for each thread are obtained using the FDM and FILT.

We investigated the computation time and accuracy of the proposed method and its usefulness for heat conduction analysis. The results obtained were compared with an analytical solution to validate the computational accuracy of the proposed method. Furthermore, time division parallel computing is used in the TD-FDM to obtain the initial temperature distribution for the explicit FDM. The proposed method is suitable for researching thermal transient states in detail because the temperature distribution for an arbitrary time can be obtained. Moreover, a specific period can be analyzed thoroughly.

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* Corresponding author: Seiya Kishimoto (kishimoto.seiya@nihon-u.ac.jp).
The authors are with the Department of Electrical Engineering, College of Science and Technology, Nihon University 1-8-14, Surugadai, Kanda, Chiyoda-ku, Tokyo 101-8308, Japan.

The remainder of this paper is organized as follows. Section 2 describes the formulation of the proposed computational method and implementation of the time-division parallel computing method. Heat transfer equations in the complex frequency domain and formulation of FILT are outlined. Section 3 presents the computational results and comparisons with the analytical solution. The computation time and accuracy of the proposed method are validated. Finally, conclusions are provided in Section 4.

2. FORMULATION

2.1. Heat Transfer Equation in Complex Frequency Domain

The heat transfer of a two-dimensional stationary medium can be described by the heat transfer equation in the time domain [8, 9]. It is written as

$$\frac{\partial \phi(t)}{\partial t} = \frac{k}{\rho c}\left(\frac{\partial^2 \phi(t)}{\partial x^2} + \frac{\partial^2 \phi(t)}{\partial y^2}\right) \tag{1}$$

where $\phi(t)$ is the heat distribution in the time domain, ρ the density of the medium, c is specific heat, and k the thermal conductivity. Equation (1) can be transformed into the complex frequency domain using the Laplace transform, as follows:

$$\Phi = \int_0^\infty \phi(t) e^{-st} dt \tag{2}$$

$$s\Phi - \phi_0 = \int_0^\infty \frac{\partial \phi(t)}{\partial t} e^{-st} dt \tag{3}$$

$$s\Phi - \phi_0 = \frac{k}{\rho c}\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) \tag{4}$$

where s is the complex frequency given by $\sigma + j\omega$, Φ the image function for heat distribution, and ϕ_0 the initial heat distribution.

The FDM is used for obtaining the heat distribution in the complex frequency domain. Fig. 1 shows the two dimensional space discretization. The heat transfer equation in the complex frequency

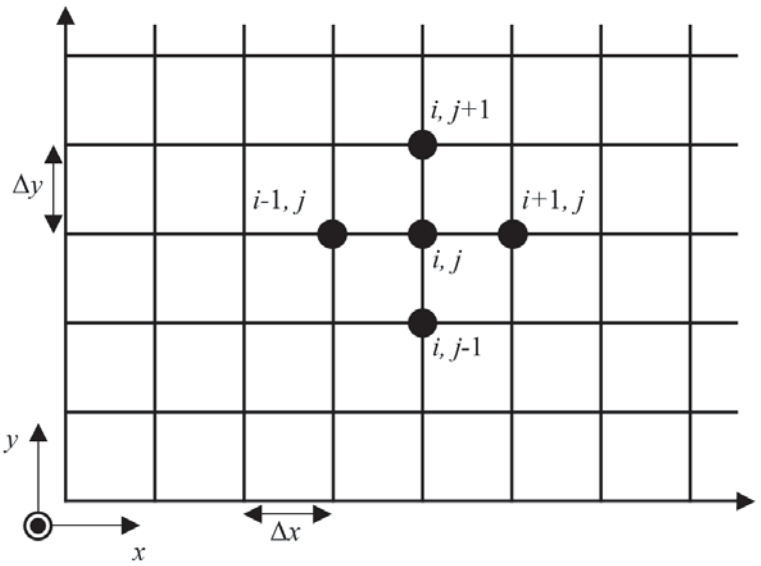

Figure 1. Space discretization for the finite difference scheme, i and j indicate the discretized points for the x and y directions, respectively. Δx and Δy represent the size of the unit cell.

domain, Equation (4), can be represented by

$$s\Phi(i,j)-\frac{k}{\rho c}\left(\frac{\Phi(i+1,j)-2\Phi(i,j)+\Phi(i-1,j)}{\Delta x^2}+\frac{\Phi(i,j+1)-2\Phi(i,j)+\Phi(i,j-1)}{\Delta y^2}\right)=\phi_0(i,j) \quad (5)$$

where i and j are the discretized points for the x and y directions, respectively, and Δx and Δy represent the size of a unit cell. Considering that Equation (5) satisfies all discretized points in the computational space, the linear equation can be obtained as $\bar{A}\mathbf{x}=\mathbf{b}$, where $\bar{A}$ is the operator matrix, $\mathbf{x}$ the unknown temperature vector, and $\mathbf{b}$ the initial temperature vector. The unknown heat distribution in the complex frequency domain can be obtained by solving the linear equation.

2.2. Fast Inverse Laplace Transform

In the proposed method, the heat distribution in the complex frequency domain is transformed into the time domain by applying FILT. Particularly, in FILT, the exponential function in the Bromwich integral is replaced by the cosine in the hyperbolic function [10]. This approximated function is substituted into the integrand, and the residue theorem is used for computing the Bromwich integral. Subsequently, the infinite series is truncated, and the alternating series is considered. Therefore, the approximated time-domain function, $f_{ec}(t,\alpha)$, can be evaluated using the following equation:

$$f_{ec}(t,a)=\frac{e^{\alpha}}{t}\left(\sum\nolimits_{n=1}^{L}F_n+\frac{1}{A_{p0}}\sum\nolimits_{q=1}^{p}A_{pq}F_{L+q}\right) \quad (6)$$

where,

$$F_n = (-1)^n \operatorname{Im}[F(s_n)] \quad (7)$$

$$s_n = \frac{\alpha+j(n-0.5)\pi}{t}, \quad (8)$$

$$A_{pp} = 1, \quad A_{p0}=2^p, \quad (9)$$

$$A_{pq} = A_{pq-1}-\frac{(p+1)!}{q!\,(p+1-q)!}, \quad (10)$$

where $F(s)$ is the image function of the original time-domain function $f(t)$, L the truncation number, p the number of terms in the Euler transformation, and n and q are the indices of summation. The second summation in Equation (6) performs the Euler transformation with the number of terms p, to achieve the rapid convergence of the alternating series. Here, the accuracy of f_{ec} can be controlled by the approximation parameter α. When calculating Equation (6), sequential calculations such as the conventional TD-FDM are not needed. The instantaneous temperature at a specific observed time can be obtained independently. Furthermore, the time step size can be selected arbitrarily to obtain a time response.

2.3. Time-Division Parallel Computing Method

Using FILT, parallel computing for the TD-FDM can be performed for the observation time. Figs. 2(a) and (b) depict the flowchart of the proposed parallel computational method. The complete observation time, t_0-t_N, is divided into the number of threads, N. To compute the temperature using TD-FDM in each thread for observation period t_n-t_{n+1} $(n=0,1,2,\ldots,N-1)$, the temperature distributions at t_n are required for the initial values. In FILT, the instantaneous temperature distribution at a specific observation time can be computed. This result can be used instead of the initial values for TD-FDM. As data exchange is unnecessary during computation, TD-FDM computation in each thread can be performed independently.

3. COMPUTATIONAL RESULTS

The results obtained using the proposed method for heat transfer analyses were compared with an analytical solution to verify the efficacy of the method. As shown in Fig. 3, the computational model

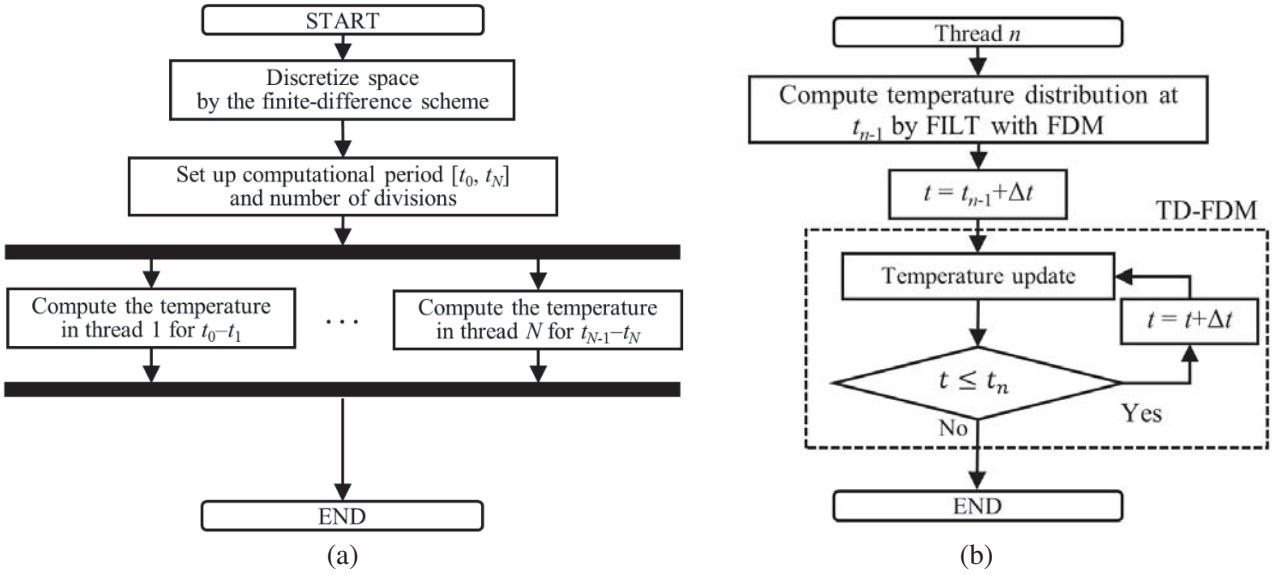

Figure 2. Flowchart for the time-division parallel computing method. (a) Distributing observation time to threads. (b) Computing the temperature distribution in each thread. The initial temperature distribution can be obtained by FILT with FDM expanded in the complex frequency domain.

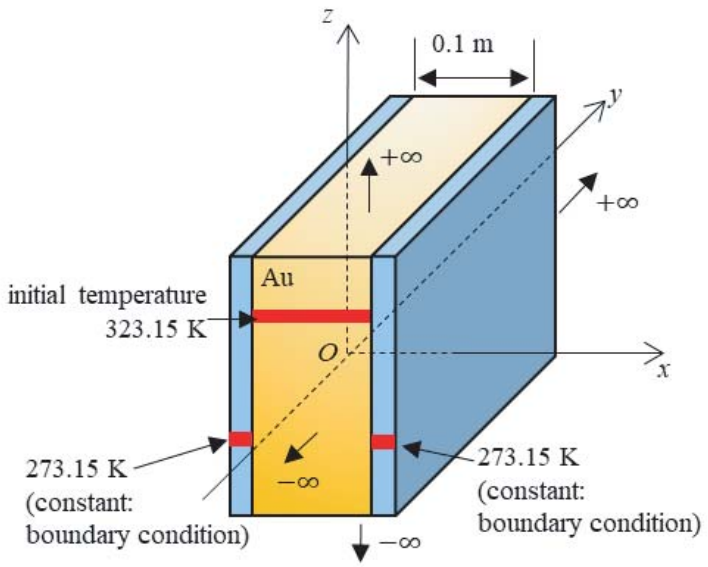

Figure 3. Computational model of a two-dimensional object for validating the proposed heat transfer computational method.

is a one dimensional object composed of gold, with a length of 0.1 m. The parameters of gold are $\rho = 19.32 \times 10^3\,\mathrm{kg/m^3}$, $k = 295\,\mathrm{W/(m \cdot k)}$, and $c = 130\,\mathrm{J/kg \cdot K}$, and the space discretization step is $\Delta x = 1.0 \times 10^{-5}$ [13]. The initial temperature is 323.15 K for the complete computational area. The Dirichlet boundary condition is assumed, where the temperature is 273.15° at $x = 0$ and 0.1 m. The observation plane is set at the center of the object. Fig. 4 shows the time response of the temperature at the observation plane. The results obtained using our method and the TD-FDM are in good agreement. The proposed method was also evaluated for computational accuracy. To this end, the convergence test

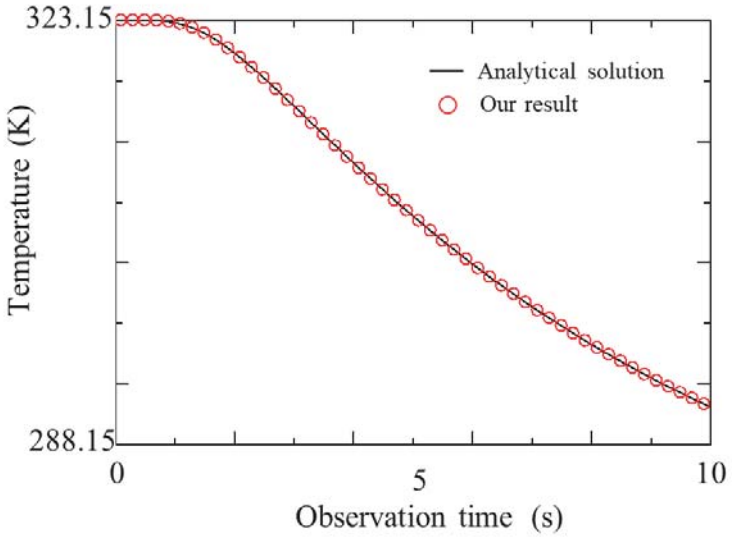

Figure 4. Temperature-time response at the observation plane. The results from the proposed method are in good agreement with the analytical solution at all observation times.

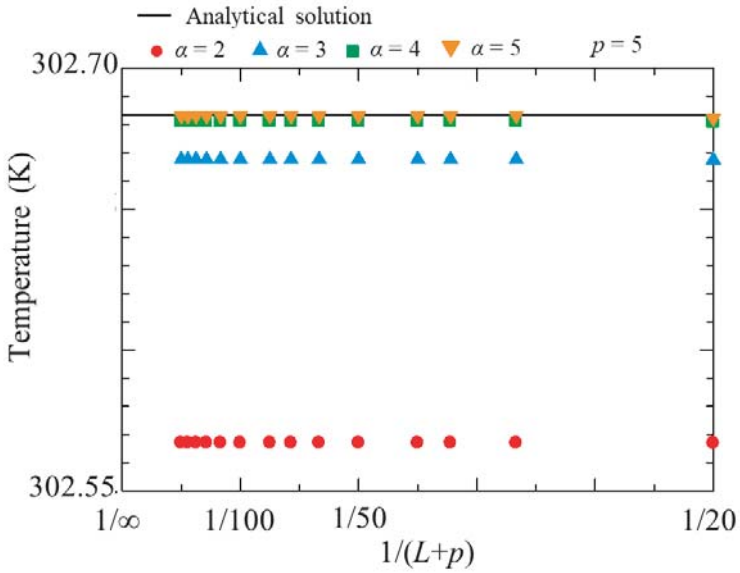

Figure 5. Convergence test for various truncation numbers, L. The result converges as the truncation number increases. The accuracy of the result increases with the truncation number.

was performed by varying the truncation number L at $t = 6.3\,\mathrm{s}$, as shown in Fig. 5. Here, the number of terms p for the Euler transformation is 10. We focused on the accuracy of the results from the proposed model for various values of the approximation parameter α. The plots for various truncation numbers converged to a certain value, approaching the analytical solution. Table 1 lists the convergence values for various approximation parameters. A convergence value can be obtained by increasing the approximation parameter, as denoted by the digits in red. The number of accuracy digits can be controlled by α.

The heat transfer problem for a two-dimensional object composed of gold as shown in Fig. 6 was studied to verify that time-division parallel computing can be performed in the TD-FDM. The size of the object is $0.1\,\mathrm{m} \times 0.1\,\mathrm{m}$. The initial temperature distribution at $t = 0\,\mathrm{s}$ is given by the

Table 1. Convergence value for approximation parameter α. The number of digits of accuracy can be controlled by α.

Approximation parameter α	Temperature (K)
2	302.5671553
3	302.6675805
4	302.6812241
5	302.6830717
6	302.6833217
Analytical solution	302.6833507

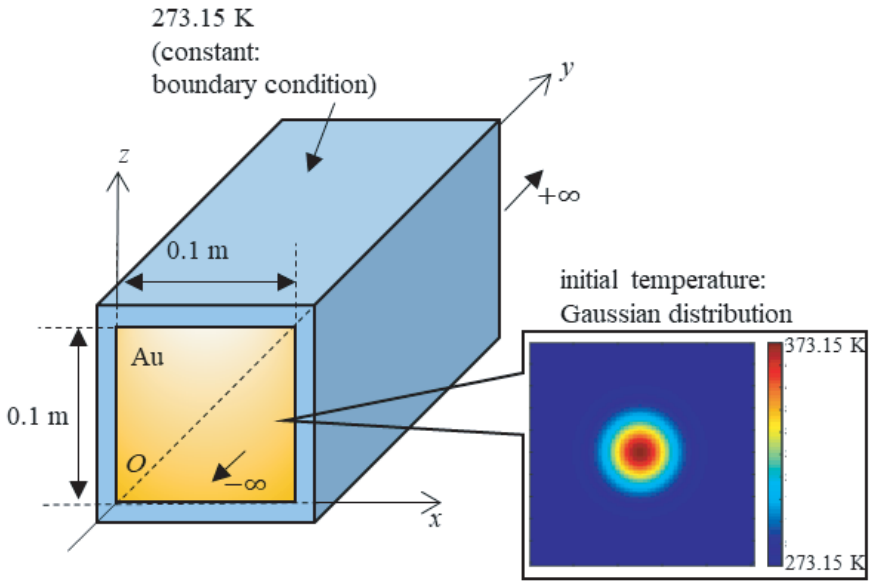

Figure 6. Computational model of a two-dimensional object made of gold for validating the proposed heat transfer computation method.

Gaussian distribution, for which the peak value is 373.15 K. The boundary condition (273.15°) for the computational area is the same as the Dirichlet boundary condition.

Figures 7(a) and (b) show the temperature distributions at $t = 6$ s, obtained by the proposed method and the TD-FDM, respectively. Fig. 7(c) shows the relative error between the results obtained by the proposed method and the conventional FDM. The computational error is less than 0.005% for the complete computational area. Therefore, our results can be used instead of the initial values for the conventional TD-FDM.

The proposed time-division parallel computing method was also compared with the conventional explicit TD-FDM. Fig. 8 shows the temperature-time response at the center of the object. Here, the parameters were set as $N = 10$, $t_0 = 0$ s, and $t_N = 10$ s. The triangle and circle indicate the initial values for each thread obtained by FILT and the computational results obtained by FDM for each thread, respectively. The results of both methods are in good agreement. Thus, parallel computing can be performed by the proposed method.

Table 2 shows the CPU time of parallel computing and the conventional TD-FDM. In the table, the longest CPU time of each thread is shown. As the number of threads increases, the CPU time for obtaining the initial distribution does not change, and the sequential calculation part by the TD-FDM

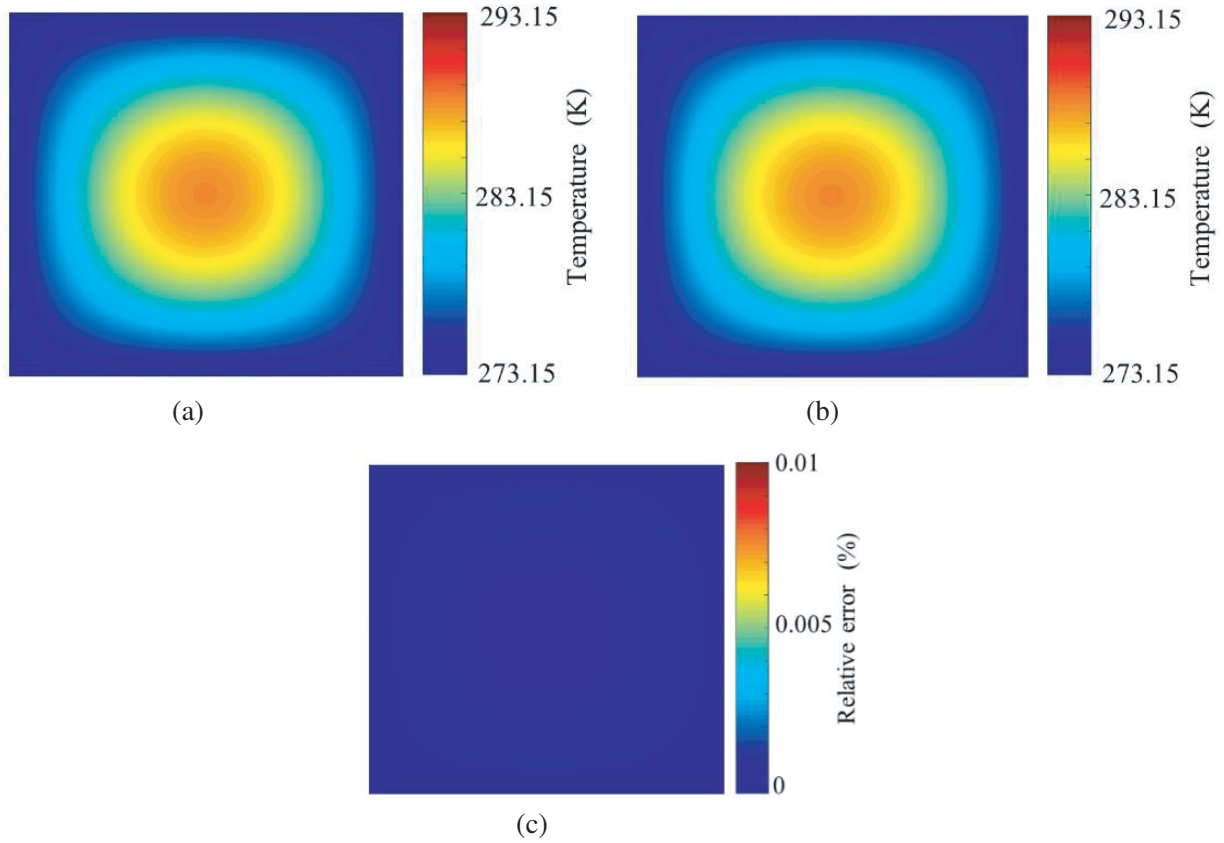

Figure 7. Temperature distribution at $t = 6\,\mathrm{s}$ obtained using (a) the proposed method and (b) conventional FDM. (c) Relative error between the proposed method and conventional FDM. The computational error is less than 0.005% for the complete computational area.

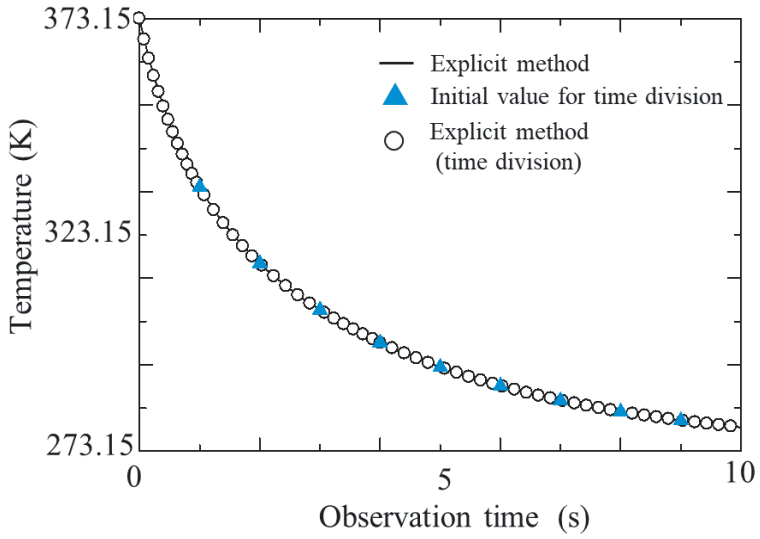

Figure 8. Temperature-time response at the center of an object. Parallel computing can be performed by the proposed method.

Table 2. Computation time of time-division FDM with FILT. The total computation time can be reduced by increasing the number of threads.

Number of threads N	CPU time for initial value computation (s)	CPU time for sequential computation (s)	Total time (s)
2	274.8	1939.2	2214.0
5	265.2	732.0	997.2
10	265.2	364.8	630.0
Conventional method ($N = 1$)	-	3870.0	3870.0

Table 3. Properties of the computational model.

Medium	Density of media ρ (kg/m^3)	Heat conductivity k (W/(m $\cdot$ k))	Specific heat c (J/kg $\cdot$ K)
PCB	1600	0.620	950.0
Cu	8800	398.0	130.0
Air	1.166	0.0272	1008.0

per thread decreases. Compared to the conventional method, the computation time for $N = 10$ is 1/6. Here, multi-threading is not adapted for the conventional TD-FDM. If multithreading is applied, the computation time of the conventional TD-FDM and the sequential computation part of our method could be reduced.

Furthermore, using the proposed method, the temperature distribution at the desired observation period can be computed using TD-FDM. Fig. 9 shows the temperature-time response for the desired observation period from 6 to 8 s. Our method provides the initial temperature of TD-FDM, and the time evolution of the temperature can be computed by TD-FDM.

To demonstrate our proposed method, the heat transfer problem of a printed circuit board (PCB) is analyzed. The computational model is shown in Fig. 10, and Table 3 shows the media parameters.

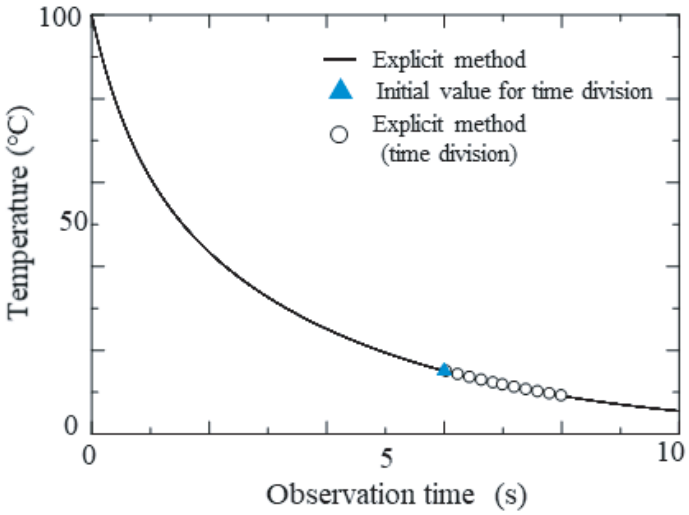

Figure 9. Temperature-time response at the desired observation period by the proposed method and TD-FDM.

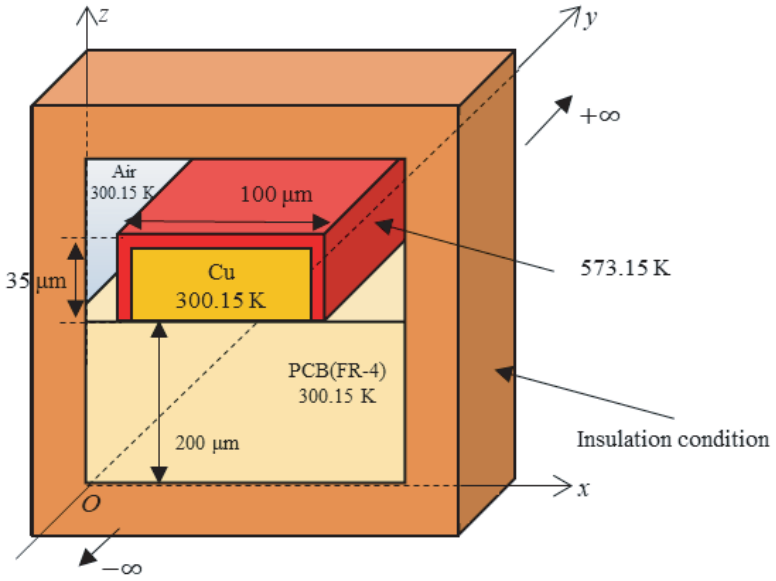

Figure 10. Computational model of heat transfer problem of PCB.

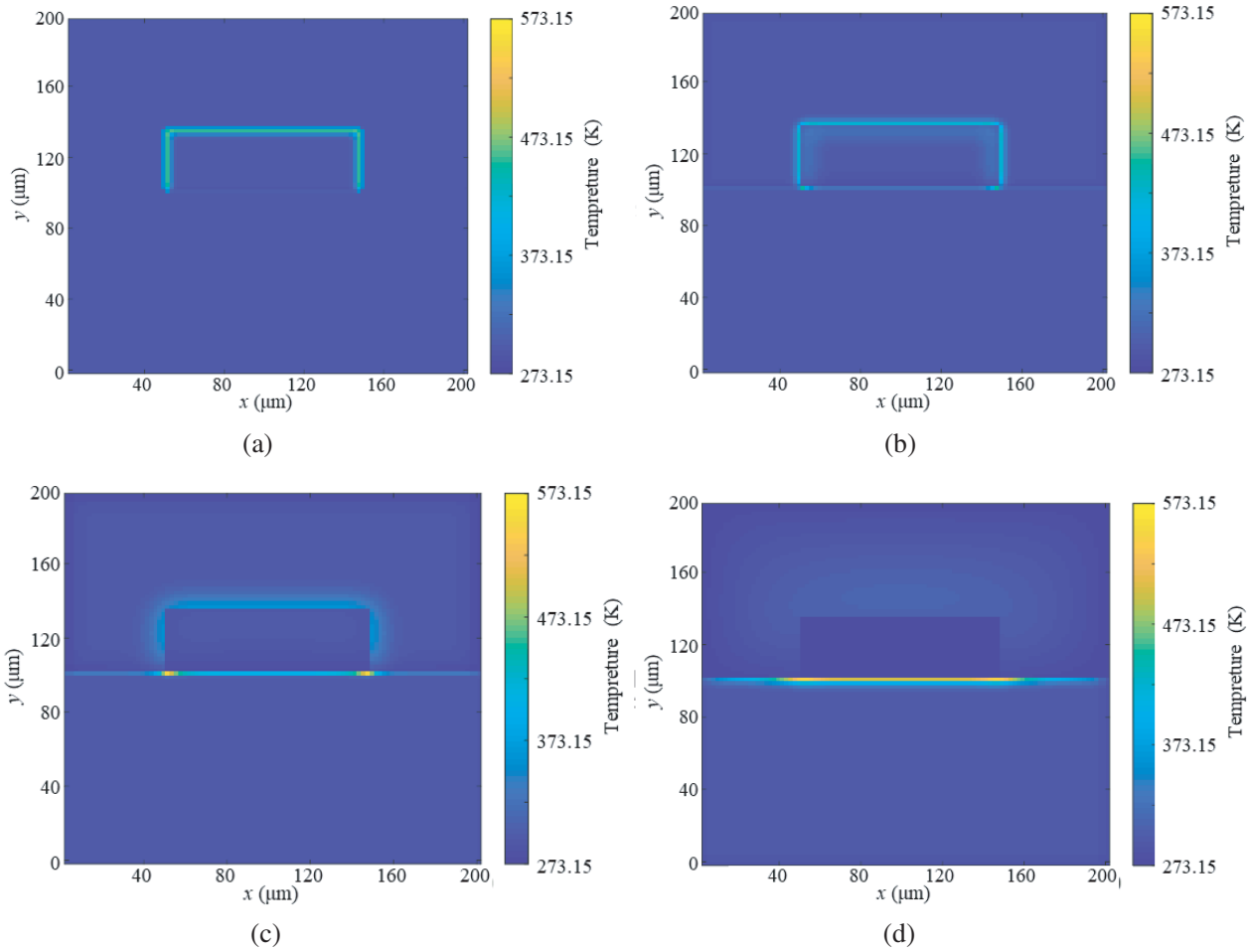

Figure 11. Temperature distributions at a specific observation time t computed by our method. (a) $t = 10$ ns, (b) $t = 100$ ns, (c) $t = 1000$ ns, and (d) $t = 10000$ ns.

The Neumann boundary condition is adopted for the computational area. The thickness and width of copper are 35 and 100 µm, respectively. The initial temperatures are assumed to be 300.15 and 573.15 K at the boundaries between copper and air. The temperature distributions for $t = 10$ ns, 100 ns, 1000 ns, and 10000 ns are shown in Fig. 11. As our method can compute the temperature at a specific observation time, the time step size is varied logarithmically. It can be confirmed that heat transfer occurs from copper to PCB with the passage of time.

4. CONCLUSION

A computational method has been proposed for the time-domain analysis of heat conduction problems. The temperature distribution in the complex frequency domain is obtained through FDM. The response in the time domain is obtained using FILT. The computational accuracy for time response analysis was evaluated by comparing the results of the proposed method with an analytical solution. Moreover, the time-division parallel computing method for TD-FDM is implemented using the initial temperature distribution computed by FILT. As each thread can be computed independently, the computation time of the proposed method is less than that of the conventional TD-FDM because of the increase in the number of threads. The proposed method is suitable for studying thermal transient states in detail because a specific period can be analyzed thoroughly.

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