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An Off-Grid Compressed Sensing Method for Synthesis of Maximally Sparse Arrays with Arbitrary Beampatterns

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Abstract—Most of the works on sparse array synthesizing via the compressed sensing (CS) approach assume that the active elements exactly lie on the predefined grids. In fact, grid-mismatch error is unavoidable when an array aperture is discretized into grids, and the synthesizing results largely depend on the density of the grids. To overcome this limitation, an innovative off-grid CS approach is proposed for jointly estimating the excitations and positions of array elements. The synthesis problem is specifically formulated using a ridge regression model based on dynamic grids. The candidate positions of elements are treated as variables rather than constants predefined by discretization. Numerical experiments are conducted to validate the effectiveness and flexibility of the proposed method in realizing several maximally sparse arrays meeting the targeted patterns, i.e., the focused and shaped beam patterns of 1-D and 2-D arrays.

1. INTRODUCTION

Array antennas have been widely used in modern radio systems, such as satellite communication [1], sonar [2], radar [3], radio astronomy [4], electronic countermeasure [5], and microwave remote sensing [6], due to their advantages of easy realizing low sidelobe and high gain patterns as well as easy reconfiguring multiple, scanning, or shaped beams. The growing demand in fields of high-resolution radar imaging, microwave remote sensing, and communications has facilitated the application of large-scale array antenna in recent years. However, the high cost and massive data processing constrain the development of array antennas [7].

In order to reduce the element number as well as the effect of mutual coupling while achieving the desired radiation performance so as to decrease both the hardware and software costs, the aperiodic arrays with unequally-spaced elements [8] have been proposed, and it is exactly the reason that sparse array has been continuously focused on in the past decade [9–18]. Compared with thinning array, sparse array has more degrees of freedom which helps for further reducing the total number of elements since the radiating elements are randomly separated instead of being confined to regular lattices with 0.5λ spacing. Naturally, the synthesis of such an array is more complex and challenging, especially when the objective is to find the minimum number of elements realizing the desired radiation performance.

Up to now, the synthesis techniques of maximally sparse arrays in open literatures can be generally classified into three categories, i.e., evolutionary optimization algorithms [9], matrix pencil methods (MPM) [10], and CS-based methods [11]. Specifically, although evolutionary optimization algorithms such as genetic algorithm (GA) [12], differential evolution algorithm (DEA) [13], and hybrid algorithm composed of PSO-SOCP [14] have congenital advantage for solving nonlinear optimization problems, they all need to set the number of active elements in advance. Therefore, those methods cannot find the minimum number of elements intelligently and easily. Besides, multiple runs have to be done

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with varying input in order to find the sparse solution with as few elements as possible. This process is time-consuming and easily falls into local optimization especially for large aperture arrays. Non-iterative MPM has been successfully introduced to synthesize linear sparse arrays for overcoming these drawbacks in [10], which first focused on the problem of reconstructing a desired pattern with the minimum number of elements. Due to its effectiveness, MPM has been extended to synthesizing shaped-beam patterns [15], multiple patterns [16], and sparse planar arrays [17, 18], respectively. It performs a complete optimization on the number of elements, element weights, as well as the element positions based on the constructed Hankel matrix. However, the singular value decomposition (SVD) of Hankel matrix becomes cumbersome with the increase of array aperture. Consequently, this kind of method is still not suitable for the sparse optimization of large arrays. To cope with this issue, CS theory has been creatively applied to the maximally sparse array synthesis, which has been addressed as a sparse signal recovery [11]. A variety of CS methods have been introduced for finding the maximally sparse representation from oversampling dictionary. They roughly fall into three categories based on the Bayesian compressive sensing (BCS) algorithms [19–23], the FOCal under-determined system solver (FOCUSS) approaches [24–26], as well as convex optimization techniques [27–34].

Loosely speaking, the vast majority of existing CS methods usually assume the elements of sparse array to lie exactly on the grids generated by discretizing the continuous array aperture in advance. Then, the synthesis problem can be formulated as a sparse signal recovery with a constant dictionary. Unfortunately, the synthesized results are extremely sensitive to the discretized grids since the grid mismatch is unavoidable during the disretizing process. More in general, a finer discretization with dense grids is required to guarantee that the synthesized array has good sparsity along with the desired radiation performance. Nevertheless, the computational load and the risk of violating the RIP condition [35] are likely to increase especially for the case of planar array. In order to overcome these drawbacks caused by discretization, the perturbation method assuming that the true element positions deviate from the nearest predefined grids with unknown shifts has been developed in [24] to synthesize the sparse array fitting the desired pattern with as few elements as possible. In this way, the true dictionary is approximated as a summation of a presumed dictionary and a structured parameterized matrix. However, it is still subjected to the problems of aperture discretization because the positions of active elements are jointly determined by the predefined grids and calculated perturbations. Moreover, the synthesis performance depends on the accuracy of Taylor approximation. Recently, several improved methods based on the minimization of the weighted l_1-norm have also been introduced in [31–34] to overcome the limitations of traditional CS methods. Although a further reduction of element number is achieved, those works are solely suitable for the synthesis of the sparse arrays with pencil beams.

In this paper, an innovative method based on off-grid compressed sensing [36] is proposed as a complement to the existing methods for the synthesis of sparse array antennas with the minimum number of elements. Unlike the previous CS methods, the elements are randomly distributed in the continuous parameter space instead of on fixed grids or supposed to deviate from the fixed grids with unknown small shifts. The proposed method directly takes the element position and excitation as variables so as to effectively avoid the aperture discretization. According to off-grid CS, the synthesis problem is firstly formulated as a sparse signal recovery problem with an unknown parametric dictionary by combining the position parameter learning and sparse excitation recovery. An alternative optimization algorithm applied to line spectral estimation in [37] is extended to the sparse array synthesis for 1-D and 2-D. It is worth noting that the proposed method is very universal and flexible, and it can be used to synthesize the sparse linear and planar arrays with focused/shaped beams. Preliminary results have been recently presented in [38] demonstrating the effectiveness of the proposed method.

The rest of the paper is organized as follows. In Section 2, the sparse array synthesis is formulated using a ridge regression model with the element positions and excitations as unknowns. Section 3 provides numerical analysis and experiments to validate the effectiveness and advantages of the proposed method. Finally, some conclusions are drawn in Section 4.

2. FORMULATION FOR SYNTHESIS PROBLEM

Without loss of generality, a planar array with N elements located in x-y plane is considered and discussed in the following since the linear array can be easily derived as a special case. The array factor is given by

$$F(u,v) = \sum_{n=1}^{N} w_n e^{j\beta(x_n u + y_n v)} \tag{1}$$

where w_n denotes the excitation of the n-th element at position (x_n, y_n); the propagation constant $\beta = 2\pi/\lambda$; λ is the wavelength; $u = \sin(\theta)\cos(\varphi)$; and $v = \sin(\theta)\sin(\varphi)$. Assume that K samples are obtained at different directions (u_k, v_k) $(k = 1, ..., K)$, and the column vector $\mathbf{a}(x_n, y_n)$ is defined as

$$\mathbf{a}(x_n, y_n) = \left[e^{j\beta(x_n u_1 + y_n v_1)}, \cdots, e^{j\beta(x_n u_K + y_n v_K)}\right]^T \tag{2}$$

Then the vector-matrix form of (3) can be expressed as

$$\mathbf{F} = \mathbf{A}(\mathbf{x}, \mathbf{y})\,\mathbf{w} \tag{3}$$

where $\mathbf{F} = [F(u_1, v_1), F(u_2, v_2), ..., F(u_K, v_K)]^T$, $\mathbf{x} = [x_1, x_2, ..., x_N]$, $\mathbf{y} = [y_1, y_2, ..., y_N]$, $\mathbf{w} = [w_1, w_2, ..., w_N]^T$, and $\mathbf{A}(\mathbf{x}, \mathbf{y}) = [\mathbf{a}(x_1, y_1), \mathbf{a}(x_2, y_2), ..., \mathbf{a}(x_N, y_N)]$. If the array is symmetrical, the entry of $\mathbf{a}(x_n, y_n)$ should be modified as $a_k(x_n, y_n) = 4\cos(\beta x_n u_k)\cos(\beta y_n v_k)$. The goal herein is to synthesize a planar array with the minimum number of elements to reconstruct a prescribed pattern $F_{ref}(u, v)$ with a small matching error ε. That is, the synthesis aims to determine the minimum element number N along with the element excitations $\{w_n\}$ and their 2-D position coordinates $\{(x_n, y_n)\}$. Obviously, this is a complex non-convex optimization problem. To solve this problem, the conventional CS-based techniques avoid the design of $\{(x_n, y_n)\}$ by discretizing the continuous array aperture with grids as dense as possible so as to put in enough elements. The element with zero excitation is equivalent to having no element in the supposed location. Therefore, the active element locations are all selected from the predefined grids. But the most optimized active element positions are not exactly situated at the discretised grids. Unlike the above-mentioned handling manner, the candidate positions $\{(x_n, y_n)\}$ related to the dictionary $\mathbf{A}(\mathbf{x}, \mathbf{y})$ are treated as unknowns rather than the predefined positions. Such a synthesis problem can be described as

$$\begin{aligned} &\min_{\mathbf{x},\mathbf{y},\mathbf{w}} \|\mathbf{w}\|_0 \\ &s.t.\ \ \|\mathbf{F}_{ref} - \mathbf{A}(\mathbf{x}, \mathbf{y})\,\mathbf{w}\|_2 \le \varepsilon \end{aligned} \tag{4}$$

where the l_0-norm of $\mathbf{w}$ represents the number of nonzero elements in $\mathbf{w}$; $\mathbf{F}_{ref} = [F_{ref}(u_1, v_1), F_{ref}(u_2, v_2), ..., F_{ref}(u_M, v_M)]^T$ is a set of M samples of the reference pattern. According to the CS theory, Eq. (3) can be treated as a sparse signal recovery problem with an unknown parametric dictionary. Because the elements can be randomly arranged in the array aperture according to the formulation herein, this approach thus not only can increase the degree of freedom but also help for finding the optimal solution with fewer dynamic grids. The optimization problem of (3) can be regularized as follows,

$$\min_{\mathbf{x},\mathbf{y},\mathbf{w}} \|\mathbf{F}_{ref} - \mathbf{A}(\mathbf{x}, \mathbf{y})\,\mathbf{w}\|_2^2 + \gamma\|\mathbf{w}\|_0 \tag{5}$$

where $\gamma > 0$ is the regularization parameter controlling the tradeoff between sample matching and the sparsity of the solution. It is well known that the problem (4) is still NP-hard. Fortunately, many sparse approximation algorithms based on iterative reweighting [39] have been proposed in signal processing. As one typical variant, the iterative reweighted l_2 minimization is adopted here for achieving sparse solutions, and then the synthesis is further transformed to

$$\min_{\mathbf{x},\mathbf{y},\mathbf{w}} \|\mathbf{F}_{ref} - \mathbf{A}(\mathbf{x}, \mathbf{y})\,\mathbf{w}\|_2^2 + \gamma\mathbf{w}^H\mathbf{P}^{(i)}\mathbf{w} \tag{6}$$

where $\mathbf{P}(i)$ is a diagonal matrix which can be updated by

$$p_n^{(i)} = \frac{1}{\left|w_n^{(i)}\right|^2 + \xi^{(i)}} \tag{7}$$

It has been experimentally validated in [40] that parameter ξ should be updated by iterations instead of being fixed so as to improve the sparse recovery performance, e.g., we can set $\xi^{(0)} = 1$, if

$$\xi^{(i-1)} > 10^{-8} \text{ and } \left\|\mathbf{w}^{(i-1)} - \mathbf{w}^{(i-2)}\right\|_2 \le \sqrt{\xi^{(i-1)}} \tag{8}$$

then

$$\xi^{(i)} = \xi^{(i-1)}/10 \tag{9}$$

else

$$\xi^{(i)} = \xi^{(i-1)} \tag{10}$$

The iterative reweighting method in [39], however, could not deal with the optimization model (5) because the overcomplete dictionary is constructed based on the unknown position parameter $\{(x_n, y_n)\}$. Fortunately, we can utilize (5) to produce a closed form for $\mathbf{w}$ updating $\{(x_n, y_n)\}$. To solve the problem efficiently, the alternative optimization strategy [37] is extended to position parameter $\{(x_n, y_n)\}$ learning and sparse excitation $\mathbf{w}$ recovery. It is to say, once $\{(x_n, y_n)\}$ is determined, the analytic solution to (5) can be obtained as

$$\mathbf{w} = \left[\gamma \mathbf{P}^{(i)} + \mathbf{A}^H(\mathbf{x},\mathbf{y})\,\mathbf{A}(\mathbf{x},\mathbf{y})\right]^{-1} \mathbf{A}^H(\mathbf{x},\mathbf{y})\,\mathbf{F}_{ref} \tag{11}$$

By substituting (10) into (5), we get

$$\begin{aligned} &\|\mathbf{F}_{ref} - \mathbf{A}(\mathbf{x},\mathbf{y})\,\mathbf{w}\|_2^2 + \gamma \mathbf{w}^H \mathbf{P}^{(i)} \mathbf{w} \\ =\ & \mathbf{F}_{ref}^H \mathbf{F}_{ref} - \mathbf{F}_{ref}^H \mathbf{A}(\mathbf{x},\mathbf{y}) \left[\mathbf{A}^H(\mathbf{x},\mathbf{y})\,\mathbf{A}(\mathbf{x},\mathbf{y}) + \gamma \mathbf{P}^{(i)}\right]^{-1} \times \mathbf{A}^H(\mathbf{x},\mathbf{y})\,\mathbf{F}_{ref} \\ =\ & \mathbf{F}_{ref}^H \mathbf{F}_{ref} + g(\mathbf{x},\mathbf{y}) \end{aligned} \tag{12}$$

Obviously, the first term of (11) is a constant while the second term is associated with $\{(x_n, y_n)\}$, and thus the synthesis is equivalent to minimizing $g(\mathbf{x},\mathbf{y})$, i.e.,

$$\min_{\mathbf{x},\mathbf{y}} g(\mathbf{x},\mathbf{y}) \tag{13}$$

In the following the gradient descent algorithm is used to solve (12) via the iterative equation,

$$\begin{bmatrix} \mathbf{x}^{(i+1)} \\ \mathbf{y}^{(i+1)} \end{bmatrix} = \begin{bmatrix} \mathbf{x}^{(i)} \\ \mathbf{y}^{(i)} \end{bmatrix} - \alpha \begin{bmatrix} \nabla g\left(\mathbf{x}, \mathbf{y}^{(i)}\right)|_{\mathbf{x}=\mathbf{x}^{(i)}} \\ \nabla g\left(\mathbf{x}^{(i)}, \mathbf{y}\right)|_{\mathbf{y}=\mathbf{y}^{(i)}} \end{bmatrix} \tag{14}$$

where the iteration coefficient α is set to 1 in our simulations. The overall synthesis procedure can be mainly divided into four steps:

1) Input K sampling points (u_k, v_k), the reference pattern $\mathbf{F}_{ref}$ and the assumed number of elements N, which represents the initial length of the unknown vectors $\mathbf{w}$, $\mathbf{x}$ and $\mathbf{y}$;
2) Initialize the element positions $\mathbf{x}^{(0)}$ and $\mathbf{y}^{(0)}$, and then the initial excitation $\mathbf{w}^{(0)} = [\mathbf{A}^H(\mathbf{x}^{(0)}, \mathbf{y}^{(0)})\mathbf{A}(\mathbf{x}^{(0)}, \mathbf{y}^{(0)})]^{-1}\mathbf{A}^H(\mathbf{x}^{(0)}, \mathbf{y}^{(0)})\mathbf{F}_{ref}$;
3) Update the positions $\mathbf{x}^{(i)}$ and $\mathbf{y}^{(i)}$ using gradient descent algorithm, then a new excitation $\mathbf{w}(i)$ is obtained by substituting $\mathbf{x}^{(i)}$ and $\mathbf{y}^{(i)}$ back into (10);
4) Repeat step 3) until $\|\mathbf{w}^{(i)} - \mathbf{w}^{(i-1)}\|_2 \le \mu$ (μ is a threshold controlling the approximation error).

The choice of γ should consider the variance of the matching error and the properties of $\mathbf{A}(\mathbf{x},\mathbf{y})$ and $\mathbf{w}$, which has been dealt with in [36, 42, 43] for sparsely solving the inverse problem. We herein adopt the adaptive update method of [36] to determine γ, i.e., $\gamma^{(i)}$ is updated as

$$\gamma^{(i)} = C_0 \frac{\left\|\mathbf{F}_{ref} - \mathbf{A}^{(i)}(\mathbf{x},\mathbf{y})\,\mathbf{w}^{(i)}\right\|_2^2}{K} \tag{15}$$

where C_0 is a constant. According to our experience, $C_0 = 5$ is generally good for sparse array synthesis. Different K and N will lead to different results. Consequently, a tradeoff between the number of active elements and the matching error should be reached depending on K and N. The iteration threshold μ controls the approximation error of $\|\mathbf{w}^{(i)} - \mathbf{w}^{(i-1)}\|_2$, herein it is set as 10^{-3}. Fig. 1 presents the flowchart of the synthesis process for summary.

As shown above, the proposed method only needs a specific reference pattern a priori, and there is no restriction on the array geometry, size, as well as the number of elements. In addition, we should point

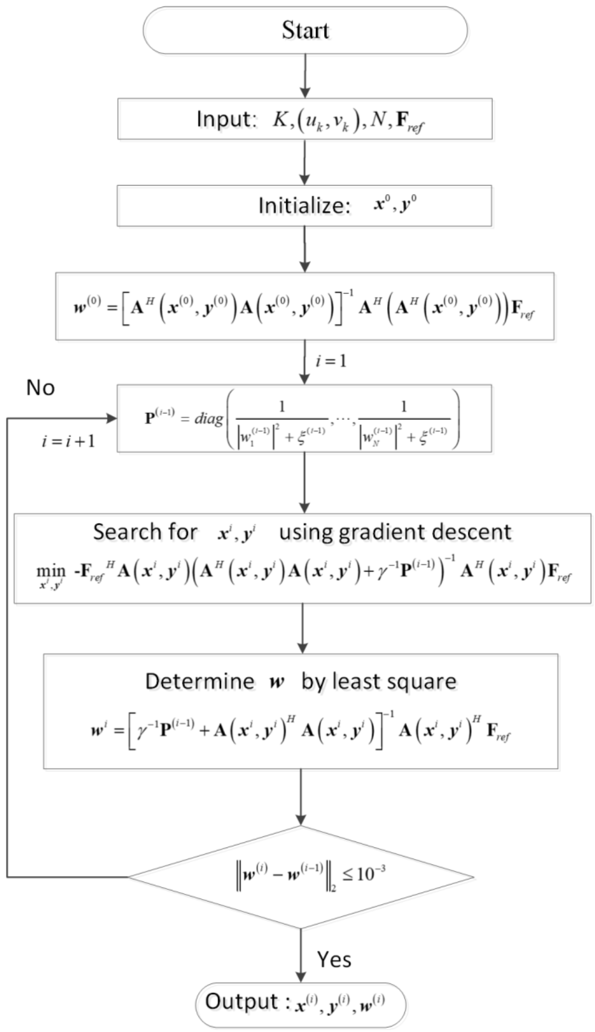


Figure 1. The flowchart of the synthesis process.

out that the proposed method is also applicable to the synthesis of scanned arrays by just extending the sampling range to $u^2 + v^2 \leq [1 + \sin(\theta_M)]^2$ in the synthesis formulation for the desired steering cone of θ_M. Therefore, the sparse scanned array can also be efficiently synthesized by modifying the value range of u-v according to the desired steering range.

3. NUMERICAL EXPERIMENTS

In this section, some representative problems related to the sparse synthesis of both linear and planar arrays are considered to validate the effectiveness and flexibility of the proposed method. It should be emphasized that the proposed method aims at synthesizing a sparse array meeting the reference pattern with as few elements as possible. Therefore, the accuracy of the pattern matching is important to evaluating the performance. To quantitatively describe the approximation degree of the synthesized pattern $F(u, v)$ to the reference pattern $F_{ref}(u, v)$, the matching error ε is usually defined as [19–22]

$$\varepsilon = \frac{\displaystyle\int_{u^2+v^2\leq 1} \left|F_{ref}\left(u, v\right) - F\left(u, v\right)\right|^2 dudv}{\displaystyle\int_{u^2+v^2\leq 1} \left|F_{ref}\left(u, v\right)\right|^2 dudv} \tag{16}$$

The calculation of ε in discrete domain is as follows

$$\varepsilon = \frac{\sum_{m=1}^{M} \left| F_{ref}\left(u_m, v_m\right) - F\left(u_m, v_m\right) \right|^2}{\sum_{m=1}^{M} \left| F_{ref}\left(u_m, v_m\right) \right|^2} \tag{17}$$

where M is the number of samples from the region $u_m^2 + v_m^2 \leq 1$ for both the synthesized pattern and reference pattern.

For all the numerical experiments, parameters γ and ξ are updated according to Eq. (14) and Eq. (7) to Eq. (9), respectively, and $\mu = 10^{-3}$ is used. All the simulations are coded in MATLAB and conducted on a 2.30 GHz/8 GB RAM PC under Windows 10.

3.1. Linear Array Synthesis with Chebyshev Pattern

In the first example, a sparse linear array for matching a Chebyshev pattern with PSLL $= -30$ dB is synthesized. Notice that such a reference pattern can be realized by a 40-element uniform linear array with equidistance $d = 0.5\lambda$ spanning an array aperture of $L = 19.5\lambda$. To synthesize a sparse array with a simplified feeding network while keeping the same symmetric geometry as the reference, the reference pattern is sampled by K points with a uniform step $u \in [0, 1]$, and N elements are randomly located in the right half aperture $[0, L/2]$. Fig. 2 shows the synthesized results by the proposed method for different values of $N \in [10, 40]$ and $K \in [15, 40]$. For the same synthesis problem, the number of unknowns has been reduced from 200 to less than 40 compared with the synthesis method in [31]. Fig. 3 presents the synthesized patterns with different numbers of elements, where the Chebyshev pattern of 40 elements is also presented for comparison. It should be stressed that the saving of elements should base on the guarantee of reconstructing the desired reference pattern reliably. As shown in Fig. 3, the solution of 19 active elements realized a poor matching ($\varepsilon = 3.18 \times 10^{-2}$), while the solution of 23 greatly improves matching error to $\varepsilon = 3.98 \times 10^{-5}$. Although the sparse array with 22 elements provides a slightly larger matching error of 2.30×10^{-3}, the mismatches mainly appear in the far sidelobe region, and their influences on the power pattern can be neglected, i.e., it is acceptable. Fig. 4 gives our realized pattern and that from [31] for detailed comparison. The proposed method has obtained the sparser solution with fewer unknowns. The total computation time T_{tot} is about 0.37 s in this case. Table 1 lists half of the reconstructed positions and excitations in consideration of symmetry.

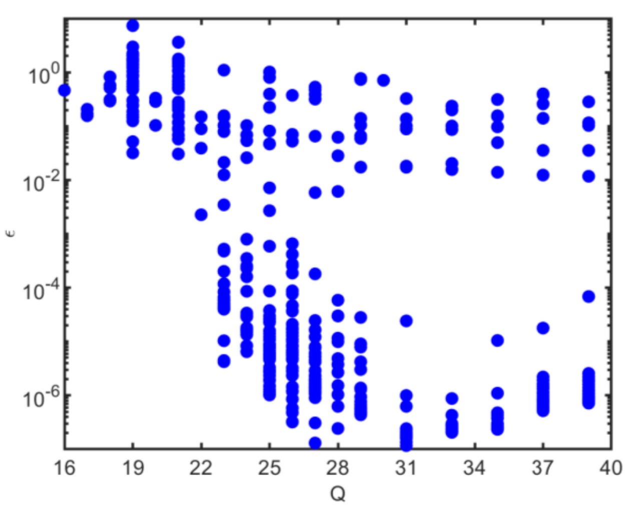


Figure 2. Pareto diagram of the results with the active element number $Q < 40$.

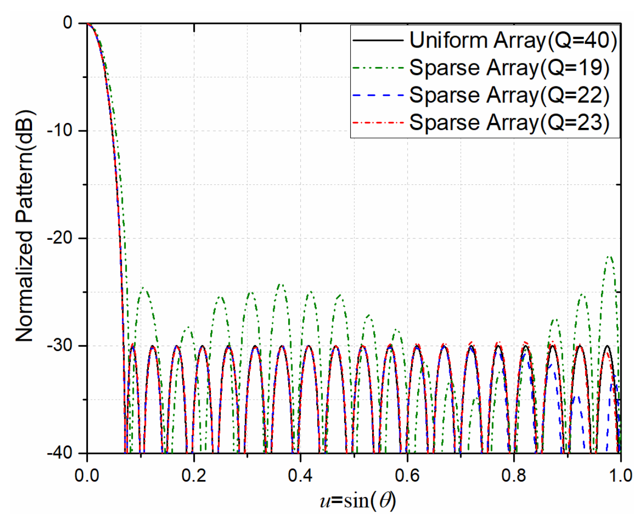


Figure 3. Power patterns of the sparse arrays with different number of elements along with the reference uniform array.

Table 1. Related parameters for the synthesized sparse array.

q	$x_q(\lambda)$	w_q	q	$x_q(\lambda)$	w_q
1	0.4645	1	7	6.0473	0.5726
2	1.3937	0.9754	8	6.9795	0.4662
3	2.3231	0.9275	9	7.9091	0.3620
4	3.2531	0.8591	10	8.8256	0.2621
5	4.1837	0.7740	11	9.7091	0.3356
6	5.1151	0.6769			

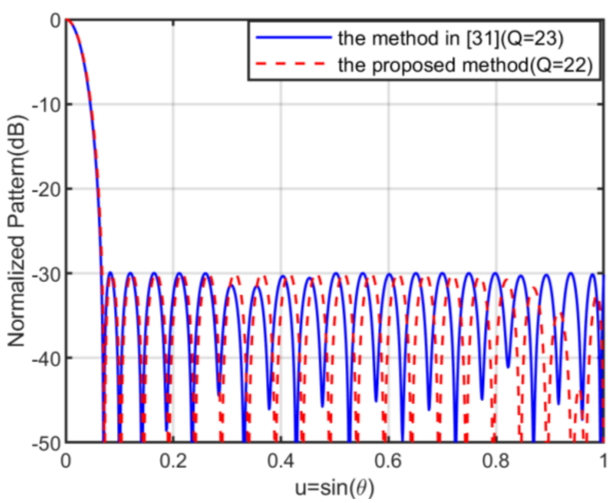


Figure 4. Power patterns synthesized by the proposed method with 22 elements and by the method in [31] with 23 elements.

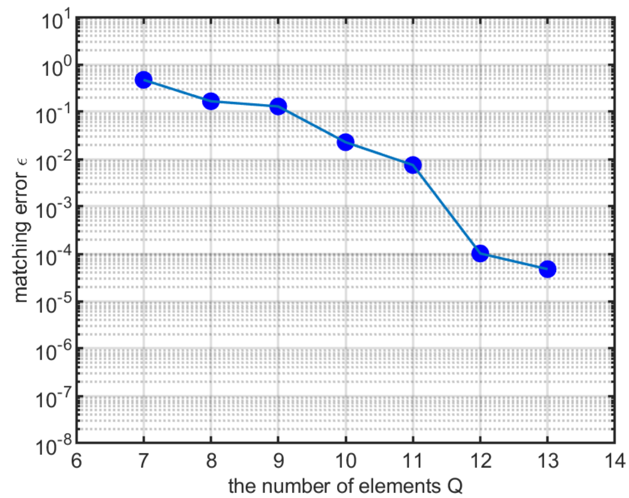


Figure 5. Variation of ε versus Q in the second example.

3.2. Linear Array Synthesis with Multiple Pattern

In this example, a multibeam pattern is considered. A 16-element nonuniformly spaced linear array spanning an aperture of 7.46λ was synthesized by the WORD method to realize two beams steering at $30°$ and $-10°$, respectively as shown in Fig. 8 of [41]. Here, we present our results obtained by using less than 16 elements. The synthesized results are presented in Fig. 5 in terms of matching error ε vs the number of elements Q. As expected, the value of ε decreases as the Q increases. However, when the number reaches a certain number, say 12, the improvement of matching error becomes negligible. Fig. 6 shows the comparison of the synthesized patterns with $Q = \{11, 12, 13\}$ and $\varepsilon = \{0.75 \times 10^{-2}, 1.5047 \times 10^{-4}, 5.7378 \times 10^{-5}\}$ along with the desired multibeam pattern in [41]. The synthesis times are 3.08, 1.17 s, and 1.12 s for $Q = \{11, 12, 13\}$. As can be seen, the matching error mainly appears in the far sidelobe region. In addition, the solution of $Q = 12$ elements is the optimal tradeoff among the sparsity, accuracy reconstruction, and time consumption. That is to say, the proposed method saves 25% of elements. Table 2 lists the corresponding element positions and excitations.

3.3. Planar Array Synthesis with Inseparable Chebyshev Pattern

This example is devoted to evaluating the effectiveness of the proposed method for sparse optimization of planar array. The reference patterns with $\text{SLL} = -30\,\text{dB}$ as shown in Fig. 7(a) are from a uniformly

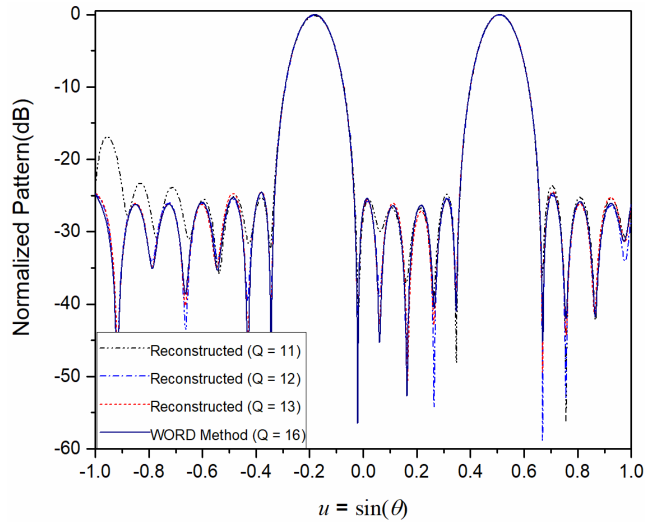
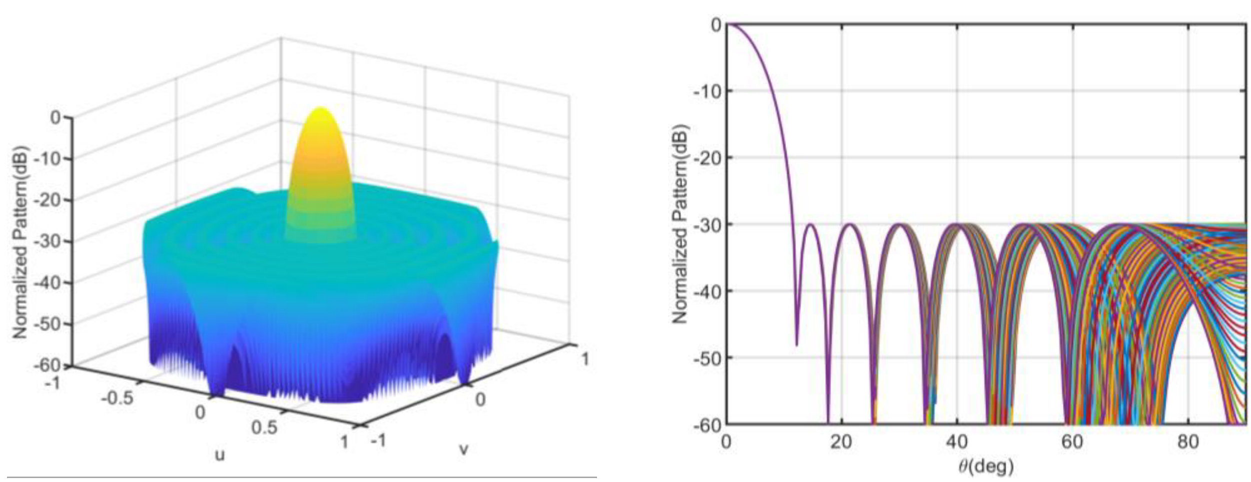


Figure 6. Synthesized multiple patterns by the proposed method and the WORD method in [41].

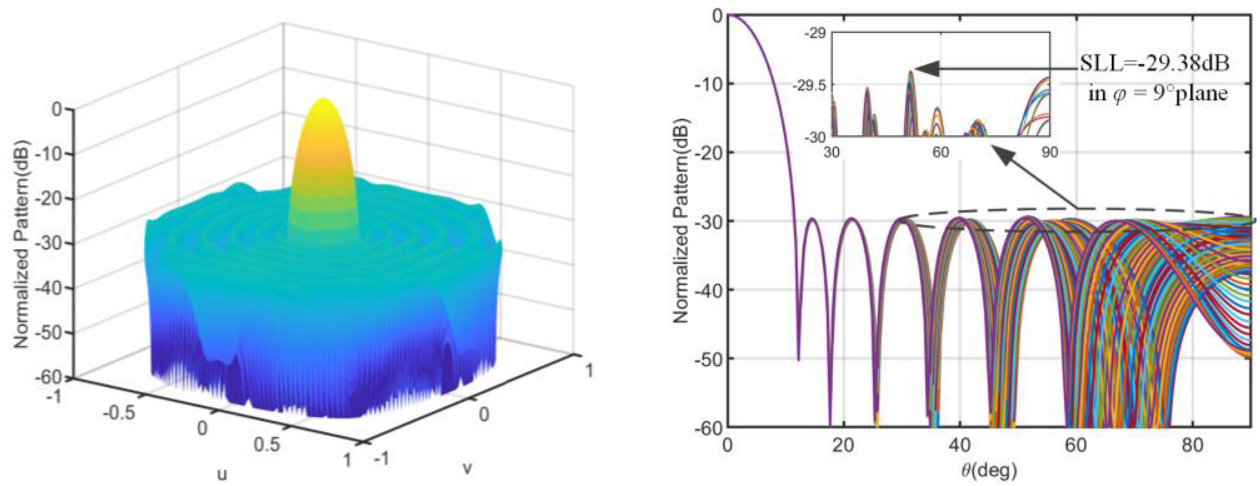
Table 2. Related parameters for the synthesized 12-element sparse array.

q	$x_q(\lambda)$	w_a	q	$x_q(\lambda)$	w_a
1	0	$0.4365e^{-j0.0812}$	7	4.1927	$0.9494e^{-j1.1895}$
2	0.4477	$0.2673e^{-j0.4309}$	8	4.6771	$0.9339e^{-j1.6734}$
3	1.2708	$0.6217e^{+j1.8297}$	9	5.6492	$0.7424e^{+j0.4383}$
4	1.7677	$0.7892e^{+j1.2673}$	10	6.1295	$0.5914e^{-j0.0244}$
5	2.7418	$0.8737e^{-j2.8343}$	11	7.0979	$0.3700e^{+j2.1780}$
6	3.2183	$1e^{+j2.9557}$	12	7.5024	$0.2696e^{+j1.6122}$

spaced Chebyshev planar array of 14×14 elements located in a square of 6.5λ side. They are symmetrical about the x and y axes, and every quadrant has 7×7 elements. The left column of Fig. 7 presents the synthesized 3-D patterns while the right column presents the φ-cut patterns corresponding to 360° with $\Delta\varphi = 1°$. In the sparse synthetization, $N_x \times N_y = 5\times5$ elements are supposed to randomly locate in the first quadrant for keeping the same symmetry of geometry as the reference. So, the sparse array



(a)



(b)

Figure 7. Synthesized patterns. (a) The desired Chebyshev pattern with 196 elements, (b) the reconstructed pattern with 81 elements.

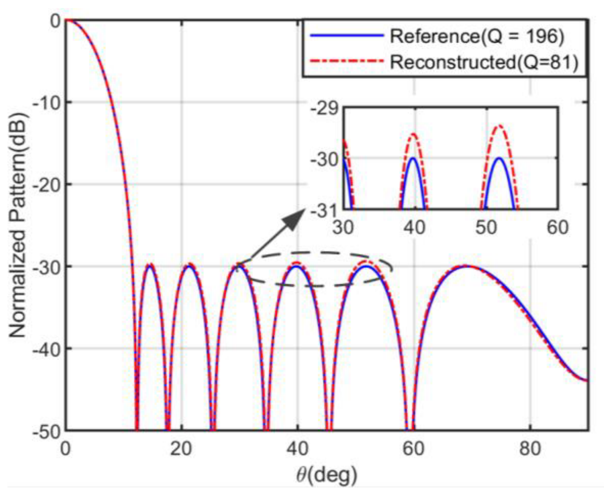


Figure 8. Synthesized patterns in $\varphi = 9^\circ$ plane.

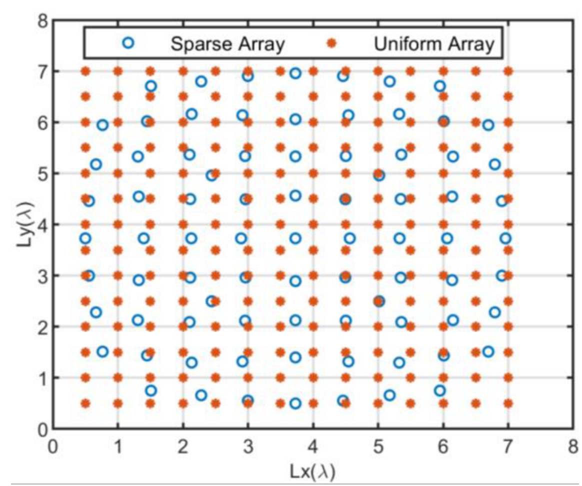


Figure 9. Layouts of the synthesized 81-element sparse array and the reference 196-element uniform array.

of 81 elements is optimized with time consumption $T_{\text{tot}} = 127.18\,\text{s}$. The synthesized radiation patterns are shown in Fig. 7(b), where the maximum SLL $= -29.38\,\text{dB}$ appears in $\varphi = 9^\circ$ plane. In this case, the proposed method saves about 58.67% of elements with the SLL deteriorated only about 0.62 dB. For clarity, Fig. 8 presents the detailed comparison of the patterns in $\varphi = 9^\circ$ plane between the realized sparse array and the reference. The layout of the sparse array obtained herein is presented in Fig. 9 as compared to the uniform array. The same synthesis problem has been solved by the MEMP method in [44], where a sparse array composed of 100 elements was synthesized to approximate the reference pattern with SLL $= -25.11\,\text{dB}$. The numerical comparison of the results obtained by the above two methods is presented in Table 3, and it is clearly shown that the proposed method outperforms the MEMP not only for the numbers of elements but also for the performances.

Table 3. Synthesis results by the proposed method and the MEMP method in [44].

Array	Uniform	The proposed	MEMP in [44]
Q	196	81	100
SLL	-30 dB	-29.38 dB	-25.11 dB
HPBW	9.1°	9.1°	9.4°

3.4. Planar Array Synthesis for Flattop Pattern

In this example, let us consider the synthesis of a sparse planar array with elements limited in a square of 5λ side for reproducing a reference flat-top pattern produced by a uniform array of 11×11 elements. As far as we know, the best result for this synthesis problem has been reported in [27], where the desired shaped pattern was achieved with only 57 elements. Now, in view of the symmetry, 5×5 elements are supposed in the first quadrant as inputs. So, a sparse array of 49 elements is synthesized taking about 481.87 s. The reconstructed patterns are presented in Fig. 10 along with that of the 57-element

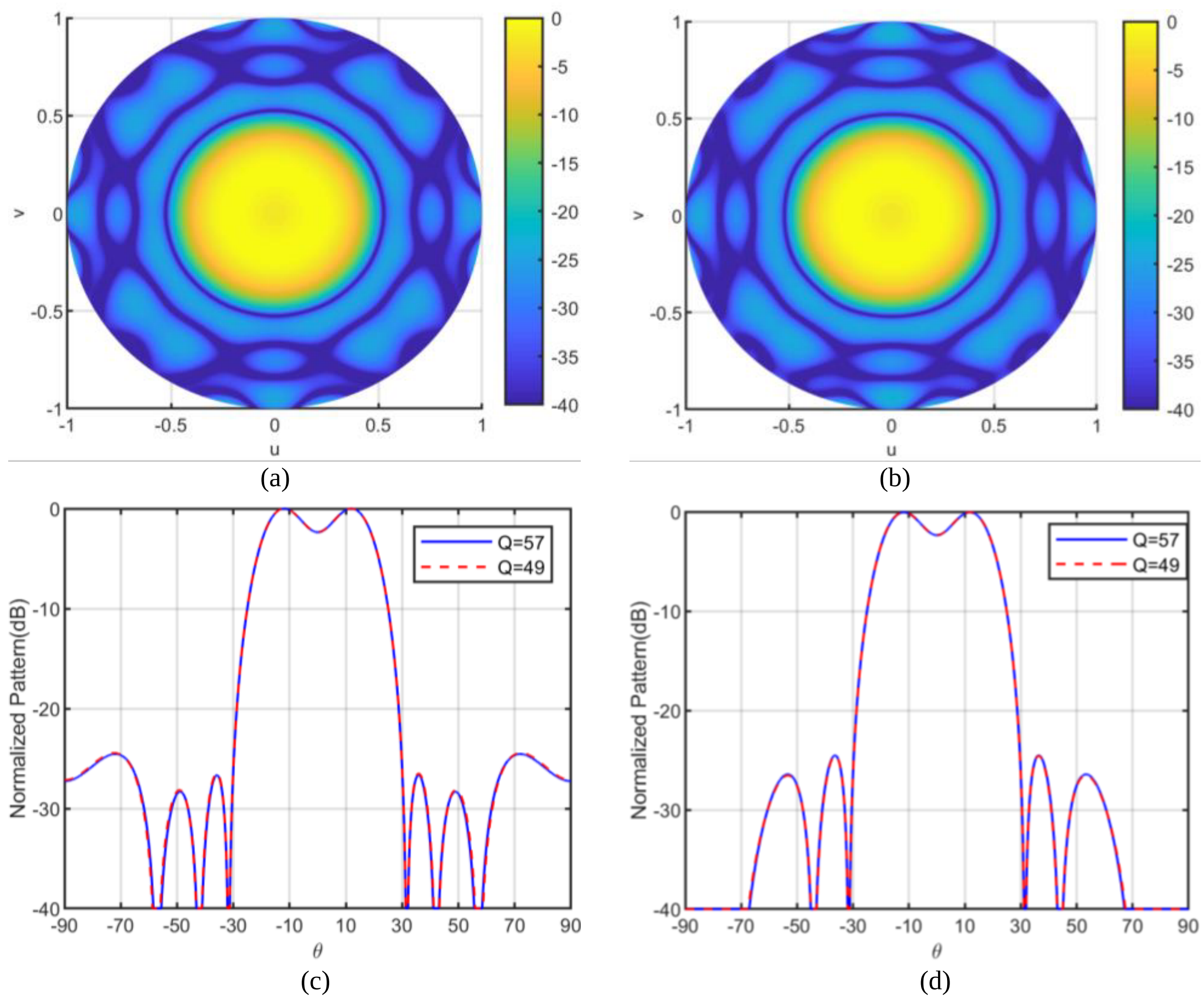


Figure 10. Comparison of the reconstructed flat-top patterns. (a) The sparse array with 57 elements in [27], (b) the sparse array with 49 elements herein, (c) $\varphi = 0°$ plane, (d) $\varphi = 45°$.

Figure 11. QR code for storing the positions and excitations of the synthesized sparse arrays.

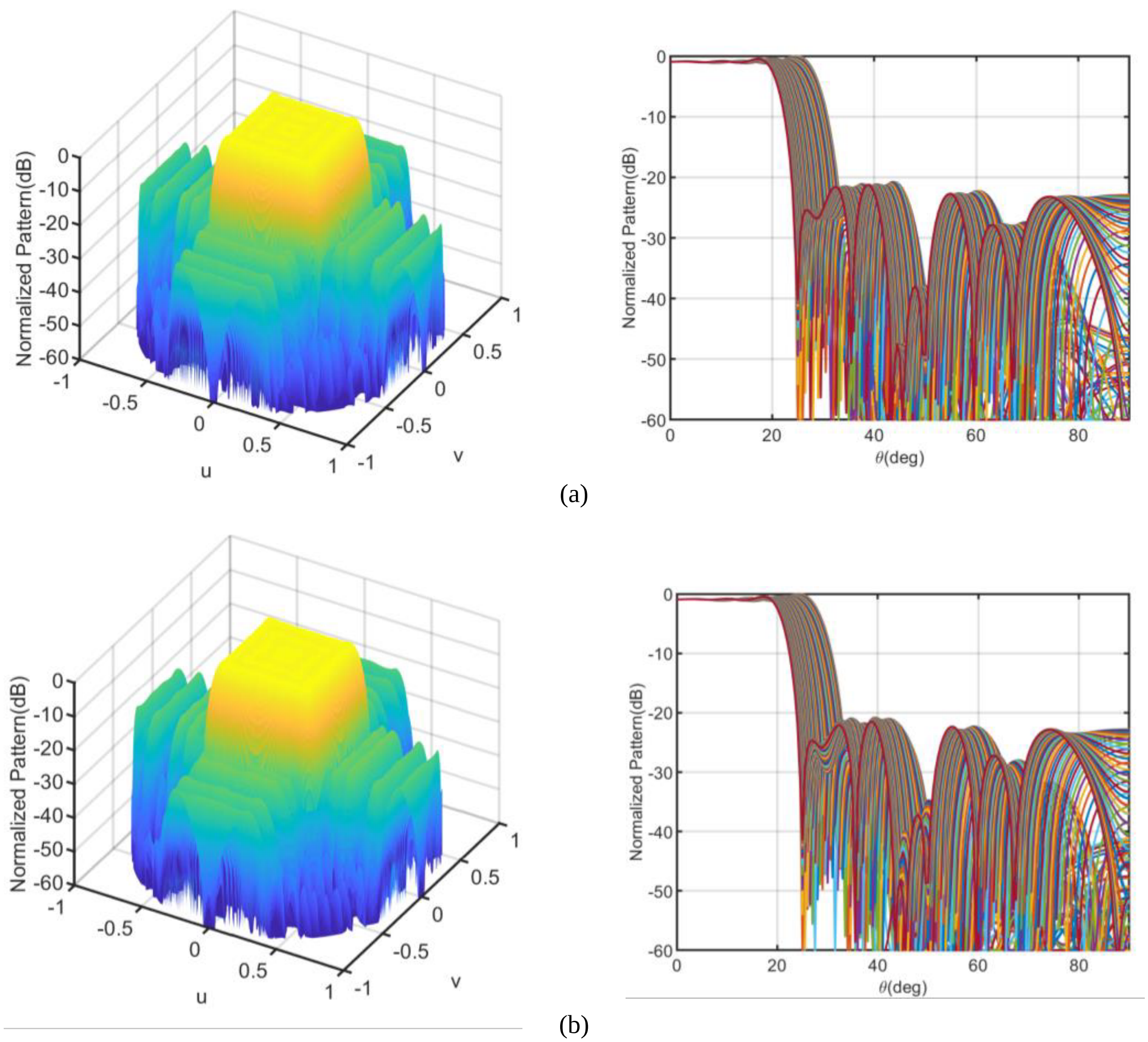


Figure 12. Synthesized patterns. (a) The desired pattern with 900 elements, (b) the reconstructed pattern with 153 elements.

sparse array [27]. Fig. 11 gives the QR code of the synthesized sparse array by the proposed method for conveniently storing and browsing the 49 sets of parameters (including the element positions and excitations). To further evaluate the efficiency of the proposed method for large array synthesis, the same array in [24] is referenced, where a 30×30-element separable uniform planar array with elements equally spaced by 0.5λ, whose pattern is formed by the product of two orthogonal linear arrays. The corresponding pattern has the maximum ripple level MRL $= 0.43$ dB in the flat-top region and the peak sidelobe level PSLL $= -20.5$ dB as shown in Fig. 12(a), where the left column presents the 3D pattern while the right column presents the φ-cut patterns. In [24], a sparse array composed of 185 elements has been synthesized by the EPCS-FOCUSS method to approximate the same flat-top mainbeam with required PSLL. By using the proposed method, only 153 elements are needed in this case, and the results are given by Fig. 12(b). Compared with EPCS-FOCUSS method, our method uses 32 fewer elements, i.e., about 17.3% elements are saved. For the sake of clarity, Fig. 13 presents the detailed

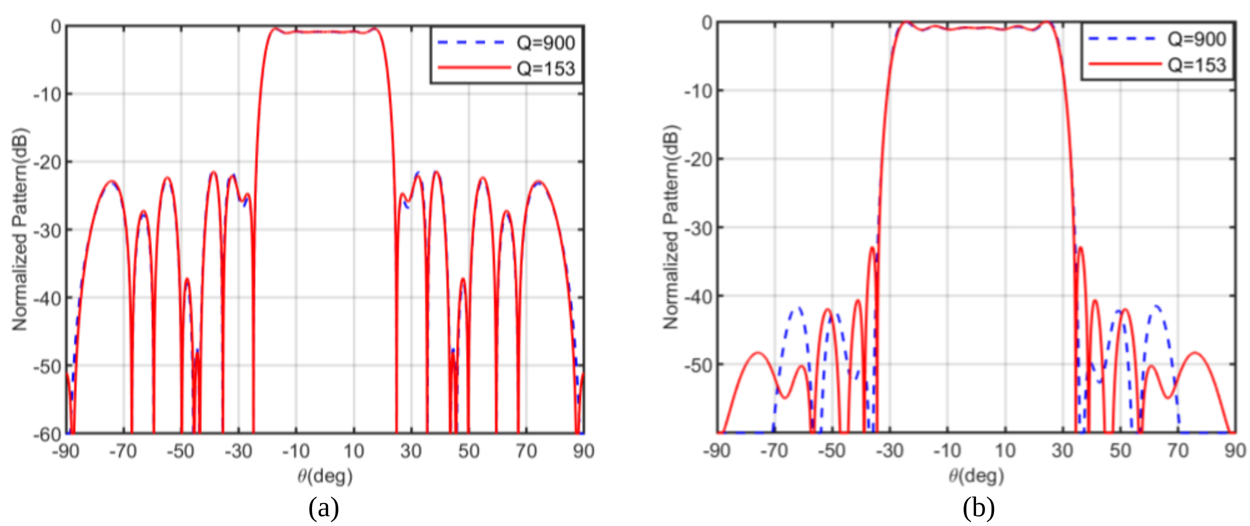


Figure 13. Detailed comparison of the desired φ-cut pattern with 900 elements and the reconstructed one with 153 elements. (a) $\varphi = 0°$, (b) $\varphi = 45°$.

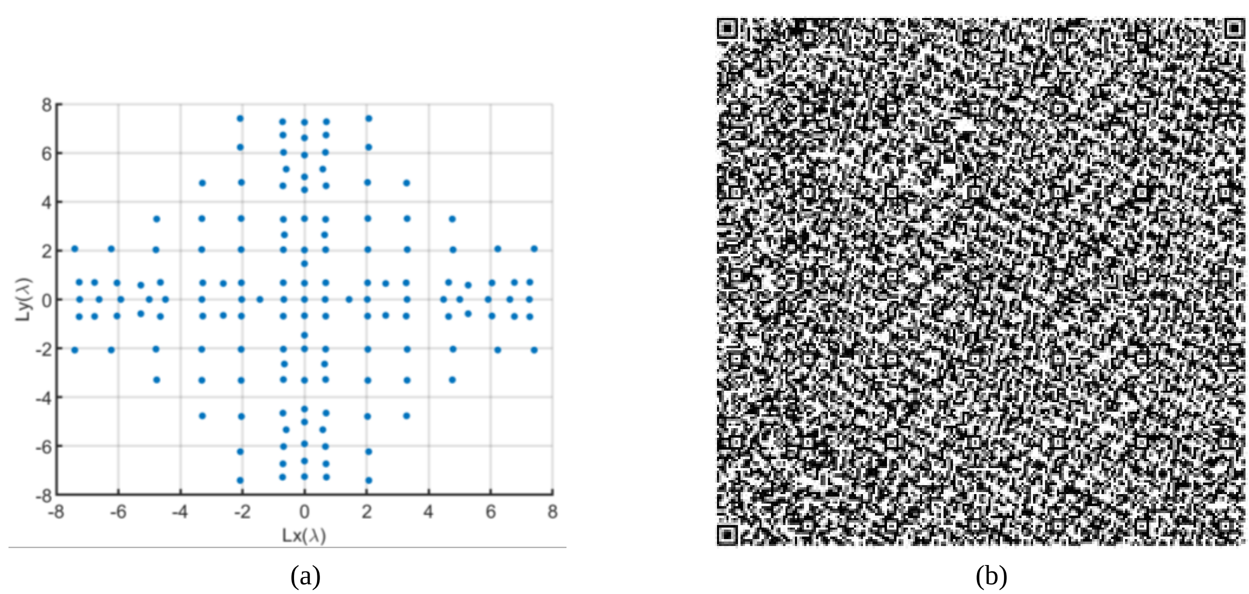


Figure 14. Layouts of (a) the synthesized 153-element sparse planar array and (b) the QR code for storing the positions and excitations.

comparison of the φ-cut patterns at $\varphi = 0°$ and $\varphi = 45°$, respectively. The parameter results of the synthesized sparse array are given by Fig. 14.

4. CONCLUSION

In this paper, a novel sparse array synthesis method has been proposed for reducing the number of elements in linear and planar arrays with focused or shaped beam. Although the proposed method still uses a grid set, different from traditional CS methods, it neither needs to discretize the aperture with dense grids to guarantee the sparsity of the solution nor confines the active elements to the prescribed grids. The sparse array synthesis is formulated using a sparse excitation recovery model taking element position as unknown. For such an optimization problem, the alternative optimization strategy is naturally adopted to obtain the excitations and positions of the active elements by jointing position parameter learning and sparse excitation recovery. Several typical 1-D and 2-D sparse arrays are synthesized demonstrating that the proposed method outperforms the other typical methods not only for the less elements but also for the reconstructed patterns. Besides, the numerical experiments also demonstrate the universal and flexible advantages of our method.

We should emphasize that although isotropic element with the pattern $g(\theta) = 1$ is considered here, the array with actual element pattern $g(\theta)$ can also be synthesized via only multiplying (2) by the n-th element pattern $g_n(\theta)$. Further study on synthesizing reconfigurable sparse arrays and sparse concentric ring arrays is underway.

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