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# The Analytical Formula for Calculating the Self-Inductance for the Circular Coil of the Rectangular Cross-Section with a Nonuniform Current Density

**Slobodan Babic[1, *], Matthew Smith[2], Nikiforos Fokas[2], Yuriy Langer[2], and Jerry Selvaggi[3]**

**Abstract**—In this article we give an analytical formula for calculating the self-inductance for circular coils of rectangular cross-section which has a nonuniform current density. Recently, the formula for calculating this important electromagnetic quantity was published in the form of the single integral whose kernel function was a sum of elementary functions. However, a new formula is obtained in the form of elementary functions, single integrals, and the complete elliptic integral of the first, second, and third kind. Although its development looks tedious, we obtain a rather user-friendly expression for the calculation. From the general case, the self-inductance of the thin disk coil (pancake coil) with the nonuniform current is obtained in a remarkably simple form. The results of this work are compared with different known methods, and all results are in the excellent agreement. Our approach has not been found in the literature.

## 1. INTRODUCTION

The computation of the electromagnetic quantities (magnetic field, self-inductance, mutual inductance, magnetic force, etc.) for the conventional circular coaxial coils with the constant azimuthal current density has been presented in many papers, books, monographs, and studies [1–16]. The analytical, semi-analytical, and numerical methods have been used to calculate these electromagnetic quantities. These calculations are used in many electromagnetic applications (tubular linear motors, magnetically controllable devices and sensors, current reactors, cochlear implants, defibrillators, instrumented orthopedic implants, in magnetic resonance imaging (MRI) systems, superconducting coils, and tokamaks, etc.). Also, there are nonconventional circular coils with nonuniform density current which are used in many technical applications such as superconducting coils and the homopolar motors [17–30]. Coils with rectangular cross-section and the nonuniform current density, which changes inversely with the cylindrical coordinate $r$ known as Bitter coils, can produce extremely high magnetic fields up to 45 T. In this paper, we give a new formula for calculating the self-inductance of the circular thick coil of the rectangular cross section with nonuniform current density. In [28], the formula for calculating the self-inductance of the thick circular coil of rectangular cross section and with a nonuniform current density is obtained as a single integral whose kernel function is a combination of elementary functions. The formula is obtained over the complete elliptic integrals of the first, second, and third kind. It is possible to obtain the self-inductance of the thin disk coil (pancake coil) with the nonuniform current density by finding the limit in the general formula when the height of the coil tends to zero [23]. All integrals, which have been solved, are the integrals which appear in the electromagnetics for calculating

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* Corresponding author: Slobodan Babic (slobobob@yahoo.com).

[1] Independent Researcher, 53 Berlioz, 101, H3E 1N2, Montréal, Québec, Canada. [2] Division of Engineering Programs at SUNY New Paltz, 1 Hawk Dr., New Paltz, New York 12561, USA. [3] Electrical and Mechanical Engineering Consultant, Schenectady, New York 12308, USA.

the magnetic field, magnetic force, mutual inductance, self-inductance, and electromagnetic energy of circular coils with different cross-sections. The new formula presented in this article is verified by employing other known methods such as semi-analytical methods and various numerical methods including Finite Elements. Examples are introduced in order to confirm the validity of the presented method.

## 2. BASIC EXPRESSIONS

Let us consider the circular coil of the rectangular cross-section, as shown in Figure 1, where

$R_1$ — the inner radius (m),
$R_2$ — the outer radius (m),
$I$ — the current in coil (A),
$J$ — the nonuniform current density (A/m$^2$),
$r_1$, $r_2$ — the coordinates which determine any radial position inside the coil (m),
$z_1$, $z_2$ — the coordinates which determine any axial position inside the coil (m),
$l = z_2 - z_1$ — the height of the coil (m).

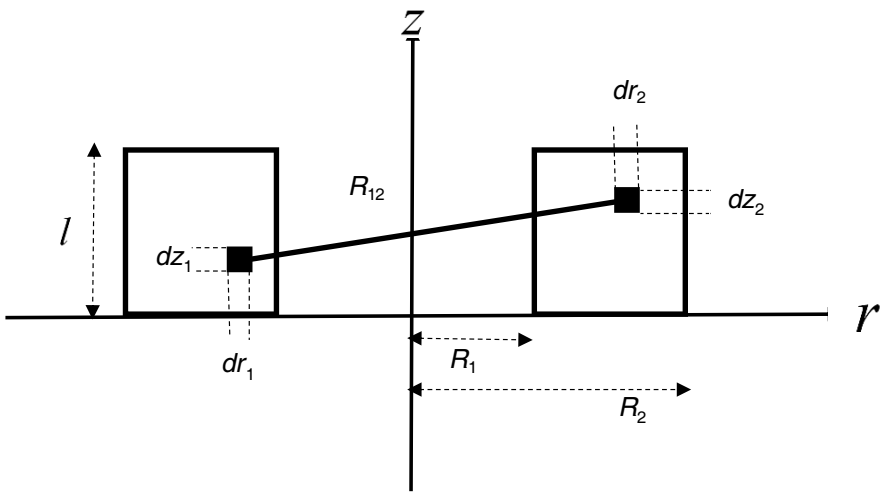


**Figure 1.** Circular thick coil of the rectangular cross section.

The nonuniform current density and the corresponding self-inductance of the coil of rectangular cross section are given by [18–21].

$$J = \frac{NI}{l \cdot \ln \dfrac{R_2}{R_1}} \cdot \frac{1}{r}, \quad R_1 \le r \le R_2 \tag{1}$$

$$L = \frac{\mu_0 N^2}{l^2 \ln^2 \dfrac{R_2}{R_1}} \int_0^l \int_0^l \int_{R_1}^{R_2} \int_{R_1}^{R_2} \int_0^{\pi} \frac{\cos(\theta) dz_1 dz_2 dr_1 dr_2 d\theta}{R_{12}} \tag{2}$$

where

$$R_{12} = \sqrt{r_1^2 + r_2^2 - 2 r_1 r_2 \cos(\theta) + (z_2 - z_1)^2}$$

$R_{12}$ is the distance between two differential elements $dz_1 dr_1$ and $dz_2 dr_2$.

The self-inductance does not depend on the current. If the emf is induced across the coil, it does depend on how the current changes with time that is not our case.

## 3. MATHEMATICAL DEVELOPMENT

Introducing the substitution $\theta = \pi - 2\beta$, $r_1 = xR_1$, $r_2 = yR_1$, $z_1 = vR_1$, $z_2 = zR_1$, $l = bR_1$, $R_2 = \alpha R_1$, $l = bR_1$ in Eq. (2), one obtains

$$L = -\frac{2\mu_0 N^2 R_1}{b^2 \ln^2(\alpha)} \int_1^{\alpha}\int_1^{\alpha}\int_0^{b}\int_1^{b}\int_0^{\frac{\pi}{2}} \frac{\cos(2\beta)dydxdvdzd\beta}{r_{12}} \tag{3}$$

where

$$r_{12} = \sqrt{y^2 + x^2 + 2yx\cos(2\beta) + (v-z)^2} = \sqrt{y^2 + x^2 + 2yx\cos(2\beta) + t^2}, \quad t = v - z$$

$1 \le x \le \alpha$, $1 \le y \le \alpha$, $0 \le z < b$, $0 \le v < b$, $b > 0$ and $\alpha > 1$ are dimensionless parameters.

In [28], the order of integrations is over the variables $x$, $y$, $v$, $z$, and $\beta$. In this paper, the order of integration is over $v$, $z$, $y$, $x$, and $\beta$. It will be interesting to compare the last expression before the last integration regarding the variable $\beta$.

The first integration in Eq. (3) gives the following:

$$I_1 = \int_0^{b} \frac{dv}{r_{12}} = \text{asinh}\frac{b-z}{\sqrt{y^2 + x^2 + 2yx\cos(2\beta)}} + \text{asinh}\frac{z}{\sqrt{y^2 + x^2 + 2yx\cos(2\beta)}}$$

The second integration in Eq. (3) gives,

$$\begin{aligned} I_2 &= \int_0^{b} I_1 dz \\ &= b\text{asinh}\frac{b}{\sqrt{y^2 + x^2 + 2yx\cos(2\beta)}} - \sqrt{y^2 + x^2 + 2yx\cos(2\beta) + b^2} - \sqrt{y^2 + x^2 + 2yx\cos(2\beta)} \end{aligned}$$

The third integration in Eq. (3) gives,

$$\begin{aligned} I_3 &= \int_1^{\alpha} I_2 dy = 2b\alpha\,\text{asinh}\left[\frac{b}{r_0(x,\alpha)}\right] + 2bx\cos(2\beta)\,\text{atanh}\left[\frac{r(x,\alpha)}{b}\right] \\ &-2bx\sin(2\beta)\,\text{atan}\left[\frac{b(\alpha + x\cos(2\beta))}{x\sin(2\beta)\,r(x,\alpha)}\right] + \left(b^2 - x^2\sin^2(2\beta)\right)\text{asinh}\left[\frac{\alpha + x\cos(2\beta)}{\sqrt{x^2\sin^2(2\beta) + b^2}}\right] \\ &+x^2\sin^2(2\beta)\,\text{asinh}\left[\frac{\alpha + x\cos(2\beta)}{x\sin(2\beta)}\right] - [\alpha + x\cos(2\beta)]\,r(x,\alpha) + [\alpha + x\cos(2\beta)]\,r_0(x,\alpha) \\ &-2b\text{asinh}\left[\frac{b}{r_0(x,1)}\right] - 2bx\cos(2\beta)\,\text{atanh}\left[\frac{r(x,1)}{b}\right] \\ &+2bx\sin(2\beta)\,\text{atan}\left[\frac{b(1 + x\cos(2\beta))}{x\sin(2\beta)\,r(x,1)}\right] - \left(b^2 - x^2\sin^2(2\beta)\right)\text{asinh}\left[\frac{1 + x\cos(2\beta)}{\sqrt{x^2\sin^2(2\beta) + b^2}}\right] \\ &-x^2\sin^2(2\beta)\,\text{asinh}\left[\frac{1 + x\cos(2\beta)}{x\sin(2\beta)}\right] + [1 + x\cos(2\beta)]\,r(x,1) - [1 + x\cos(2\beta)]\,r_0(x,\alpha) \end{aligned}$$

where

$$\begin{aligned} r(x,\alpha) &= \sqrt{x^2 + 2\alpha x\cos(2\beta) + \alpha^2 + b^2}, \quad r(x,1) = \sqrt{x^2 + 2x\cos(2\beta) + 1 + b^2} \\ r_0(x,\alpha) &= \sqrt{x^2 + 2\alpha x\cos(2\beta) + \alpha^2}, \quad r_0(x,1) = \sqrt{x^2 + 2x\cos(2\beta) + 1+} \end{aligned}$$

The fourth integration in Eq. (3) gives,

$$
\begin{aligned}
I_4 \;=\; & \int_1^{\alpha} I_3 dx = \frac{8}{3}\left(\alpha^3+1\right)\cos^3(\beta) - \frac{4}{3}\cos^2(\beta)\left(\alpha^2 r_2 + r_1\right) + \frac{2}{3}\left[\left(\alpha^2+1\right)\cos(2\beta)\,2\alpha\right](r-r_0) \\
& + 2b\alpha\,\mathrm{asinh}\left[\frac{b}{2\alpha\cos(\beta)}\right] + 2b\,\mathrm{asinh}\left[\frac{b}{2\cos(\beta)}\right] - 4b\alpha\,\mathrm{asinh}\left[\frac{b}{r_0}\right] \\
& + \frac{2}{3}\alpha\left[3b^2 - \alpha^2\sin^2(2\beta)\right]\left\{\mathrm{asinh}\left[\frac{\alpha+\alpha\cos(2\beta)}{\sqrt{\alpha^2\sin^2(2\beta)+b^2}}\right] - \mathrm{asinh}\left[\frac{1+\alpha\cos(2\beta)}{\sqrt{\alpha^2\sin^2(2\beta)+b^2}}\right]\right\} \\
& + \frac{2}{3}\left[3b^2 - \sin^2(2\beta)\right]\left\{\mathrm{asinh}\left[\frac{1+\cos(2\beta)}{\sqrt{\sin^2(2\beta)+b^2}}\right] - \mathrm{asinh}\left[\frac{\alpha+\cos(2\beta)}{\sqrt{\sin^2(2\beta)+b^2}}\right]\right\} \\
& + \frac{2}{3}\left(\alpha^3+1\right)\sin^2(2\beta)\,\mathrm{asinh}\left[\cot(\beta)\right] - \frac{2}{3}\alpha^3\sin^2(2\beta)\,\mathrm{asinh}\left[\frac{1+\alpha\cos(2\beta)}{\alpha\sin(2\beta)}\right] \\
& - \frac{2}{3}\sin^2(2\beta)\,\mathrm{asinh}\left[\frac{\alpha+\cos(2\beta)}{\sin(2\beta)}\right] - 2b\alpha^2\sin(2\beta)\,\mathrm{atan}\left[\frac{b\left(\alpha+\alpha\cos(2\beta)\right)}{\alpha\sin(2\beta)\,r_2}\right] \\
& - 2b\sin(2\beta)\,\mathrm{atan}\left[\frac{b\left(1+\cos(2\beta)\right)}{\sin(2\beta)\,r_1}\right] + 2b\alpha^2\sin(2\beta)\,\mathrm{atan}\left[\frac{b\left(1+\alpha\cos(2\beta)\right)}{\alpha\sin(2\beta)\,r}\right] \\
& + 2b\sin(2\beta)\,\mathrm{atan}\left[\frac{b\left(\alpha+\cos(2\beta)\right)}{\sin(2\beta)\,r}\right] + b\alpha^2\cos(2\beta)\ln\left[\frac{r_2+b}{r_2-b}\right] + b\cos(2\beta)\ln\left[\frac{r_1+b}{r_1-b}\right] \\
& - b\left(\alpha^2+1\right)\cos(2\beta)\ln\left[\frac{r+b}{r-b}\right] + \frac{b^3}{3\sin(2\beta)}\left\{2\mathrm{atan}\left[\frac{\alpha\sin^2(2\beta) - b^2\cos(2\beta)}{b\sin(2\beta)\,r}\right]\right. \\
& \left. - \mathrm{atan}\left[\frac{\sin^2(2\beta) - b^2\cos(2\beta)}{b\sin(2\beta)\,r_1}\right] - \mathrm{atan}\left[\frac{\alpha^2\sin^2(2\beta) - b^2\cos(2\beta)}{b\sin(2\beta)\,r_1}\right]\right\}
\end{aligned}
$$

where

$$
\begin{aligned}
r \;&=\; \sqrt{\alpha^2 + 2\alpha\cos(2\beta) + 1 + b^2}, \quad r_0 = \sqrt{\alpha^2 + 2\alpha cos(2\beta) + 1} \\
r_1 \;&=\; \sqrt{2 + 2\cos(2\beta) + b^2}, \quad r_2 = \sqrt{2\alpha^2 + 2\alpha^2\cos(2\beta) + b^2}
\end{aligned}
$$

The self-inductance is

$$
L = -\frac{2\mu_0 N^2 R_1}{b^2\ln^2(\alpha)} \int_0^{\pi/2} \cos(2\beta)\, I_4 d\beta = -\frac{2\mu_0 N^2 R_1}{b^2\ln^2(\alpha)} \sum_{i=1}^{22} J_i \tag{4}
$$

where $I_4$ is previously given. Comparing the kernel functions $I_4$ in Eq. (4) and $T_n$ in [28], one can see that they differ in some terms. It is possible because we changed the order of the integration. Even though the self-inductance is a physical quantity, the different orders of integration must give the same result. All expressions before the last integration regarding the variable $\beta$ are analytical functions, and the numerical integration is simple. All expressions will be verified numerically.

Doing the last integration in Eq. (4), one obtains

$$
\begin{aligned}
J_1 \;&=\; \frac{8}{3}\left(\alpha^3+1\right)\int_0^{\pi/2} \cos(2\beta)\cos^3(\beta)\,d\beta = \frac{16}{15}\left(\alpha^3+1\right) \\
J_2 \;&=\; -\frac{4}{3}\int_0^{\pi/2} \cos(2\beta)\cos^2(\beta)\left[\alpha^2 r_2 + r_1\right]d\beta = -\frac{8\alpha^3}{45k_2^5}\left\{\left[7k_2^4 - 11k_2^2 + 4\right]K(k_2)\right.
\end{aligned}
$$

$$+\left[k_2^4+9k_2^2-4\right]E\left(k_2\right)\}-\frac{8}{45k_1^5}\left\{\left[7k_1^4-11k_1^2+4\right]K\left(k_1\right)+\left[k_1^4+9k_1^2-4\right]E\left(k_1\right)\right\}$$

$$J_3 = \frac{2}{3}\int_0^{\frac{\pi}{2}}\cos\left(2\beta\right)\left[\left(\alpha^2+1\right)\cos\left(2\beta\right)+2\alpha\right]\left[r-r_0\right]d\beta=\frac{4\alpha}{45k^5\sqrt{\alpha}}\left\{\left[(4\{\alpha\}^2+20\alpha+4)k^4\right.\right.$$
$$-\left(12\alpha^2+20\alpha+12\right)k^2+8\left(\alpha^2+1\right)\left.\right]K\left(k\right)+\left[\left(7\alpha^2-10\alpha+7\right)k^4+\left(8\alpha^2+20\alpha+8\right)k^2\right.$$
$$\left.-8\left(\alpha^2+1\right)\right]E\left(k\right)\}-\frac{4\alpha}{45k_0^5\sqrt{\alpha}}\left\{\left[\left(4\alpha^2+20\alpha+4\right)k_0^4-\left(12\alpha^2+20\alpha+12\right)k_0^2\right.$$
$$\left.+8(\alpha^2+1\ )\right]\left.\right]K\left(k_0\right)+\left[\left(7\alpha^2-10\alpha+7\right)k_0^4+\left(8\alpha^2+20\alpha+8\right)k_0^2-8\left(\alpha^2+1\right)\right]E\left(k_0\right)\}$$

$$J_4 = 2b\alpha^2\int_0^{\pi/2}\cos(2\beta)\text{asinh}\frac{b}{2\alpha\cos(\beta)}d\beta=-\frac{\alpha b^2}{k_2}\left[K\left(k_2\right)-E\left(k_2\right)\right]$$

$$J_5 = 2b\int_0^{\pi/2}\cos(2\beta)\text{asinh}\frac{b}{2\cos(\beta)}d\beta=-\frac{b^2}{k_1}\left[K\left(k_1\right)-E\left(k_1\right)\right]$$

$$J_6 = -4b\alpha\int_0^{\frac{\pi}{2}}\cos\left(2\beta\right)\text{asinh}\frac{b}{r_0}d\beta=\frac{b^2}{2k\sqrt{\alpha}}\left[\left(\left(\alpha^2+1-2\alpha\right)k^2+4\alpha\right)K\left(k\right)-4\alpha E\left(k\right)\right]$$
$$-\frac{kb^2}{2\sqrt{\alpha}}\left(\alpha-1\right)^2\Pi\left(h,k\right)$$

$$J_7 = \frac{2}{3}\alpha\int_0^{\frac{\pi}{2}}\cos\left(2\beta\right)\left[3b^2-\alpha^2\text{sin}^2\left(2\beta\right)\right]\text{asinh}\left[\frac{\alpha+\alpha\cos\left(2\beta\right)}{\sqrt{\alpha^2\text{sin}^2\left(2\beta\right)+b^2}}\right]d\beta$$
$$= \frac{8\alpha^3}{135k_2^5}\left\{\left[150\ k_2^6-289k_2^4+127k_2^2+12\right]K\left(k_2\right)+\left[-157k_2^4+167k_2^2-12\right]E\left(k_2\right)\right\}$$
$$-\frac{5b^2k_2}{9a}\left[\left(\sqrt{\alpha^2+b^2}-\alpha\right)^2\Pi\left(m_1,k_2\right)+\left(\sqrt{\alpha^2+b^2}+\alpha\right)^2\Pi\left(m_2,k_2\right)\right]$$

$$J_8 = \frac{2}{3}\int_0^{\frac{\pi}{2}}\cos\left(2\beta\right)\left[3b^2-\sin^2\left(2\beta\right)\right]\text{asinh}\left[\frac{1+\cos\left(2\beta\right)}{\sqrt{\sin^2\left(2\beta\right)+b^2}}\right]d\beta$$
$$= \frac{8}{135k_1^5}\left\{\left[150\ k_1^6-289k_1^4+127k_1^2+12\right]K\left(k_1\right)+\left[-157k_1^4+167k_1^2-12\right]E\left(k_1\right)\right\}$$
$$-\frac{5b^2k_1}{9}\left\{\left(\sqrt{1+b^2}-1\right)^2\Pi\left(m_3,k_1\right)+\left(\sqrt{1+b^2}+1\right)^2\Pi\left(m_4,k_1\right)\right\}$$

$$J_9 = -\frac{2}{3}\alpha\int_0^{\frac{\pi}{2}}\cos\left(2\beta\right)\left[3b^2-\alpha^2\text{sin}^2\left(2\beta\right)\right]\text{asinh}\left[\frac{1+\alpha\cos\left(2\beta\right)}{\sqrt{\alpha^2\text{sin}^2\left(2\beta\right)+b^2}}\right]d\beta$$
$$= -\frac{k}{540\alpha^2\sqrt{\alpha}}\left\{\left[4\alpha^2\left(150b^4+\alpha^2+141b^2\alpha^2-9\alpha^4\right)K\left(k\right)-2\alpha^2(-2+3b^4+7\alpha^2\right.\right.$$
$$\left.\left.+3\alpha^4-149b^2+6b^2\alpha^2)\left[\left(b^2+(1+a)^2\right)E\left(k\right)-\left(1+b^2+a^2\right)K\left(k\right)\right]\right\}$$

$$+\frac{5b^2k}{9\sqrt{\alpha}}\left\{\left(\sqrt{\alpha^2+b^2}-\alpha\right)\left(\sqrt{\alpha^2+b^2}-1\right)\Pi\left(m_1,k\right)+\right.$$
$$\left.+\left(\sqrt{\alpha^2+b^2}+\alpha\right)\left(\sqrt{\alpha^2+b^2}+1\right)\Pi\left(m_1,k\right)\right\}$$

$$J_{10} = -\frac{2}{3}\int_0^{\frac{\pi}{2}}\cos\left(2\beta\right)\left[3b^2-\sin^2\left(2\beta\right)\right]\operatorname{asinh}\left[\frac{\alpha+\cos\left(2\beta\right)}{\sqrt{\sin^2\left(2\beta\right)+b^2}}\right]d\beta$$
$$= -\frac{k}{540a^2\sqrt{\alpha}}\left\{\left[4a^2\left(-9+141b^2+150b^4+a^2\right)K\left(k\right)-2(3+3b^2+7a^2-2a^4\right.\right.$$
$$\left.+6b^2-149a^2b^2)\left[\left(b^2+(\alpha+1)^2\right)E\left(k\right)+\left(1+\alpha^2+b^2\right)K(k)\right]\right\}$$
$$+\frac{5b^2k}{9\sqrt{\alpha}}\left\{\left(\sqrt{1+b^2}-\alpha\right)\left(\sqrt{1+b^2}-1\right)\Pi\left(m_3,k\right)+\left(\sqrt{1+b^2}+\alpha\right)\left(\sqrt{1+b^2}+1\right)\Pi\left(m_4,k\right)\right\}$$

$$J_{11} = \frac{2}{3}\left(\alpha^3+1\right)\int_0^{\pi/2}\cos(2\beta)\sin^2\left(2\beta\right)\operatorname{asinh}\left[\cot(\beta)\right]d\beta = \frac{16}{135}\left(\alpha^3+1\right)$$

$$J_{12} = -\frac{2}{3}\alpha^3\int_0^{\frac{\pi}{2}}\cos\left(2\beta\right)\sin^2\left(2\beta\right)\operatorname{asinh}\left[\frac{1+\alpha\cos\left(2\beta\right)}{\alpha\sin\left(2\beta\right)}\right]d\beta$$
$$= -\frac{4\alpha^3}{135\sqrt{\alpha}k_0^5}\left\{\left[(10\alpha+8)\ k_0^4-(10\alpha+24)\,k_0^2+16\right]K\left(k_0\right)+\left[-\left(5\alpha+1\right)\,k_0^4\right.\right.$$
$$\left.+(10\alpha+16)\,k_0^2-\ 16\right]E(k_0)\}$$

$$J_{13} = -\frac{2}{3}\int_0^{\frac{\pi}{2}}\cos\left(2\beta\right)\sin^2\left(2\beta\right)\operatorname{asinh}\left[\frac{\alpha+\cos\left(2\beta\right)}{\sin\left(2\beta\right)}\right]d\beta = -\frac{4}{135\sqrt{\alpha}k_0^5}\left\{\left[(8\alpha+10)\ k_0^4\right.\right.$$
$$\left.-(24\alpha+10)\,k_0^2+16\alpha\right]K\left(k_0\right)+\left[-\left(\alpha+5\right)\,k_0^4+(16\alpha+10)\,k_0^2-\ 16\alpha\right]E(k_0)\}$$

$$J_{14} = -2b\alpha^2\int_0^{\frac{\pi}{2}}\cos\left(2\beta\right)\sin\left(2\beta\right)\operatorname{atan}\left[\frac{b\left(\alpha+\alpha\cos\left(2\beta\right)\right)}{\alpha\sin\left(2\beta\right)r_2}\right]d\beta$$
$$= -\frac{\alpha b^2}{3k_2^3}\left\{\left[-3\ k_2^4+k_2^2+2\right]K\left(k_2\right)+\left[4k_2^2-2\right]\,E\left(k_2\right)\right\}$$
$$+\frac{k_2b^2}{4}\left\{\left(\sqrt{\alpha^2+b^2}-\alpha\right)^2\Pi\left(m_1,k_2\right)+\left(\sqrt{\alpha^2+b^2}+\alpha\right)^2\Pi\left(m_2,k_2\right)\right\}$$

$$J_{15} = -2b\int_0^{\frac{\pi}{2}}\cos\left(2\beta\right)\sin\left(2\beta\right)\operatorname{atan}\left[\frac{b\left(1+\cos\left(2\beta\right)\right)}{\sin\left(2\beta\right)r_1}\right]d\beta$$
$$= -\frac{b^2}{3k_1^3}\left\{\left[-3\ k_1^4+k_1^2+2\right]K\left(k_1\right)+\left[4k_1^2-2\right]\,E\left(k_1\right)\right\}$$
$$+\frac{k_1b^2}{4}\left\{\left(\sqrt{1+b^2}-1\right)^2\Pi\left(m_3,k_1\right)+\left(\sqrt{1+b^2}+1\right)^2\Pi\left(m_4,k_1\right)\right\}$$

$$J_{16} = 2b\alpha^2 \int_0^{\frac{\pi}{2}} \cos(2\beta)\sin(2\beta)\,\mathrm{atan}\left[\frac{b\,(1+\alpha\cos(2\beta))}{\alpha\sin(2\beta)\,r}\right] d\beta$$

$$= \frac{b^2}{48k^3\sqrt{\alpha}}\left\{\left[3\left(\alpha^4-2\alpha^3+8\alpha^2+10\alpha-1+8b^2\right)k^4+\left(12\alpha^3-60\alpha-32\alpha^2\right)k^2+32\alpha^2\right]K(k)\right.$$

$$\left.-\left[\left(12\alpha^3-60\alpha-16\alpha^2\right)k^2+32\alpha^2\right]E(k)\right\}-\frac{kb^2\left(\alpha^2-1\right)(\alpha-1)^2}{16\sqrt{\alpha}}\Pi(h,k)$$

$$-\frac{kb^2}{4\sqrt{\alpha}}\left\{\left(\sqrt{\alpha^2+b^2}-1\right)\left(\sqrt{\alpha^2+b^2}-\alpha\right)\Pi(m_1,k)\right.$$

$$\left.+\left(\sqrt{\alpha^2+b^2}+1\right)\left(\sqrt{\alpha^2+b^2}+\alpha\right)\Pi(m_2,k)\right\}$$

$$J_{17} = 2b\int_0^{\frac{\pi}{2}} \cos(2\beta)\sin(2\beta)\,\mathrm{atan}\left[\frac{b\,(\alpha+\cos(2\beta))}{\sin(2\beta)\,r}\right] d\beta$$

$$= \frac{b^2}{48\alpha^2k^3\sqrt{\alpha}}\left\{\left[3\left(-\alpha^4+10\alpha^3+8\alpha^2-2\alpha+1+8b^2\alpha^2\right)k^4+\left(12\alpha-60\alpha^3-32\alpha^2\right)k^2\right.\right.$$

$$\left.\left.+32\,\alpha^2\right]K(k)-\left[\left(12\alpha-60\alpha^3-16\alpha^2\right)k^2+32\alpha^2\right]E(k)\right\}+\frac{kb^2\left(\alpha^2-1\right)(\alpha-1)^2}{16\alpha^2\sqrt{\alpha}}\Pi(h,k)$$

$$-\frac{kb^2}{4\sqrt{\alpha}}\left\{\left(\sqrt{1+b^2}-1\right)\left(\sqrt{1+b^2}-\alpha\right)\Pi(m_3,k)+\right.$$

$$\left.+\left(\sqrt{1+b^2}+1\right)\left(\sqrt{1+b^2}+\alpha\right)\Pi(m_4,k)\right\}$$

$$J_{18} = \frac{b\alpha^2}{2}\int_0^{\pi/2} \cos(4\beta)\ln\left[\frac{r_2+b}{r_2-b}\right] d\beta = \frac{\alpha b^2}{6k_2^3}\left\{\left[4-k_2^2\right]K(k_2)-\left[4+k_2^2\right]E(k_2)\right\}$$

$$J_{19} = \frac{b}{2}\int_0^{\pi/2} \cos(4\beta)\ln\left[\frac{r_1+b}{r_1-b}\right] d\beta = \frac{b^2}{6k_1^3}\left\{\left[4-k_1^2\right]K(k_1)-\left[4+k_1^2\right]E(k_1)\right\}$$

$$J_{20} = -\frac{b(\alpha^2+1)}{2}\int_0^{\pi/2} \cos(4\beta)\ln\left[\frac{r+b}{r-b}\right] d\beta$$

$$= -\frac{b^2(\alpha^2+1)}{48\alpha^2k^3\sqrt{\alpha}}\left\{\left[3\left(\alpha^4-2\alpha^3+2\alpha^2-2\alpha+1\right)k^4+\left(12\alpha^3+12\alpha-32\alpha^2\right)k^2\right.\right.$$

$$\left.\left.+32\alpha^2\right]K(k)-\left[\left(12\alpha^3+12\alpha-16\alpha^2\right)k^2+32\alpha^2\right]E(k)\right\}+\frac{kb^2(\alpha^2+1)^2(\alpha-1)^2}{16\alpha^2\sqrt{\alpha}}\Pi(h,k)$$

$$J_{21} = \frac{b^3}{2}\int_0^{\pi/2} \frac{\cos(2\beta)}{\sin(2\beta)}\left[2\mathrm{arctg}(q)-\mathrm{arctg}(q_{11})-\mathrm{arctg}(q_{22})\right] d\beta$$

$$J_{22} = \frac{b}{2}\int_0^{\pi/2} \left[\alpha^2\ln\frac{r_2+b}{r_2-b}+\ln\frac{r_1+b}{r_1-b}-\left(\alpha^2+1\right)\ln\frac{r+b}{r-b}\right] d\beta$$

The last two terms must be integrated using numerical integration.

$$\begin{aligned}
q &= \frac{\alpha \sin^2(2\beta) - b^2 \cos(2\beta)}{b \sin(2\beta) r} \\
q_1 &= \frac{\alpha^2 \sin^2(2\beta) - b^2 \cos(2\beta)}{b \sin(2\beta) r_2}, \quad q_2 = \frac{\sin^2(2\beta) - b^2 \cos(2\beta)}{b \sin(2\beta) r_1} \\
k^2 &= \frac{4\alpha}{(\alpha+1)^2 + b^2}, \quad k_1^2 = \frac{4}{4+b^2}, \quad k_2^2 = \frac{4\alpha^2}{4\alpha^2 + b^2}, \quad h = \frac{4\alpha}{(\alpha+1)^2} = k_0^2 \\
0 < m_1 &= \frac{2\alpha}{\alpha + \sqrt{b^2+\alpha^2}} < 1, \quad m_2 = \frac{2\alpha}{\alpha - \sqrt{b^2+\alpha^2}} < 0, \\
0 < m_3 &= \frac{2}{1 + \sqrt{b^2+1}} < 1, \quad m_4 = \frac{2}{1 - \sqrt{b^2+1}} < 0,
\end{aligned}$$

Finally, the simplest form of $V$ in Eq. (5) is,

$$L = -\frac{2\mu_0 N^2 R_1}{b^2 \ln^2(\alpha)} V \tag{5}$$

with

$$V = \sum_{i=1}^{22} J_i$$

Thus, the new formula for the self-inductance of the circular coil with rectangular cross section and a nonuniform current density can be obtained by Eq. (5) using the complete integral of the first, second, and third kind in [31–33]. We believe that this formula appears for the first time in the literature. The special case of Equation (5) is the self-inductance of the thin disk coil with the nonuniform current [22] and [23]. This self-inductance can be obtained from Eq. (5) by finding the limit as $b \to 0$, or doing three integrations such as in [23].

The self-inductance $L_{DISK}$ is obtained in analytical form as follows:

$$L_{DISK} = \frac{4\mu_0 N^2 R_1 (\alpha+1)}{\ln^2 \alpha} \left[ E(k_0) - 1 \right] \tag{6}$$

where

$$k_0^2 = \frac{4\alpha}{(\alpha+1)^2}$$

$K(k)$ and $E(k)$ are the complete elliptical integrals of the first and second kind, respectively. $\Pi(h,k)$ is the complete elliptic integral of the third kind [31–33].

## 4. NUMERICAL VALIDATION

To verify the new formula for the self-inductances, $L$, we consider the following set of examples. Also, special cases are discussed. We compare the results of our approach with those which are found in the known literature.

### 4.1. Example 1.

Calculate the self-inductance of a thick Bitter circular coil of rectangular cross section. The coil dimensions and the number of turns are as follows:

$$R_1 = 1\,(m), \quad R_2 = 2\,(m), \quad l = 2\,(m), \quad N = 100$$

Applying single numerical integration in Eq. (4), the self-inductance is

$$L_{New} = 17.81533309115452\,(\mathrm{mH})$$

In [28], the-self-inductance is.

$$L_{[28]} = 17.81533309115452\,(\mathrm{mH})$$

Thus, we have shown that regardless of the order of the integration, the same result is obtained.

By using the Conway's method [18], the self-inductance is,

$$L_{Conway} = 17.81533308115452\,(\mathrm{mH})$$

Finally, from the exact analytical expression (5), in the form of the elliptical integrals, we obtained the same results for the self-inductance as Conway [18].

$$L_{New\,Elliptic} = 17.81533309115452\,(\mathrm{mH})$$

### 4.2. Example 2.

Calculate the self-inductance of the thick Bitter circular coil of rectangular cross section. The coil dimensions and the number of turns are as follows:

$$R_1 = 0.025\,(m)\,, \quad R_2 = 0.035\,(m)\,, \quad l = 0.04\,(m)\,, \quad N = 100$$

In [28], the self-inductance is obtained by the single integration,

$$L_{[28]} = 0.438398854271743\,(\mathrm{mH})$$

From Eq. (4), the self-inductance is,

$$L_{New} = 0.438398854271743\,(\mathrm{mH})$$

Again, we have shown that regardless of the order of the integration, the same result is obtained.

From the exact analytical expression, in the form of the elliptical integrals, we obtained the same results for the self-inductance,

$$L_{New\,Elliptic} = 0.438398854271743\,(\mathrm{mH})$$

By using the Ren's method [19–21], the self-inductance is,

$$L_{Ren} = 0.4383978\,(\mathrm{mH})$$

This self-inductance is obtained by double integration.

Using the software ANSYS (FEM) [19–21] the self-inductance is,

$$L_{Ren} = 0.44528\,(\mathrm{mH})$$

All results are in good agreement.

### 4.3. Example 3.

In Table 1, for different $\alpha$ and $b$, the self-inductance is calculated by our method employing Equations (5) and (6), and the one found in [28].

### 4.4. Example 4.

Calculate the self-inductance of the thin Bitter disk (pancake) [22]. The coil dimensions and the number of turns are as follows:

$$R_1 = 0.3\,(m)\,, \quad R_2 = 0.4\,(m)\,, \quad N = 100$$

Equation (6) gives,

$$L_{DISK} = \mathbf{12.36243889748211}\,(\mathrm{mH})$$

Substituting directly into Eq. (7) [22] with $z_1 = z_2 = 0$, $R_1 = 0.3(m)$, $R_2 = 0.4(m)$, and $N = 100$ (formula for calculating the mutual inductance between two coaxial thin Bitter thin disk coils). We have two identical disk coils which overlap, and their mutual inductance becomes the self-inductance,

$$L_{DISK} = M_{DISKS\text{-}overlap} = \mathbf{12.36243889748207}\,(\mathrm{H})$$

In Table 2, we give the self-inductance as $b$ approaches zero, and when it is applied in Eq. (2).

When $b = 0$, Eq. (5) gives infinity. Thus, finding the limit as $b \to 0$, Eq. (5) gives Eq. (6) or [23].

All results are in exceptionally good agreement with the result obtained in the analytical form in Eq. (6). The numbers that agree are given in boldface.

**Table 1.** The self-inductance obtained by the different calculation (In all calculations $R_1 = 1\,\mathrm{m}$).

| $\alpha$ | $b$ | This Work (5) (mH) | This Work (6) (mH) | [28] (mH) |
|---|---|---|---|---|
| 1.2 | 0.1 | 39.66072688287898 | 39.66072688287898 | 39.66072688287898 |
| 1.2 | 0.5 | 28.23658515441310 | 28.23658515441310 | 28.23658515441310 |
| 1.2 | 1.0 | 21.37780922245035 | 21.37780922245035 | 21.37780922245035 |
| 1.2 | 2.0 | 14.54320327001611 | 14.54320327001611 | 14.54320327001611 |
| 2.0 | 0.1 | 33.94607203219047 | 33.94607203219047 | 33.94607203219047 |
| 2.0 | 0.5 | 28.53208224277337 | 28.53208224277337 | 28.53208224277337 |
| 2.0 | 1.0 | 23.80800600196159 | 23.80800600196159 | 23.80800600196159 |
| 2.0 | 2.0 | 17.81533309115452 | 17.81533309115452 | 17.81533309115452 |
| 2.5 | 0.1 | 33.71444715932103 | 33.71444715932103 | 33.71444715932103 |
| 2.5 | 0.5 | 29.30151236356844 | 29.30151236356844 | 29.30151236356844 |
| 2.5 | 1.0 | 25.16974995917750 | 25.16974995917750 | 25.16974995917750 |
| 2.5 | 2.0 | 19.53694103357572 | 19.53694103357572 | 19.53694103357572 |

**Table 2.** The self-inductance when $b \to 0$.

| $b$ | This Work (4) or (5) (mH) | [22] (mH) or (6) |
|---|---|---|
| $10^{-6}$ | **12.36243**432318282 | **12.36243**889748211 |
| $10^{-7}$ | **12.362438**44005104 | **12.362438**89748211 |
| $10^{-8}$ | **12.3624388**5173889 | **12.3624388**9748211 |
| $10^{-9}$ | **12.36243889**290775 | **12.36243889**748211 |
| $10^{-10}$ | **12.362438897**01627 | **12.362438897**48211 |
| $10^{-11}$ | **12.362438897**43836 | **12.362438897**48211 |
| $10^{-12}$ | **12.362438897**49853 | **12.362438897**48211 |

### 4.5. Example 5.

Calculate the self-inductance of the thin Bitter disk (pancake coil). The coil dimensions and the number of turns are as follows:

$$R_1 = 1\,(m)\,, \quad R_2 = 2\,(m)\,, \quad N = 1000$$

Equation (6) gives,

$$L_{DISK} = \mathbf{3.569912886724}816214467637\,(\mathrm{H})$$

For two thin disk Bitter coils which overlap [22, 23], we obtain,

$$L_{DISK} = M_{DISKS\text{-}overlap} = \mathbf{3.569912886724815}770378427\,(\mathrm{H})$$

Using Eq. (5) or (6) of this work and $b = 10^{-12}$

$$L_{This\,Work(5)-(6)} = \mathbf{3.569912886724}12196236686038363\,(\mathrm{H})$$

The figures that agree are in boldface. All results are in an excellent agreement.

In these examples, we confirmed the validity of Eqs. (5) and (6) for calculating the self-inductance of the thick circular cross-section coil with the nonuniform current density.

## 5. CONCLUSION

A new analytical self-inductance formula for a circular thick coil of the rectangular cross section and with a nonuniform current density is given. The formula is obtained in the form of the complete elliptical integrals of the first, second, and third kind. We believe that this formula appears for the first time in the literature. The special case of this formula gives the self-inductance for a thin disk coil in a very simple form. Our method can be helpful to engineers, physicists, and anyone who works in similar fields. All computations are easily carried out within the Mathematica or MATLAB environment.

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