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Generalized Kronecker Array Transform

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Abstract—Fast evaluation of the array response matrix and its vector or matrix products play a central role in several applied electromagnetics and array processing applications. In this context, the Kronecker Array Transform (KAT) has been introduced by Ribeiro and Nascimento as an efficient factorization technique that can be applied when the elements of a planar array and the wavevectors exhibit separability. The computational savings leverage on the decomposition of the full array response matrix in the Kronecker product of two smaller array response matrices. In this contribution we extend and apply the generalized Kronecker product introduced by Fino and Algazi to the array response matrix decomposition problem. The resulting Generalized Kronecker Array Transform (GKAT) broadens the class of problems that can be addressed while achieving the same computational savings. The complexity of GKAT is compared with Non-Uniform Fast Fourier Transform (NUFFT), and optimal integration of the two techniques is elaborated.

1. INTRODUCTION

The Array Response Matrix is the mathematical tool that represents the response of an array of sensors to impinging plane waves. Considering the incoming waves as input signals, the matrix provides a Linear Time Invariant (LTI) Multiple-Input-Multiple-Output (MIMO) representation of the input/output relationship [1]. Computationally efficient methods for its evaluation and manipulation are fundamental tools that find application in several areas such as: array geometry optimization [2], spatial response evaluation [1], multiple beam beamforming [3], tapering/apodization [4], etc. Interpreted as a Discrete Fourier Transform (DFT) between a sampled spatial domain and a sampled Fourier domain, the problem is of broad interest for applied electromagnetics, for example in computing radiated fields [5], for antenna analysis/synthesis [6] and testing [7], in Fourier imaging systems ranging from synthetic aperture interferometric radiometry for Earth observation [8] or radio astronomy [9], to Magnetic Resonance Imaging (MRI) [10] and tomography [11].

A broad set of matrix-multiplication techniques has been developed which is agnostic to the matrix structure [12] or takes advantage of the symmetries of the matrices [13], nevertheless they are sub-optimal in terms of reduction of the number of multiplications with respect to specialised techniques that benefit of the known structure of the matrix [14]. Exploitation of symmetry for array response matrix decomposition has been presented in [15], but the solution is limited to symmetric elements and/or wavevectors (exhibiting center and/or axis symmetry). The equivalence between the array response matrix of planar arrays, when the wavevectors correspond to the reciprocal lattice of the array elements, and the bidimensional Discrete Fourier Transform (2D-DFT) is detailed in [3] together with the application of the bidimensional Fast Fourier Transform (2D-FFT) for complexity reduction. When the array elements and wavevectors are non-uniformly distributed, an approximated solution to the multiplicative complexity reduction problem is offered by the Non-Uniform Fast Fourier Transform (NUFFT) [16, 17]. For a summary of the NUFFT theory applied to antenna problems, refer to [18].

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Generally speaking, when the array response matrix can be decomposed into the product of smaller matrices, exact computational simplification can be obtained compared to direct use of the full matrix. The Kronecker product [19, 20] offers a broad and rich algebraic framework for representing and exploiting matrix patterns. The Kronecker product representation also leads to efficient ("Fast") algorithmic implementations of various discrete unitary transforms [21–24]. Thus, it can be expected that casting the array response matrix in some form of Kronecker product may provide an effective representation for deriving efficient processing algorithms. The classes of non-uniform arrays' and wavenumbers' geometries that allow to recognize a pattern reducible to the Kronecker product in the array response matrix structure have been progressively enlarged. Planar arrays with separable element positions along two principal axes allow representing the bi-dimensional steering vector as the Kronecker product of two one-dimensional steering vectors [1] (§9.9). This result has been extended by Ribeiro and Nascimento to planar arrays with non-periodic but separable element positions and wavevectors which are separable along the same axes [25]. Proper organization of the array elements and wavevectors allows writing the array response matrix as the Kronecker product of two one-dimensional array response matrices. In recognition of the Kronecker matrix product structure, the procedure for the evaluation of the array response matrix has been termed Kronecker Array Transform (KAT) [25]. In a recent contribution, Masiero and Nascimento [26] extended the KAT to quasi-separable geometries where the otherwise separable planar array or wavenumber grid exhibits missing elements or wavevectors, which can be reduced to a thinning of the active array elements or to a pruning of the wavevectors.

Inspired by a generalization of the Kronecker matrix product introduced by Fino and Algazi [24] we extend the class of efficient decompositions to "semi-separable" planar arrays and wavenumbers geometries. This newly defined class requires only that the array elements are organized in columns and the wavenumbers are organized in rows. The array columns can be non-uniformly spaced and the elements of each column can be independently non-uniformly spaced. Similarly, the wavenumber rows can be non-uniformly spaced and wavenumbers in each row can be independently non-uniformly spaced (refer to Fig. 1). The generalized Kronecker product decomposition of the array response matrix for semi-separable geometries will be shortly referred to as Generalized Kronecker Array Transform (GKAT). Finally, it is worth noting that the semi-separable geometries that can be tackled with the GKAT cover also, as sub-case, the results of Bagchi and Mitra for periodic bi-dimensional arrays and nonuniform sampling of the wavenumber space on parallel lines [27].

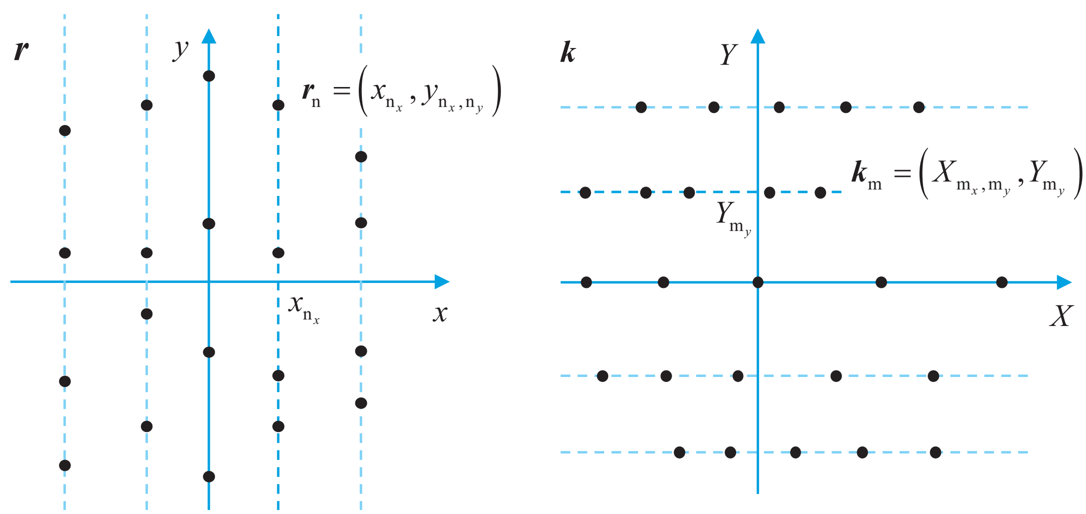

Figure 1. Semi-separable array and wavenumber geometries.

In the following section we will shortly describe the array processing context and mathematical notation, and recall key aspects of Kronecker product and the KAT. The generalized Kronecker product and its application to semi-separable geometries will be detailed in Section 3, where the GKAT will be also described. In Section 4 we will analyze GKAT implementation, computational aspects, and its integration with NUFFT techniques. Conclusions will follow.

2. ARRAY RESPONSE MATRIX AND KRONECKER ARRAY TRANSFORM (KAT)

The set of frequency responses of an N-element array, with element positions $\mathbf{r}_n$, $n = 0, \ldots, N-1$, to a monochromatic source of unit power with angular frequency ω and angular wavenumber vector $\mathbf{k}$ (wavevector) can be collected into a N-dimensional complex vector known as the array response vector (also known as steering or array manifold vector [1]),

$$\mathbf{v}(\mathbf{k}) = \begin{bmatrix} v_0(\mathbf{k}) \\ \vdots \\ v_n(\mathbf{k}) \\ \vdots \\ v_{N-1}(\mathbf{k}) \end{bmatrix} = \begin{bmatrix} \exp(-j\mathbf{k}\cdot\mathbf{r}_0) \\ \vdots \\ \exp(-j\mathbf{k}\cdot\mathbf{r}_n) \\ \vdots \\ \exp(-j\mathbf{k}\cdot\mathbf{r}_{N-1}) \end{bmatrix}. \tag{1}$$

The array response vectors corresponding to M different wavevectors,

$$\mathbf{K} = [\mathbf{k}_0, \ldots, \mathbf{k}_m, \ldots, \mathbf{k}_{M-1}], \tag{2}$$

can then be arranged together into an array response matrix,

$$\mathbf{V}(\mathbf{K}) = [\mathbf{v}(\mathbf{k}_0), \ldots, \mathbf{v}(\mathbf{k}_m), \ldots, \mathbf{v}(\mathbf{k}_{M-1})], \tag{3}$$

whose entries are,

$$[\mathbf{V}(\mathbf{K})]_{n|m} = v_{n|m} = \exp(-j\mathbf{k}_m \cdot \mathbf{r}_n). \tag{4}$$

The entry of the array response matrix ($v_{n|m}$) represents the LTI-MIMO response at the n-th element output port with respect to a plane-wave excitation of the m-th wavevector input, where $m = 0, \ldots, M-1$. The notation using the column separator between rows (outputs) and columns (inputs), introduced in [3], emphasises the multidimensional indexing of input and outputs. Evaluation of the array response matrix, its vector or matrix product, conjugate transpose and inverse (or pseudo-inverse) play a central role in several array applications [1–4].

The KAT [25] exploits the additiveness of the Fourier kernel and separability of the array element positions and wavevectors along identically oriented orthogonal axes. Separability of the element positions and wavenumber samples along the same axes requires that the elements of the planar array and the samples in the wavenumber space can be described in each space as the Cartesian product of two one-dimensional positions along the separable axes (i.e., X-Y for reference) equally oriented in the two domains.

Let the planar array be composed of $N = N_x N_y$ elements disposed in N_x columns with x-axis projection x_{n_x} (where $n_x = 0, \ldots, N_x - 1$) and N_y rows with y-axis projection y_{n_y} (where $n_y = 0, \ldots, N_y - 1$),

$$\mathbf{r}_n = \mathbf{r}_{n_x,n_y} = x_{n_x}\hat{\mathbf{x}} + y_{n_y}\hat{\mathbf{y}} \quad \text{with} \quad n = n_x + N_y n_y. \tag{5}$$

We can first note that separability of element positions along two principal axes allows us to represent the bi-dimensional steering vector as the Kronecker product of two one-dimensional steering vectors [1] (§9.9),

$$\mathbf{v}(\mathbf{k}) = \mathbf{v}_x(\mathbf{k}) \otimes \mathbf{v}_y(\mathbf{k}), \tag{6}$$

where,

$$[\mathbf{v}_x(\mathbf{k})]_{n_x} = \exp(-j\mathbf{k}\cdot\hat{\mathbf{x}}x_{n_x}) \tag{7}$$

$$[\mathbf{v}_y(\mathbf{k})]_{n_y} = \exp(-j\mathbf{k}\cdot\hat{\mathbf{y}}y_{n_y}). \tag{8}$$

The hat indicates the versor operator, and Bellman's Kronecker product notation [19] is adopted throughout this paper. Accordingly, given a $K \times L$ matrix $\mathbf{A} \in \mathbb{C}^{K\times L}$, and a $P \times Q$ matrix $\mathbf{B} \in \mathbb{C}^{P\times Q}$, their Kronecker product, $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$, is a $KP \times LQ$ matrix, $\mathbf{C} \in \mathbb{C}^{KP\times LQ}$, with the elements defined by,

$$[\mathbf{C}]_{i|j} = [\mathbf{A}]_{k|l}[\mathbf{B}]_{p|q}, \tag{9}$$

where,

$$i = Pk + p \begin{cases} i = 0, \ldots, KP - 1 \\ k = 0, \ldots, K - 1 \\ p = 0, \ldots, P - 1 \end{cases} \tag{10}$$

$$j = Ql + q \begin{cases} j = 0, \ldots, LQ - 1 \\ l = 0, \ldots, L - 1 \\ q = 0, \ldots, Q - 1. \end{cases} \tag{11}$$

The indexing in Eqs. (10)–(11) can also be interpreted as a bi-dimensionalization of the i and j indexes:

$$i \,|\, j = p, k \,|\, q, l, \tag{12}$$

$$[\mathbf{C}]_{p,k|q,l} = [\mathbf{A}]_{k|l}[\mathbf{B}]_{p|q}, \tag{13}$$

This interpretation of the Kronecker product can be used to extend the result in Eq. (6) to a grid of wavevectors separable along the same X-Y axes.

Let the grid of wavevectors be composed of $M = M_x M_y$ wavevectors $\mathbf{k}_m$ disposed along M_x columns with x-axis projection X_{m_x} (where $m_x = 0, \ldots, M_x - 1$), and along M_y rows with y-axis projection Y_{m_y} (where $m_y = 0, \ldots, M_y - 1$),

$$\mathbf{k}_m = \mathbf{k}_{m_x,m_y} = X_{m_x}\hat{\mathbf{x}} + Y_{m_y}\hat{\mathbf{y}}, \quad m = m_x + M_y m_y \tag{14}$$

We can now observe that, due to the additiveness of the exponential function,

$$\begin{aligned} v_{n|m} &= v_{n_x,n_y|m_x,m_y} \\ &= \exp(-j\mathbf{k}_m \cdot \mathbf{r}_n) \\ &= \exp(-jX_{m_x}x_{n_x})\exp(-jY_{m_y}y_{n_y}). \end{aligned} \tag{15}$$

which is readable in terms of Eq. (13) as,

$$[\mathbf{V}(\mathbf{K})]_{n_x,n_y|m_x,m_y} = [\mathbf{V}_y(\mathbf{K}_y)]_{n_y|m_y}[\mathbf{V}_x(\mathbf{K}_x)]_{n_x|m_x}, \tag{16}$$

where,

$$[\mathbf{V}_x(\mathbf{K}_x)]_{n_x|m_x} = \exp(-jX_{m_x}x_{n_x}) \tag{17}$$

$$[\mathbf{V}_y(\mathbf{K}_y)]_{n_y|m_y} = \exp(-jY_{m_y}y_{n_y}), \tag{18}$$

and, in summary,

$$\mathbf{V}(\mathbf{K}) = \mathbf{V}_y(\mathbf{K}_y) \otimes \mathbf{V}_x(\mathbf{K}_x). \tag{19}$$

This constitutes the definition of the Kronecker Array Transform [25]. It is worth noting that the order of X and Y components is exchanged with respect to [26] as we prefer to use an ordering that follows the order of the Cartesian axes.

3. GENERALIZED KRONECKER PRODUCT AND GENERALIZED KRONECKER ARRAY TRANSFORM (GKAT)

Fino and Algazi [24] have shown that a wide class of discrete unitary transforms can be represented as a generalization of the Kronecker product. The starting point of their generalization is the interpretation of the Kronecker product in terms of Mason's signal flow graph [28, 29] which is shown in Fig. 2. The matrices $\mathbf{B}$ and $\mathbf{A}$ play the role of block multipliers interconnected by perfect shuffle permutation stages $\mathbf{P}_{L,P}$ and $\mathbf{P}_{P,K}$, where the permutation matrix $\mathbf{P}_{N,M}$ corresponds to a symmetric distribution of N packs of M cards into M packs of N cards. It can now be observed that by preserving the same block structure, additional freedom can be obtained by allowing each block matrix to be different (refer to Fig. 3).

The Fino-Algazi definition can be extended to non-square matrices. Given two sets of matrices, $\{\mathbf{A}^p\}$ of $K \times L$ matrices $\mathbf{A}^p \in \mathbb{C}^{K\times L}$, with $p = 0, \ldots, P - 1$, and $\{\mathbf{B}^l\}$ of $P \times Q$ matrices $\mathbf{B}^l \in \mathbb{C}^{P\times Q}$,

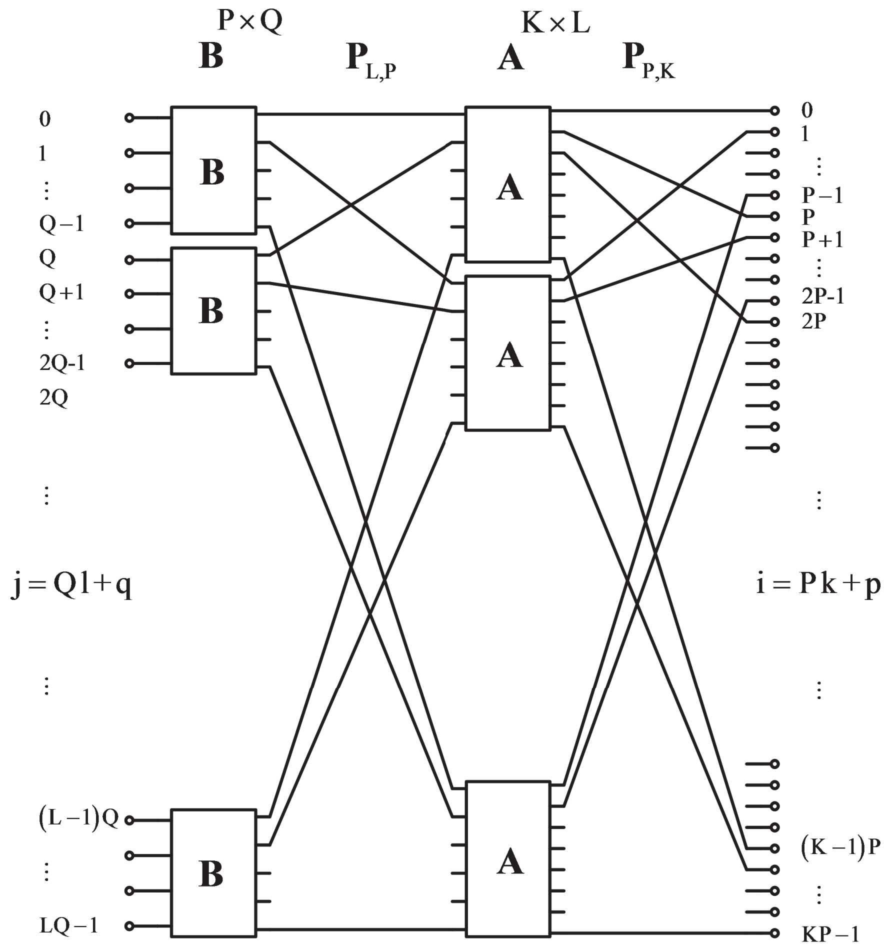

Figure 2. Kronecker product flow graph.

with $l = 0, \ldots, L-1$, their Generalized Kronecker Product (GKP), $\mathbf{C} = \{\mathbf{A}^p\} \otimes \{\mathbf{B}^l\}$, is an $KP \times LQ$ matrix $\mathbf{C} \in \mathbb{C}^{KP \times LQ}$ with the elements defined by,

$$[\mathbf{C}]_{i|j} = [\mathbf{C}]_{p,k|q,l} = [\mathbf{A}^p]_{k|l} \left[\mathbf{B}^l\right]_{p|q}, \tag{20}$$

and indexes running as per Eqs. (10)–(11). When the matrices of the set $\{\mathbf{A}^p\}$ are all the same, $\mathbf{A}^p = \mathbf{A}$, and the matrices of the set $\{\mathbf{B}^l\}$ are all the same, $\mathbf{B}^l = \mathbf{B}$, then the GKP corresponds to the standard Kronecker product.

We can now apply this generalization to "semi-separable" geometries (refer to Fig. 1). Let the planar array be composed of $N = N_x N_y$ elements disposed in N_x columns with x-axis projection x_{n_x} (where $n_x = 0, \ldots, N_x - 1$), and let each column include N_y elements with y-axis projection y_{n_y,n_y} (where $n_y = 0, \ldots, N_y - 1$),

$$\mathbf{r}_n = \mathbf{r}_{n_x,n_y} = x_{n_x}\hat{\mathbf{x}} + y_{n_x,n_y}\hat{\mathbf{y}}, \quad n = n_x + N_y n_y. \tag{21}$$

Let also the grid of wavevectors be composed of $M = M_x M_y$ wavevectors disposed along M_y rows with y-axis projection Y_{m_y} (where $m_y = 0, \ldots, M_y - 1$), and let each row include M_x wavevectors with x-axis projection X_{m_x,m_y} (where $m_x = 0, \ldots, M_x - 1$),

$$\mathbf{k}_m = \mathbf{k}_{m_x,m_y} = X_{m_x,m_y}\hat{\mathbf{x}} + Y_{m_y}\hat{\mathbf{y}}, \quad m = m_x + M_y m_y. \tag{22}$$

By the additiveness of the exponential function,

$$\begin{aligned} v_{n|m} &= v_{n_x,n_y|m_x,m_y} = \exp(-j\mathbf{k}_m \cdot \mathbf{r}_n) \\ &= \exp(-jX_{m_x,m_y} x_{n_x})\exp(-jY_{m_y} y_{n_x,n_y}), \end{aligned} \tag{23}$$

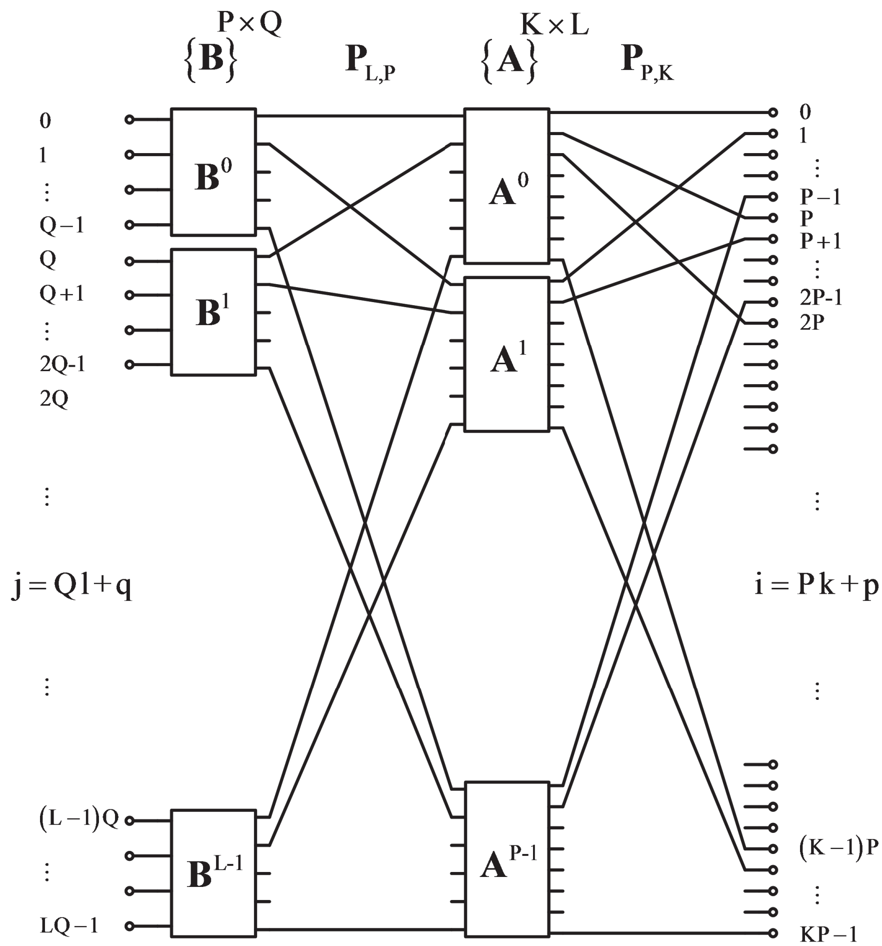

Figure 3. Fino-Algazi Generalized Kronecker product flow graph.

which is expressible in terms of Eq. (20) as,

$$[\mathbf{V}(\mathbf{K})]_{n_x,n_y|m_x,m_y} = \left[\mathbf{V}_y^{n_x}(\mathbf{K}_y)\right]_{n_y|m_y}\left[\mathbf{V}_x(\mathbf{K}_x^{m_y})\right]_{n_x|m_x},$$

and, in summary,

$$\mathbf{V}(\mathbf{K}) = \left\{\mathbf{V}_y^{n_x}(\mathbf{K}_y)\right\}\otimes\left\{\mathbf{V}_x(\mathbf{K}_x^{m_y})\right\}, \tag{24}$$

where,

$$\left[\mathbf{V}_x(\mathbf{K}_x^{m_y})\right]_{n_x|m_x} = \exp(-jX_{m_x,m_y}x_{n_x}) \tag{25}$$

$$\left[\mathbf{V}_y^{n_x}(\mathbf{K}_y)\right]_{n_y|m_y} = \exp(-jY_{m_y}y_{n_x,n_y}), \tag{26}$$

with $\mathbf{V}_x(\mathbf{K}_x^{m_y}) \in \mathbb{C}^{N_x\times M_x}$, and $\mathbf{V}_y^{n_x}(\mathbf{K}_y) \in \mathbb{C}^{N_y\times M_y}$.

Equation (24) constitutes the definition of the Generalized Kronecker Array Transform (GKAT).

4. IMPLEMENTATION AND COMPUTATIONAL ASPECTS

The GKAT, as many other *divide and conquer* methods, solves large, complex problems by partitioning them into a set of smaller sub-problems of lower complexity. A non-factorized evaluation of the $N_xN_y \times M_xM_y$ full array response matrix $\mathbf{V}(\mathbf{K})$ would require $N_xN_yM_xM_y$ complex multiplications, and would be equivalent to a bi-dimensional Non-uniform Discrete Fourier Transforms (2D-NUDFT). Exploiting the Generalized Kronecker Product structure, the number of multiplication is reduced to M_y

evaluations of $\mathbf{V}_x(\mathbf{K}_x^{m_y})$ and N_x evaluations of $\mathbf{V}_y^{n_x}(\mathbf{K}_y)$ with a total number of multiplications equal to:

$$M_y N_x M_x + N_x N_y M_y = (M_x + N_y) N_x M_y \tag{27}$$

which is equivalent to a computational saving of an order of magnitude. Additionally, it must be noted that the array response matrices $\mathbf{V}_x(\mathbf{K}_x^{m_y})$ and $\mathbf{V}_y^{n_x}(\mathbf{K}_y)$ correspond to those of linear arrays and can be recast as one-dimensional Non-uniform Discrete Fourier Transforms (1D-NUDFT) with non-equispaced elements and/or wavenumbers, for which the 1D-NUFFT is a fast approximation solution [17].

For the sake of assessing the multiplicative computational advantage of GKAT and its hybrid integration with 1D-NUFFTs, we will assume that the number of elements and wavevectors are the same and a square of N_x ($N_x = N_y = M_x = M_y$). The number of multiplications required for the 2D-NUDFT, the GKAT (based on 1D-NUDFTs), the 2D-NUFFT (Type 3), and the hybrid GKAT with 1D-NUFFTs (Type 3), is reported in Table 1. Parameters K and c respectively govern the correlation/interpolation window radius and the FFT oversizing in the NUFFT [18]. A radix-2 FFT has been assumed for the multiplicative complexity of the FFT within the NUFFT [30], and relevant twiddle factors needed for the zero-padded version. The number of multiplications as a function of the one-dimensional parameter N_x is shown in Fig. 4 (in a log-log scale). Together with the improvement of the GKAT with respect to the 2D-NUDFT (both exact), one observes the improvement of the hybrid GKAT+1D-NUFFTs with respect to the 2D-NUFFT due to the different effects of the FFT oversizing in the one-dimensional and bi-dimensional cases (i.e, $2c$ vs c^2).

In developing and describing the GKAT we made use of array columns with identical number of elements N_y and wavevectors rows with identical number of samples M_x. Similarly to the procedure described in Masiero and Nascimento [26] we can further extend the GKAT to geometries with unequal number of elements and/or wavevectors, which can be reduced to a thinning of the inputs and/or pruning of the outputs of the signal flow graph of Fig. 3). Finally, the class of semi-separable geometries has been described according to bi-dimensional Cartesian axes but the reader should consider that solid rotation of the array and wavenumber space as well as affine transformations of the two spaces [31] are trivial extensions of the addressable semi-separable geometries.

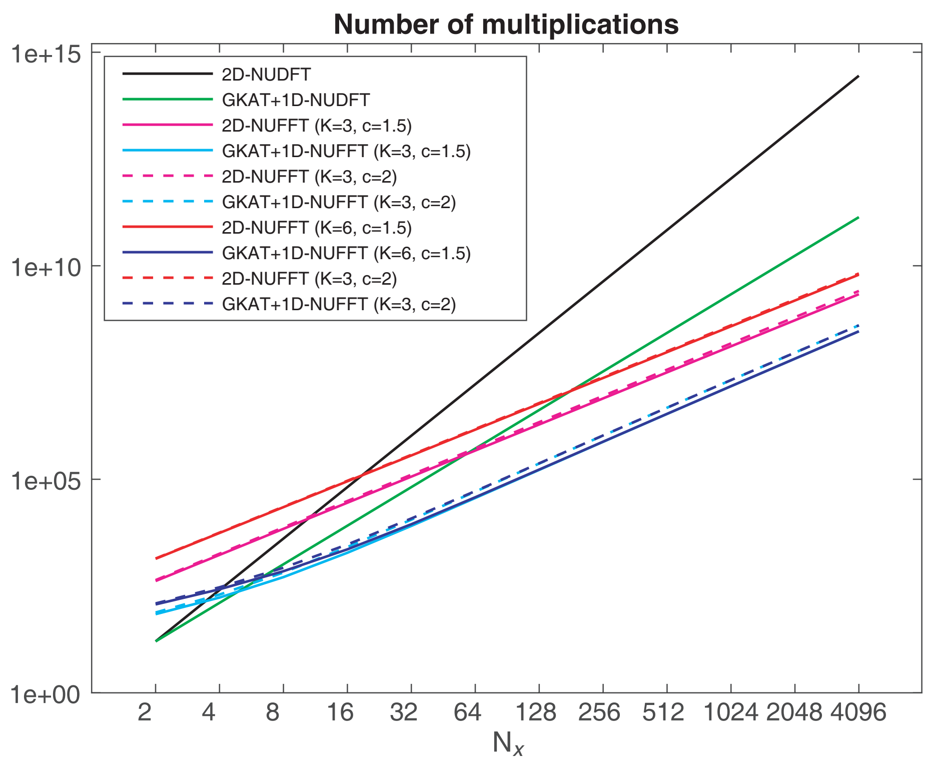

Figure 4. Multiplicative complexity comparison.

Table 1. Multiplicative complexity comparison.

Technique	Number of Multiplications
2D-NUDFT	${N_x}^4$
GKAT+1D-NUDFTs	${N_x}^3$
2D-NUFFT	$\left[2(2K+1)^2+c^2+1\right]N_x^2+c^2N_x^2\log_2\left(\frac{cN_x}{2}\right)$
GKAT+1D-NUFFTs	$2\left[(2K+1)+c+1\right]N_x^2+cN_x^2\log_2\left(\frac{cN_x}{2}\right)$

5. CONCLUSION AND FUTURE WORK

In this paper we first extended the generalized Kronecker product introduced by Fino and Algazi to non-square block matrices, and then we applied it to the array response matrix decomposition problem. The introduced Generalized Kronecker Array Transform (GKAT) broadens the class of array and wavenumber geometries that can be addressed as semi-separable geometries. This approach can be considered a generalization of the Kronecker Array Transform which is applicable to separable and quasi-separable geometries. An analysis of the Mason's signal flow graph of the GKAT demonstrates that the same computational complexity savings of the KAT are achieved. Fast transform techniques can also be applied to the block matrices composing the GKAT, further speeding up the overall array response matrix computation, which constitutes a central problem in many array processing applications.

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