*Progress In Electromagnetics Research M, Vol. 112, 55–65, 2022*

# Spatiotemporal Localized Waves and Accelerating Beams in a Uniformly Moving Dielectric Medium

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**Abstract**—A study is presented of several types of nondiffracting and slowly diffracting spatiotemporally localized waves supported by a simple dielectric medium moving uniformly with speed smaller or larger than the phase speed of light in the rest frame of the medium. The Minkowski material relations are not independent in the case that the speed of motion equals the phase speed of the medium; hence, the electric displacement and magnetic induction vectors cannot be uniquely determined from them. Following, however, a waveguide-theoretic approach, separate equations can be written for the longitudinal and transverse (with respect to the direction of motion) electromagnetic field intensities. The fundamental solutions associated with these equations provide a uniform transition between the cases of ordinary and Čerenkov-Vavilov radiation. The equation satisfied by the longitudinal field components in the absence of sources is examined in detail. In the temporal frequency domain one has an exact parabolic equation which supports accelerating beam solutions. The space-time equation supports several types of nondiffracting and slowly diffracting spatiotemporally localized waves. Comparisons are also made with the acoustic pressure equation in the presence of a uniform flow.

## 1. INTRODUCTION

Within the framework of the Minkowski [1] formulation of electrodynamics of uniformly moving media one considers Maxwell's equations

$$\begin{aligned} \nabla \times \vec{E}\left(\vec{r},t\right) &= -\frac{\partial}{\partial t}\,\vec{B}\left(\vec{r},t\right), \\ \nabla \times \vec{H}\left(\vec{r},t\right) &= \frac{\partial}{\partial t}\,\vec{D}\left(\vec{r},t\right) + \vec{J}\left(\vec{r},t\right), \\ \nabla \cdot \vec{D}\left(\vec{r},t\right) &= \rho\left(\vec{r},t\right), \\ \nabla \cdot \vec{B}\left(\vec{r},t\right) &= 0, \end{aligned} \tag{1}$$

together with the constitutive relations

$$\begin{aligned} \vec{D} + \frac{1}{c^2}\,\vec{v}\times\vec{H} &= \varepsilon'\left(\vec{E} + \vec{v}\times\vec{B}\right), \\ \vec{B} - \frac{1}{c^2}\,\vec{v}\times\vec{E} &= \mu'\left(\vec{H} - \vec{v}\times\vec{D}\right). \end{aligned} \tag{2}$$

Here, $\rho$ and $\vec{J}$ denote, respectively, the externally applied electric charge and current volume density distributions; $\varepsilon'$ and $\mu'$ are the permittivity and permeability of the medium in its rest frame; $\vec{v}$ is the uniform velocity of the moving frame; and $c$ is the speed of light *in vacuo*.

*Received 9 May 2022, Accepted 1 July 2022, Scheduled 25 July 2022*

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For convenience, the velocity of the medium is taken along the positive $z$-direction, i.e., $\vec{v} = v\vec{a}_z$. (This restriction does not detract from the generality of the problem because a coordinate transformation of the final solution can be used to treat the more general case.) If this is borne in mind, the constitutive relations in the laboratory frame are found to be [2]

$$\begin{aligned} \vec{D} &= \varepsilon' \bar{a} \cdot \vec{E} + \vec{\Omega} \times \vec{H}, \\ \vec{B} &= -\vec{\Omega} \times \vec{E} + \mu' \bar{a} \cdot \vec{H}, \end{aligned} \tag{3}$$

accompanied by the following definitions:

$$\begin{aligned} &\bar{a} = a\bar{I}_t + \vec{a}_z\vec{a}_z; \quad \bar{I}_t = \vec{a}_x\vec{a}_x + \vec{a}_y\vec{a}_y, \\ &a = \left(1-\beta^2\right)/\left(1-n^2\beta^2\right), \quad \vec{\Omega} = \Omega\vec{a}_z; \\ &\Omega = \beta\left(n^2-1\right)/c\left(1-n^2\beta^2\right); \quad \beta = v/c, \\ &n = c/v_0; \quad v_0 \,(\text{phase velocity}) = 1/\sqrt{\varepsilon'\mu'}. \end{aligned} \tag{4}$$

Here, $\bar{a}$ is a uniaxially anisotropic dyadic; $\bar{I}_t$ is the transverse (with respect to $z$) idemfactor; and the parameters $a$ and $\Omega$ depend on $\beta$, the ratio of the velocity of the medium and the speed of light *in vacuo*, and the index of refraction $n$, i.e., the ratio of the speed of light *in vacuo* and the phase velocity of light in the rest frame of the medium.

The constitutive relations in Eq. (3) indicate that the 'effective medium" in the laboratory frame is magnetoelectric in the sense that $\vec{D}$ and $\vec{B}$ depend on both $\vec{E}$ and $\vec{H}$. This is an induced anisotropy, however, arising from the motion and should be contrasted to the natural magnetoelectricity exhibited by several materials, even in the absence of motion.

Substituting the constitutive relations given in Eq. (3) into Maxwell's equations [cf. Eq. (1)], one obtains a definite form of the Maxwell-Minkowski equations which are written in the following convenient form:

$$\begin{aligned} &\vec{D}_0 \times \vec{E} + (\partial/\partial t)\,\mu'\bar{a} \cdot \vec{H} = 0, \\ &\vec{D}_0 \times \vec{H} - (\partial/\partial t)\,\varepsilon'\bar{a} \cdot \vec{E} = \vec{J}, \\ &\vec{D}_0 \cdot \left(\varepsilon'\bar{a} \cdot \vec{E}\right) = \rho + \vec{\Omega} \cdot \vec{J}, \\ &\vec{D}_0 \cdot \left(\mu'\bar{a} \cdot \vec{H}\right) = 0. \end{aligned} \tag{5}$$

$\vec{D}_0$ stands for the differential operator $\nabla - \vec{\Omega}(\partial/\partial t)$.

It is convenient at this stage to express the electromagnetic fields $\vec{E}$ and $\vec{H}$ in terms of the scalar potential $\Phi$ and vector potential $\vec{A}$ as follows: $\vec{E} = -\partial\vec{A}/\partial t - \vec{D}_0\Phi$ and $\mu'\bar{a} \cdot \vec{H} = \vec{D}_0 \times \vec{A}$. Using, then, the operator $\vec{D}_a = (1/a)\bar{a} \cdot \vec{D}_0$ and the definition $\vec{A}_0 = \bar{a} \cdot \vec{A}$, the "extended" Lorentz gauge $\vec{D}_0 \cdot \vec{A}_0 + (a/v_0^2)(\partial\Phi/\partial t) = 0$ yields the following uncoupled equations for the potential functions:

$$\begin{aligned} &\left[\vec{D}_a \cdot \vec{D}_0 - \left(a/v_0^2\right)\left(\partial^2/\partial t^2\right)\right]\Phi = -\left(\rho + \vec{\Omega} \cdot \vec{J}\right)/\left(a\varepsilon'\right), \\ &\left[\vec{D}_a \cdot \vec{D}_0 - \left(a/v_0^2\right)\left(\partial^2/\partial t^2\right)\right]\vec{A}_0 = -\mu' a\vec{J}. \end{aligned} \tag{6}$$

These equations and the corresponding electromagnetic fields have been examined extensively in the literature (see, e.g., [2, 3]). For $0 < v < v_0$, both $a$ and $\Omega$ are positive, and one has *ordinary radiation*. The support of the Green's function corresponding to a unit source located at the origin at some initial time, say $t = 0$, consists of an expanding wavefront which is an oblate spheroid with respect to the $z$-axis. The center of the wavefront moves away from the origin with a constant velocity which is smaller than the velocity of light in the medium. This shell always encloses the source point, i.e., the source radiates in all directions. For $v > v_0$, both $a$ and $\Omega$ are negative. Although the basic equations satisfied by the fields remain hyperbolic, a change occurs in the space-time signature, resulting in a new type of radiation known as the Čerenkov-Vavilov radiation. In this case, the entire wavefront, which is still an oblate spheroid, is "dragged" away from the source point. The point source is outside the

shell, and the shell is at all times tangent to a circular cone of interior half-angle $\theta$ specified by the relationship $\cos\theta = [(n^2\beta^2 - 1)/\beta^2(n^2 - 1)]^{1/2}$. An observer outside the cone does not experience any radiation effects. On the other hand, someone located within the conical region detects, in general, two discontinuities caused first by one side of the expanding and moving wavefront, and then the other.

It is our specific intent in this article to present a detailed study of spatiotemporally localized waves in a simple dielectric medium moving uniformly with speeds $v < v_0$, $v > v_0$ and $v = v_0$, where $v_0$ is the speed of light in the medium at rest.

*Luminal*, or focus wave modes (FWM), *superluminal*, or X waves (XWs), and *subluminal* spatiotemporally localized solutions to various hyperbolic equations governing acoustic, electromagnetic and quantum wave phenomena have been studied intensively during the past few years (see [4–17] for pertinent literature). Further details can be found in two recent edited monographs on the subject [18, 19]. In general, both linear and nonlinear LW pulses exhibit distinct advantages in comparison to conventional quasi-monochromatic signals. Their spatiotemporal confinement and extended field depths render them especially useful in diverse physical applications.

## 2. SPATIOTEMPORALLY LOCALIZED WAVES I. $v < v_0$

Consider the expanded form of the equation for the scalar potential $\Phi$ in the absence of sources; specifically,

$$\left[\nabla_t^2 + \frac{1}{a}\frac{\partial^2}{\partial z^2} - 2\frac{\Omega}{a}\frac{\partial^2}{\partial z \partial t} - \left(\frac{a}{v_0^2} - \frac{\Omega^2}{a}\right)\frac{\partial^2}{\partial t^2}\right]\Phi(\vec{r}, t) = 0. \tag{7}$$

It will be convenient to introduce the new wave function $\phi(\vec{r}, t) = \Phi(\vec{r}, \tau)$; $\tau = t + \Omega z$. The latter obeys the simpler equation

$$\left(a\nabla_t^2 + \frac{\partial^2}{\partial z^2} - \frac{1}{v_{ph}^2}\frac{\partial^2}{\partial \tau^2}\right)\phi(\vec{r}, \tau) = 0; \quad v_{ph} = \frac{v_0}{|a|}. \tag{8}$$

If the speed of motion of the medium $v$ is smaller from the phase velocity $v_0$ of the light in the medium at rest, it follows from Eq. (4) that $a$ is positive. Eq. (8), then, corresponds to that associated with a uniaxially anisotropic medium. Three types of nondiffracting, spatiotemporally localized waves are associated with this equation.

The simplest "superluminal" localized wave is the X wave [8] given below:

$$\begin{aligned} &\phi(\vec{r}, \tau) = \left[\frac{1}{a}\rho^2 + (a_1 + i\gamma_s\xi)^2\right]^{-1/2}; \quad \rho = \sqrt{x^2 + y^2}, \\ &\xi = z - V\tau, \quad \gamma_s = \left[(V/v_{ph})^2 - 1\right]^{-1/2}; \quad V > v_{ph}. \end{aligned} \tag{9}$$

The *effective group velocity* is defined as the speed at which the complex envelope travels. It can be determined by using the definition of the variable $\tau$.

$$z - V\tau = z - V(t + \Omega z) = (1 - V\Omega)\left(z - \frac{V}{1 - V\Omega}t\right). \tag{10}$$

Thus,

$$V_{gr}^{eff} = \frac{V}{1 - V\Omega} = \frac{V}{1 + \dfrac{vV}{c^2}\dfrac{c^2 - v_0^2}{v^2 - v_0^2}}. \tag{11}$$

This velocity can range from $-\infty$ to $+\infty$. For a normalized speed $c = 1$ and with $v_0 = 2/3$, $v = 1/3$, the phase speed $v_{ph}$ equals 9/16. The effective group speed approaches the minimum positive speed 9/11 as $V \to v_{ph} = 9/16$ from above, $V_{gr}^{eff} \to \infty$ as $V \to 9/5$ from below, $V_{gr}^{eff} \to -\infty$ as $V \to 9/5$ from above, and tends to zero as $V \to \infty$. It should be noted that if the medium in the stationary frame is vacuum (free space), $a = 1$, $\Omega = 0$, $v_{ph} = v_0 = c$. Then, $V_{gr}^{eff} = V > c$.

An elementary "subluminal" nondiffracting localized wave is the MacKinnon wavepacket (cf. [11] and references therein)

$$\phi\left(\vec{r},\tau\right)=\exp\left[-ik\gamma_b\frac{V}{v_{ph}}\left(z-\frac{v_{ph}^2}{V}\tau\right)\right]\frac{\sin\left[k\sqrt{\rho^2/a+\gamma_b^2\xi^2}\right]}{\sqrt{\rho^2/a+\gamma_b^2\xi^2}};$$
$$\xi=z-V\tau,\quad \gamma_b=\left[1-\left(V/v_{ph}\right)^2\right]^{-1/2};\quad V<v_{ph}, \tag{12}$$

where $k$ is a positive parameter. The effective group velocity $V_{gr}^{eff}$ of the wavepacket is the same as that in Eq. (11), with the difference that the velocity $V$ must be smaller from the phase velocity $v_{ph}$. For positive values of $V, V_{gr}^{eff}$ ranges from zero to a maximum positive value as $V\to v_{ph}$ from below. For a normalized speed $c=1$ and with $v_0=2/3$, $v=1/3$, the phase speed $v_{ph}$ equals $9/16$. The effective group speed approaches the maximum positive speed $9/11$ as $V\to v_{ph}=9/16$ from below, and becomes zero for $V=0$. It should be noted that if the medium in the stationary frame is vacuum (free space), $a=1$, $\Omega=0$, $v_{ph}=v_0=c$. Then, $V_{gr}^{eff}=V<c$.

In addition to the "subluminal" group velocity of the envelope, the MacKinnon wavepacket has a "superluminal" *effective phase velocity*. This is the speed appearing in the exponential term multiplying the complex envelope. It is determined by using the definition of the variable $\tau$.

$$z-\frac{v_{ph}^2}{V}\tau=z-\frac{v_{ph}^2}{V}\left(t+\Omega z\right)=\left(1-\Omega\frac{v_{ph}^2}{V}\right)\left[z-t\left(\frac{V}{v_{ph}^2}-\Omega\right)^{-1}\right]. \tag{13}$$

Thus,

$$V_{ph}^{eff}=\left(\frac{V}{v_{ph}^2}-\Omega\right)^{-1}=\left(\frac{V}{v_{ph}^2}+\frac{v}{c^2}\frac{c^2-v_0^2}{v^2-v_0^2}\right)^{-1}. \tag{14}$$

For a normalized speed $c=1$ and with $v_0=2/3$, $v=1/3$, the phase speed $v_{ph}$ equals $9/16$. For positive values of $V$, the effective phase speed has negative values and tends to $-\infty$ as $V\to 45/256$ from below. $V_{ph}^{eff}\to+\infty$ as $V\to 45/256$ from above and becomes zero as $V\to v_{ph}=9/16$ from below. It should be noted that if the medium in the stationary frame is vacuum (free space), $a=1$, $\Omega=0$, $v_{ph}=v_0=c$. Then, $V_{ph}^{eff}=c^2/V>c$.

The simplest "luminal" localized wave is the focus wave mode [4]

$$\phi\left(\vec{r},\tau\right)=\frac{1}{a_1+i\left(z-v_{ph}\tau\right)}e^{i\beta\left(z+v_{ph}\tau\right)}\exp\left[-\frac{\beta}{a}\frac{\rho^2}{a_1+i\left(z-v_{ph}\tau\right)}\right], \tag{15}$$

where $a_1$ and $\beta$ are positive free parameters. The effective group speed of the envelope and the effective phase are determined as follows:

$$z\mp v_{ph}\tau=z\mp v_{ph}\left(t+\Omega z\right)=\left(1\mp\Omega v_{ph}\right)\left(z\mp\frac{v_{ph}}{1\mp\Omega v_{ph}}t\right). \tag{16}$$

Thus,

$$V_{gr}^{eff}=\frac{v_{ph}}{1-\dfrac{vv_{ph}}{c^2}\dfrac{c^2-v_0^2}{v_0^2-v^2}},\quad V_{ph}^{eff}=\frac{v_{ph}}{1+\dfrac{vv_{ph}}{c^2}\dfrac{c^2-v_0^2}{v_0^2-v^2}}. \tag{17}$$

It is clear that $V_{ph}^{eff}$ is positive and smaller than $v_{ph}$. For a normalized speed $c=1$ and with $v_0=2/3$, $v=1/3$, the phase speed $v_{ph}$ equals $9/16$. In this case, $V_{ph}^{eff}=3/7<v_{ph}$ and $V_{gr}^{eff}=9/11>v_{ph}$. If the medium in the stationary frame is vacuum (free space), $a=1$, $\Omega=0$, $v_{ph}=v_0=c$. Then, $V_{ph}^{eff}=-c$ and $V_{gr}^{eff}=c$.

## 3. SPATIOTEMPORALLY LOCALIZED WAVES II. $v > v_0$

It follows from Eq. (4) that $a < 0$ if the speed of motion $v$ is smaller than the speed of light in vacuum but $v > v_0$. Under these conditions, the character of the equation governing the wave function in Eq. (18) changes since the transverse variables become time-like, and it is given by

$$\left(-\bar{a}\nabla_t^2 + \frac{\partial^2}{\partial z^2} - \frac{1}{v_{ph}^2}\frac{\partial^2}{\partial \tau^2}\right)\phi\left(\vec{r},\tau\right) = 0; \quad v_{ph} = \frac{v_0}{|a|}, \quad \bar{a} = |a|\,. \tag{18}$$

The simplest localized wave X wave to this equation assumes the form

$$\begin{aligned} \phi\left(\vec{r},\tau\right) &= \left[\frac{1}{\bar{a}}\rho^2 + \left(a_1 + i\gamma_b\xi\right)^2\right]^{-1/2}, \\ \xi &= z - V\tau, \quad \gamma_b = \left[1 - \left(V/v_{ph}\right)^2\right]^{-1/2}; \quad V < v_{ph}. \end{aligned} \tag{19}$$

The effective group velocity of the wavepacket is given by

$$V_{gr}^{eff} = \frac{V}{1 - V\Omega} = \frac{V}{1 + \dfrac{vV}{c^2}\dfrac{c^2 - v_0^2}{v^2 - v_0^2}}. \tag{20}$$

It is clear from this expression that $V_{gr}^{eff} < V < v_{ph}$.

A MacKinnon wavepacket solution to Eq. (18) is given by

$$\begin{aligned} \phi\left(\rho,z,\tau\right) &= \exp\left[-ik\frac{V}{v_{ph}}\gamma_s\left(z - \frac{v_{ph}^2}{V}\tau\right)\right]\frac{\sin\left[k\sqrt{\rho^2/\bar{a} + \gamma_s^2\xi^2}\right]}{\sqrt{\rho^2/\bar{a} + \gamma_s^2\xi^2}}; \\ \xi &= z - V\tau, \quad \gamma_s = \left[\left(V/v_{ph}\right)^2 - 1\right]^{-1/2}; \quad V > v_{ph}. \end{aligned} \tag{21}$$

The form is identical to that in Eq. (12), except that $\gamma_b$ is replaced by $\gamma_s$ with $V > v_{ph}$. The effective group velocity of the wavepacket is given by

$$V_{gr}^{eff} = \frac{V}{1 - V\Omega} = \frac{V}{1 + \dfrac{vV}{c^2}\dfrac{c^2 - v_0^2}{v^2 - v_0^2}}. \tag{22}$$

It follows, then, that $V_{gr}^{eff} < V > v_{ph}$. On the other hand, the effective phase speed is given by

$$v_{ph}^{eff} = \frac{v_{ph}^2/V}{1 + \dfrac{v_{ph}^2}{Vc}\dfrac{c^2 - v_0^2}{v^2 - v_0^2}}. \tag{23}$$

It follows from this expression that $V_{ph}^{eff} < v_{ph}^2/V > v_{ph}$.

A simple "luminal" localized wave to Eq. (18) is the FWM

$$\phi\left(\vec{r},\tau\right) = \frac{1}{a_1 - i\left(z - v_{ph}\tau\right)}e^{i\beta\left(z + v_{ph}\tau\right)}\exp\left[-\frac{\beta}{\bar{a}}\frac{\rho^2}{a_1 - i\left(z - v_{ph}\tau\right)}\right]. \tag{24}$$

The only difference from the expression in Eq. (15) is a complex conjugation in the term $a_1 + i(z - v_{ph}\tau)$. The effective group and phase speeds are given by

$$V_{gr}^{eff} = \frac{v_{ph}}{1 + \dfrac{vv_{ph}}{c^2}\dfrac{c^2 - v_0^2}{v^2 - v_0^2}}, \quad V_{ph}^{eff} = \frac{v_{ph}}{1 - \dfrac{vv_{ph}}{c^2}\dfrac{c^2 - v_0^2}{v^2 - v_0^2}}. \tag{25}$$

The effective group speed is positive and smaller than the phase speed $v_{ph}$. On the other hand, the effective phase speed assumes negative values. For a normalized speed $c = 1$ and $v_0 = 1/3$, $v = 2/3$, the phase speed $v_{ph}$ equals $9/5$. In this case, $V_{ph}^{eff} = -9/11$ and $V_{gr}^{eff} = 3/7 < v_{ph}$.

## 4. SPATIOTEMPORALLY LOCALIZED WAVES III. $v = v_0$

### 4.1. Waveguide-Theoretic Approach: Equations for the Longitudinal and Transverse Fields

The conventional Minkowski formalism cannot be used in this case because of inherent singularities; specifically, both $a$ and $\Omega$ become unbounded as $v \to v_0$. As a consequence, the material equations (3) are not independent and cannot be used to determine $\vec{D}$ and $\vec{B}$ uniquely in terms of $\vec{E}$ and $\vec{H}$. This means, in turn, that in contradistinction to the sublumimal ($v < v_0$) and superluminal ($v > v_0$) cases, one is unable in this situation to obtain separate differential equations for the electromagnetic field intensities $\vec{E}$ and $\vec{H}$. Following, however, a waveguide-theory approach, separate equations can be derived for the longitudinal and transverse (with respect to the direction of motion) electromagnetic field intensities. This possibility has been pointed out by Sen Gupta [20].

Owing to the uniaxial anisotropy of space induced by the motion of the medium, it will be useful to separate the variables and operators in Maxwell's equations (1) into their axial and transverse parts according to the conventional waveguide-theory approach [21]. The following identities will be used:

$$\begin{aligned} \nabla \times \vec{F} &= \nabla_t \times \vec{F}_t - \vec{a}_z \times \nabla_t F_z + \vec{a}_z \times \frac{\partial}{\partial z}\vec{F}_t, \\ \nabla \cdot \vec{F} &= \nabla_t \cdot \vec{F}_t + \frac{\partial}{\partial z} F_z. \end{aligned} \tag{26}$$

The subscripts "$z$" and "$t$" indicate the axial and transverse directions. The first two equations in (1) will now be written in a separate form as follows:

$$\begin{aligned} &- \vec{a}_z \times \nabla_t E_z + \vec{a}_z \times \frac{\partial}{\partial z}\,\vec{E}_t = -\frac{\partial}{\partial t}\vec{B}_t, \\ &\nabla_t \times \vec{E}_t = -\vec{a}_z \frac{\partial}{\partial t} B_z, \\ &- \vec{a}_z \times \nabla_t H_z + \vec{a}_z \times \frac{\partial}{\partial z}\,\vec{H}_t = \frac{\partial}{\partial t}\vec{D}_t + \vec{J}_t, \\ &\nabla_t \times \vec{H}_t = \vec{a}_z \frac{\partial}{\partial t}\, D_z + \vec{a}_z J_z. \end{aligned} \tag{27}$$

(The separate equations for the divergence equations in (1) are not included in this list because they will play no role at all in the case $v = v_0$). The constitutive relations (2) similarly yield

$$\begin{aligned} &\vec{D}_t - \varepsilon' v \vec{a}_z \times \vec{B}_t = \varepsilon' \vec{E}_t - \frac{1}{c^2}\, v \vec{a}_z \times \vec{H}_t, \\ &D_z = \varepsilon' E_z, \\ &\vec{B}_t + \mu' v \vec{a}_z \times \vec{D}_t = \mu' \vec{H}_t + \frac{1}{c^2}\, v \vec{a}_z \times \vec{E}_t, \\ &B_z = \mu' H_z. \end{aligned} \tag{28}$$

Also, as a direct consequence of the condition $v = v_0$, one has the relations

$$\begin{aligned} \vec{a}_z \times \vec{E} &= -\beta \mu' c \vec{a}_z \times \left( \vec{a}_z \times \vec{H} \right), \\ \vec{a}_z \times \vec{H} &= \beta \varepsilon' c \vec{a}_z \times \left( \vec{a}_z \times \vec{E} \right), \end{aligned} \tag{29}$$

or, equivalently,

$$\begin{aligned} \vec{a}_z \times \vec{E}_t &= -\beta \mu' c \vec{H}_t, \\ \vec{a}_z \times \vec{H}_t &= -\beta \varepsilon' c \vec{E}_t. \end{aligned} \tag{30}$$

According to Eqs. (28) and (29), $\vec{E}$ and $\vec{H}$, as well as $\vec{E}_t$ and $\vec{H}_t$, are always orthogonal to each other in the case of the "transition" radiation ($v = v_0$).

The separated forms of Maxwell's equations and the constitutive relations can now be used to obtain the following equations for the longitudinal fields $E_z, H_z$ and the transverse fields $\vec{E}_t, \vec{H}_t$:

$$\begin{aligned}
&\left(\nabla_t^2 - \frac{2}{v_0}\frac{\partial^2}{\partial t \partial z} - \frac{1+\beta^2}{v_0^2}\frac{\partial^2}{\partial t^2}\right) E_z = \mu' v_0 \left(\nabla \cdot \vec{J} + \frac{\partial J_z}{\partial z} + \frac{1+\beta^2}{v_0^2}\frac{\partial J_z}{\partial t}\right), \\
&\left(\nabla_t^2 - \frac{2}{v_0}\frac{\partial^2}{\partial t \partial z} - \frac{1+\beta^2}{v_0^2}\frac{\partial^2}{\partial t^2}\right) H_z = \nabla \cdot \left(\vec{a}_z \times \vec{J}\right), \\
&\left(2\frac{\partial}{\partial z} + \frac{1+\beta^2}{v_0}\frac{\partial}{\partial t}\right) \vec{E}_t = \left(-\beta c \varepsilon'\right)^{-1} \left(\vec{J}_t + \vec{a}_z \times \nabla_t H_z - \varepsilon' v_0 \nabla_t E_z\right), \\
&\left(2\frac{\partial}{\partial z} + \frac{1+\beta^2}{v_0}\frac{\partial}{\partial t}\right) \vec{H}_t = \left(\beta c \mu'\right)^{-1} \left(-\mu' v_0 \vec{a}_z \times \vec{J}_t + \vec{a}_z \times \nabla_t E_z + \mu' v_0 \nabla_t H_z\right).
\end{aligned} \tag{31}$$

The equations for the transverse electromagnetic field intensities are first order in both space and time. Therefore, with knowledge of the axial components [obtained by integrating the first two equations in (31)] and the current density distribution, they are reduced to quadratures.

### 4.2. The Time-Dependent Green's Function for the Longitudinal Fields

On the basis of the findings in the previous section one need only investigate the equations for the longitudinal fields $E_z$ and $H_z$. The causal time-dependent fundamental solution, denoted by $G(\vec{r}, t)$, associated with these fields is defined as follows:

$$\begin{aligned}
&\left(\nabla_t^2 - \frac{2}{v_0}\frac{\partial^2}{\partial t \partial z} - \frac{1+\beta^2}{v_0^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{r}, t) = \delta(\vec{r})\,\delta(t), \quad t \geq 0, \\
&G(\vec{r}, t) = 0, \quad t < 0.
\end{aligned} \tag{32}$$

(By virtue of the space-time translational invariance of Eq. (32), the Green's function $G(\vec{r}, t/\vec{r}', t')$ corresponding to the source $\delta(\vec{r} - \vec{r}')\delta(t - t')$ is given by $G(\vec{r}, t/\vec{r}', t') = G(\vec{r} - \vec{r}', t - t')$, $t \geq t'$, with $G(\vec{r}, t/\vec{r}, t') = 0$, $t < 0$ in order to satisfy causality).

Equation (32) can be brought into a canonical form by means of the Galilean-type transformation

$$\xi = \frac{v_0 t}{\sqrt{1+\beta^2}}, \quad \varsigma = \sqrt{1+\beta^2}\left(z - \frac{v_0 t}{1+\beta^2}\right). \tag{33}$$

In the process of this transformation the source term in Eq. (32) must be multiplied by the inverse of the determinant of the Jacobian of the transformation (33), which in this case equals $v_0$. Therefore,

$$\left(\nabla_t^2 + \frac{\partial^2}{\partial \varsigma^2} - \frac{\partial^2}{\partial \xi^2}\right) G\left(\vec{R}, \xi\right) = v_0 \delta\left(\vec{R}\right)\delta(\xi); \quad \vec{R} = x\vec{a}_x + y\vec{a}_y + \varsigma\vec{a}_z. \tag{34}$$

The solution to this equation is well known:

$$G\left(\vec{R}, \xi\right) = -\frac{v_0}{4\pi R}\delta(R - \xi); \quad R = \left|\vec{R}\right|. \tag{35}$$

In terms of the original variables one has

$$\begin{aligned}
G(\vec{r}, t) \;=\; &-\frac{1}{4\pi}\sqrt{1+\beta^2}\left[x^2 + y^2 + \left(1+\beta^2\right)\left(z - \frac{v_0 t}{1+\beta^2}\right)^2\right]^{-1/2} \\
&\times \delta\left(t - \frac{\sqrt{1+\beta^2}}{v_0}\left[x^2 + y^2 + \left(1+\beta^2\right)\left(z - \frac{v_0 t}{1+\beta^2}\right)^2\right]\right).
\end{aligned} \tag{36}$$

The fundamental solution (36) can be interpreted as an expanding wavefront that arrives at $R = ut$; $u = v_0(1+\beta^2)^{-1/2}$ diminished by the geometrical factor $1/R$. $R$ can be taken as the radial distance from the point $\left\{0, 0, t/(1+\beta^2)\right\}$ and the observation point $\vec{r}(x, y, z)$, with a scaling of the $z$-axis dimension by the factor $\sqrt{1+\beta^2}$ due to the Lorentz contraction. The equation obeyed by the wavefront is determined

by setting the argument of the Dirac delta function to zero; thus, $R = ut$. For constant values of $t$ the wavefront is an oblate spheroid with semiaxes $ut, ut, ut(1+\beta^2)^{-1/2}$ along the directions of the $x$, $y$, and $z$ axes, respectively. The wavefront center moves along the $z$-direction at the speed $v_0/(1+\beta^2)$ which is seen to be smaller than $v_0$ and, hence, $v$. It is important to note in this case that the $z$-direction semiaxis is equal to the distance of the center of the shell from the position of the source. This means that the wavefront is always tangent to the $x$-$y$ plane at the source point.

### 4.3. Accelerating Monochromatic Beam

Consider, for simplicity, the $(2+1)D$ version of the homogeneous (in the absence of sources) equation (32), viz.,

$$\left(\frac{\partial^2}{\partial x^2} - \frac{2}{v_0}\frac{\partial^2}{\partial t \partial z} - \frac{1+\beta^2}{v_0^2}\frac{\partial^2}{\partial t^2}\right) u(x,z,t) = 0. \tag{37}$$

The ansatz

$$f(x,z,t) = g(x,z) \exp\left[i\omega \frac{1+\beta^2}{2v_0}\left(z - \frac{2v_0}{1+\beta^2}t\right)\right] \tag{38}$$

gives rise to the equation

$$\left(i\frac{\omega}{v_0}\frac{\partial}{\partial z} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) g(x,z) = 0. \tag{39}$$

This is an *exact parabolic equation* in contradistinction to the paraxial approximation of the Helmholtz equation associated with the temporal Fourier transform of the ordinary scalar wave equation. In addition to the well-known beam solutions of the usual paraxial equation, Eq. (18) has the following "accelerating" one [22, 23]:

$$g(x,z) = \exp\left[-\frac{1}{12}(2a+iz)\left(2a^2 - i4az + z^2 - 6x\sqrt{\omega/v_0}\right)\right] Ai\left(x\sqrt{\omega/v_0} - \frac{z^2}{4} + iaz\right). \tag{40}$$

Here, $Ai(\cdot)$ denotes the Airy function, and the positive parameter $a$ ensures finite energy for the monochromatic solution. The beam follows a parabolic trajectory upon propagation. Finite-energy broadband solutions can be obtained by using the solution (40) together with the ansatz (38) and undertaking appropriate superpositions with respect to the frequency $\omega$ (see, e.g., Ref. [24]).

### 4.4. Nondiffracting Spatiotemporally Localized Waves

Consider, next, Eq. (34) in the absence of sources, viz.,

$$\left(\nabla_t^2 + \frac{\partial^2}{\partial \varsigma^2} - \frac{\partial^2}{\partial \xi^2}\right) f(x,y,\varsigma,\xi) = 0, \tag{41}$$

with the variables $\varsigma$ and $\xi$ defined in Eq. (33). This expression has the form of a scalar wave equation, which is known to have a variety of nondiffracting localized wave solutions (see collective monographs [18, 19]. One such solution is the focus wave mode (FWM)

$$f(x,y,\varsigma,\xi) = \frac{1}{a + i(\varsigma - \xi)} \exp\left[ib(\varsigma + \xi)\right] \exp\left[-b\frac{x^2+y^2}{a+i(\varsigma-\xi)}\right], \tag{42}$$

where $a$ and $b$ are positive free parameters. Written in terms of the original variables, (42) becomes

$$f(x,y,z,t) = \frac{\exp\left(ib\sqrt{1+\beta^2}z\right)}{a + i\left(\sqrt{1+\beta^2}\left(z - \frac{2v_0}{1+\beta^2}t\right)\right)} \exp\left[-b\frac{x^2+y^2}{a+i\left(\sqrt{1+\beta^2}\left(z-\frac{2v_0}{1+\beta^2}t\right)\right)}\right]. \tag{43}$$

In contradistinction to the FWM solution of the ordinary scalar wave equation, which is *bidirectional* — it involves the terms $z \pm v_0 t$, this is an infinite-energy *unidirectional* nondiffracting localized wave

solution. Superposition over the free parameter $b$ can result in finite-energy wavepackets. On such solution is a variation of the splash wave mode [11]; specifically,

$$f(x,y,z,t) = \frac{1}{a_1 + i\left(\sqrt{1+\beta^2}\left(z - \dfrac{2v_0}{1+\beta^2}t\right)\right)} \times \left[a_2 - i\sqrt{1+\beta^2}z + \frac{x^2+y^2}{a_1 + i\left(\sqrt{1+\beta^2}\left(z - \dfrac{2v_0}{1+\beta^2}t\right)\right)}\right]^{-q}. \tag{44}$$

Here, $a_{1,2}$ and $q$ are positive parameters. The solution is confined both transversely and longitudinally (in time). As it propagates along the $z$-direction with speed $2v_0/(1+\beta^2)$, it sustains distortion; the latter can be moderated by choosing the condition $a_2 \gg a_1$.

Another nondiffracting localized solution to Eq. (41) is the infinite-energy unidirectional X wave

$$f(x,y,\varsigma,\xi) = \frac{1}{\sqrt{x^2+y^2+\left[a+i\gamma(\varsigma-\lambda\xi\right]^2}}; \quad \gamma = \frac{1}{\sqrt{\lambda^2-1}}, \quad \lambda > 1, \tag{45}$$

with $a > 0$. Written in terms of the original variables, one has

$$f(x,y,z,t) = \frac{1}{\sqrt{x^2+y^2+\left[a+i\gamma\sqrt{1+\beta^2}\left(z - \dfrac{(1+\lambda)\,v_0}{1+\beta^2}\,t\right)\right]^2}}. \tag{46}$$

### 4.5. Comparisons with the Convective Equation in Acoustics

The equation of acoustic pressure under conditions of uniform flow is given as follows:

$$\left[\nabla^2 - \frac{1}{u_0^2}\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)^2\right] p(\vec{r},t) = 0. \tag{47}$$

Here, $u_0$ is the speed of sound in the rest frame of the medium, and $\vec{u}$ is the uniform velocity of the background flow. In the special case where $\vec{u} = u\vec{a}_z$ and $u = u_0$ the resulting equation for the acoustic pressure is isomorphic to the equation for the longitudinal components $E_z$ and $H_z$ in the absence of sources [see Eq. (31)].

Under the assumption that $\vec{u} = u\vec{a}_z$ Eq. (47) has several exact analytical infinite-energy nondiffracting and finite-energy slowly nondiffracting spatiotemporally localized wave solutions. One such solution is the finite-energy splash mode

$$p(\rho, Z_+, Z_-) = \frac{1}{a_1 + i\Gamma\left(1-u/u_0\right)Z_+}\left(a_2 - i\Gamma\left(1+u/u_0\right)Z_- + \frac{\rho^2}{a_1 + i\Gamma\left(1-u/u_0\right)Z_+}\right)^{-q}; \quad Z_\pm = z - (u_0 \mp u)t, \quad \Gamma = 1/\sqrt{1-(u/u_0)^2}, \quad \rho = \sqrt{x^2+y^2}. \tag{48}$$

Here, $a_{1,2}$ are positive parameters. This solution is valid for $u < u_0$.

An X-shaped infinite-energy nondiffracting solution is given

$$p(\rho,z,t) = \frac{1}{\sqrt{\rho^2 + \dfrac{(a \pm i\,(z-Vt))^2}{\left(\dfrac{u-V}{u_0}\right)^2 - 1}}}, \tag{49}$$

with the condition $(u-V)^2 > u_0^2$. If the speeds $u$ and $V$ are assumed to be positive, this means that the wavepacket moves along the $z$-direction with a speed $V$ which is smaller from the flow speed $u$, and the difference $u - V$ is larger from the speed of sound $u_0$ in the rest frame of the medium.

## 5. CONCLUDING REMARKS

A detailed study has been undertaken of spatiotemporally localized waves in a simple dielectric medium moving uniformly with speeds $v < v_0$, $v > v_0$, and $v = v_0$, where $v_0$ is the speed of light in the medium at rest. A uniform transition has been provided between the cases of ordinary and Čerenkov-Vavilov radiation. As $v \to v_0^-$, the omnidirectional radiation in the former case is concentrated in the region $z \geq 0$, whereas, as $v \to v_0^+$, Čerenkov-Vavilov radiation takes place in a conical region with half-angle equal to $\pi/2$. The equation satisfied by the longitudinal components in the absence of sources has been examined in detail. In the temporal frequency domain the latter becomes an exact parabolic equation which supports accelerating beam solutions. The space-time equation supports several types of nondiffracting and slowly diffracting spatiotemporally localized waves. Comparisons have also made with the acoustic pressure equation in the presence of a uniform flow.

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