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Equations of Motion of Interacting Classical Charged Particles and the Motion of an Electron outside a Long Solenoid

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Abstract—The equation of motion for a test particle moving in given fixed external fields is analyzed and compared to the corresponding equation of motion derived from the Darwin Lagrangian for a system of interacting charged particles. The two approaches agree as long as the part of the electric field that arises from the partial time derivative of the vector potential is taken into account. It is, however, only via the Darwin approach that the origin of this field can be understood as arising from a breakdown of the test particle approximation. Applying the formalism to an electron moving outside a long solenoid results in a classical analog of the Aharonov-Bohm effect.

1. INTRODUCTION

The magnetic field outside a long solenoid can normally be neglected and so can the electric field. From this Aharonov and Bohm [1] concluded that a classical charged particle should move there as a free particle while quantum mechanics predicts a shift in the interference pattern when a solenoid is placed between two slits. Already 1949 Ehrenberg and Siday [2] showed that the vector potential influences how electrons move in the absence of external fields. This effect is of fundamental interest and is therefore presented in many textbooks [3–5]. Ever since 1959 the general opinion has been that this well documented [6–9] quantum mechanical shift has no classical analog. Indeed, a straightforward application of the standard formalism of analytical mechanics to a charged test particle fails to reveal such an effect as found by Essén and Stén [10]. Here it will be shown that there exists a classical analog but that it requires one to go beyond the test particle approximation.

Over the years various authors, Boyer in particular, have pointed out that subtle effects due to classical electrodynamics in fact can be responsible [11–20]. This article investigates the problem of the classical motion of charged particles starting from the basic equations and approximations. We first investigate the problem starting from the basic Lagrangian, see Eq. (1), that describes the interaction of a classical charged particle with given external electromagnetic field. This formalism leads to a field that is given by the partial time derivative of the vector potential. Normally this term is considered to be part of the electric field and is then forgotten. We eventually find that this is where the effect comes form.

After studying the test particle approximation concerned with the motion in given external fields, we turn to the study of a system of electrically and magnetically interacting charged particles as described by the Darwin Lagrangian. In this formalism there are no external fields; all particles interact and influence each other. The advantage of this is that the meaning of the $\sim \partial \mathbf{A}/\partial t$ field can be calculated.

The conclusion is that classical physics predicts that electrostatic and magnetic forces induce a time rate of change in the canonical momentum, not just the kinetic momentum as is normally assumed. The canonical momentum is the sum of the kinetic momentum and the electromagnetic momentum. The latter can be expressed in terms of the vector potential. This implies a classical Aharonov-Bohm effect,

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and it arises from the often missed contribution to the electric field concealed in the time derivative of the vector potential.

In order to go beyond the test particle approximation in which the charge moves in given fixed external fields one must consider the passing charged particle as part of a system of electrically and magnetically interacting particles. In principle this entails starting from the, surprisingly often ignored, Darwin Lagrangian formalism [21–26] where magnetic interactions are taken into account explicitly and where there are no external fields. There are in fact no fields at all since the expansion of relevant quantities to order $(v/c)^2$ leads to an action-at-a-distance theory. The Darwin approach leads to equations of motion in which the motions of all the particles are coupled. This invalidates the test particle idea.

Below we first present the derivation of the equation of motion of a charged particle from the Lagrangian formalism and stress that subtle effects are hidden in the electrodynamic field $\mathbf{E}_A$ of Eq. (5).

2. LAGRANGIAN FORMALISM FOR A CHARGED PARTICLE IN GIVEN EXTERNAL FIELDS

Consider the standard Lagrangian given in many advanced textbooks [27–30]

$$L(\mathbf{r},\dot{\mathbf{r}}) = \frac{1}{2}m\dot{\mathbf{r}}^2 + \frac{e}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t) - e\phi(\mathbf{r},t) \tag{1}$$

for a non-relativistic charged test particle of mass m and charge e moving in given external electric,

$$\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \nabla\phi, \tag{2}$$

and magnetic,

$$\mathbf{B} = \nabla\times\mathbf{A}, \tag{3}$$

fields. Here we should comment on the two parts of the electric field in (2). When the Coulomb gauge is used ϕ is the ordinary electrostatic Coulomb potential, and

$$\mathbf{E}_C = -\nabla\phi \tag{4}$$

is the electrostatic (Coulomb) field. The often neglected electrodynamic field given by the partial time derivative (keeping $\mathbf{r}$ constant) of $\mathbf{A}(\mathbf{r},t)$,

$$\mathbf{E}_A = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}, \tag{5}$$

accounts for velocity and acceleration dependent corrections to the forces. It will figure prominently in the subsequent study.

Here the generalized (canonical) momentum is[†]

$$\mathbf{p} = \frac{\partial L}{\partial\dot{\mathbf{r}}} = m\dot{\mathbf{r}} + \frac{e}{c}\mathbf{A}(\mathbf{r},t) \tag{6}$$

and the canonical force is

$$\mathbf{\Phi} = \frac{\partial L}{\partial\mathbf{r}} = \nabla\left[\frac{e}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t) - e\phi(\mathbf{r},t)\right]. \tag{7}$$

From (6) and (7) we get the Lagrange's equations of motion,

$$\dot{\mathbf{p}} = \mathbf{\Phi}. \tag{8}$$

Using,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}(\mathbf{r},t) = \frac{\partial\mathbf{A}}{\partial t} + (\dot{\mathbf{r}}\cdot\nabla)\mathbf{A} \tag{9}$$

to get $\dot{\mathbf{p}}$ from (6) and then

$$\nabla(\dot{\mathbf{r}}\cdot\mathbf{A}) = (\dot{\mathbf{r}}\cdot\nabla)\mathbf{A} + \dot{\mathbf{r}}\times(\nabla\times\mathbf{A}) \tag{10}$$

[†] Here, and below, we use the notation $\nabla L = (\partial L/\partial x, \partial L/\partial y, \partial L/\partial z) \equiv \partial L/\partial\mathbf{r}$ and correspondingly for the velocity space gradient: $(\partial L/\partial\dot{x}, \partial L/\partial\dot{y}, \partial L/\partial\dot{z}) \equiv \partial L/\partial\dot{\mathbf{r}}$.

to get $\mathbf{\Phi}$ from (7) we find explicitly

$$m\ddot{\mathbf{r}} + \frac{e}{c}\left[\frac{\partial \mathbf{A}}{\partial t} + (\dot{\mathbf{r}}\cdot\nabla)\mathbf{A}\right] = \frac{e}{c}\left[(\dot{\mathbf{r}}\cdot\nabla)\mathbf{A} + \dot{\mathbf{r}}\times(\nabla\times\mathbf{A})\right] - e\nabla\phi. \tag{11}$$

Cancelling $(e/c)(\dot{\mathbf{r}}\cdot\nabla)\mathbf{A}$ from both sides gives

$$m\ddot{\mathbf{r}} + \frac{e}{c}\frac{\partial \mathbf{A}}{\partial t} = \frac{e}{c}\left[\dot{\mathbf{r}}\times(\nabla\times\mathbf{A})\right] - e\nabla\phi. \tag{12}$$

This equation shows that not only the curl of $\mathbf{A}$ but also its time derivative determines the motion of the particle.

With the notation in (3), (4) and (5) this gives us

$$m\ddot{\mathbf{r}} = e\mathbf{E}_A + \frac{e}{c}\dot{\mathbf{r}}\times\mathbf{B} + e\mathbf{E}_C. \tag{13}$$

Denoting $\mathbf{E}_A + \mathbf{E}_C$ by $\mathbf{E}$ results in the standard textbook equation

$$m\ddot{\mathbf{r}} = e\mathbf{E} + \frac{e}{c}\dot{\mathbf{r}}\times\mathbf{B}. \tag{14}$$

Here we have chosen to stress that $\mathbf{E}$ has *two* contributions of very different origin and nature. The classical analog of an Aharonov-Bohm effect proposed below arises from $\mathbf{E}_A$.

3. HAMILTONIAN FORMALISM OF CHARGED PARTICLE MOTION, ENERGY CONSERVATION, AND QUANTUM MECHANICS

Since the quantum mechanics of a particle can be obtained by canonical quantization of the classical Hamiltonian we should investigate what the Hamiltonian formalism has to say about the problem.

The classical Hamiltonian corresponding to the Lagrangian (1) is obtained by means of a Legendre transformation and is given by $H = \mathbf{p}\cdot\mathbf{v} - L$ expressed in terms of $\mathbf{p} = \partial L/\partial\mathbf{v} = m\mathbf{v} + (e/c)\mathbf{A}$ using $\mathbf{v} = (1/m)[\mathbf{p} - (e/c)\mathbf{A}]$. This results in the expression,

$$H(\mathbf{r},\mathbf{p}) = \frac{1}{2m}\left[\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{r})\right]^2 + e\phi(\mathbf{r}) \tag{15}$$

$$= \frac{\mathbf{p}^2}{2m} - \frac{e}{mc}\mathbf{p}\cdot\mathbf{A} + \frac{e^2}{2mc^2}\mathbf{A}^2 + e\phi, \tag{16}$$

for the Hamiltonian.

If $\mathbf{A}$ and ϕ are time independent the Hamiltonian represents a conserved energy, $H(\mathbf{r},\mathbf{p}) = E$. Using that $\mathbf{p} = m\mathbf{v} + (e/c)\mathbf{A}$ this means that,

$$E(\mathbf{r},\mathbf{v}) = \frac{1}{2}m\mathbf{v}^2 + e\phi(\mathbf{r}), \tag{17}$$

is constant. If there is no electrostatic field this means that the magnetic field can not change $|\mathbf{v}| = \sqrt{2E/m}$; only the direction of the velocity changes. This seems a bit strange since when a charged particle moves near a current distribution its magnetic field should add to or subtract from the field from that current distribution and thus change the total magnetic energy. What is wrong?

The resulting Hamilton's equations of motion, $\dot{\mathbf{r}} = \partial H/\partial\mathbf{p}$, $\dot{\mathbf{p}} = -\partial H/\partial\mathbf{r}$, become,

$$\dot{\mathbf{r}} = \frac{1}{m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right) \tag{18}$$

$$\dot{\mathbf{p}} = \frac{e}{mc}\frac{\partial}{\partial\mathbf{r}}\left[\left(\mathbf{p} - \frac{e}{2c}\mathbf{A}\right)\cdot\mathbf{A}\right] - e\frac{\partial\phi}{\partial\mathbf{r}}. \tag{19}$$

Using the first of these, Eq. (18), the second, Eq. (19), gives,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{\mathbf{r}} + \frac{e}{c}\mathbf{A}\right) = \frac{e}{c}\left[\left(\dot{\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{r}}\right)\mathbf{A} + \dot{\mathbf{r}}\times\left(\frac{\partial}{\partial\mathbf{r}}\times\mathbf{A}\right)\right] - e\frac{\partial\phi}{\partial\mathbf{r}}, \tag{20}$$

which is seen to be the same as (12) since the total time derivative of $(e/c)\mathbf{A}$ gives a term on the left hand side identical to the first term on the right hand side so these cancel each other. This means that the time rate of change of $\mathbf{A}$ due to the motion of the particle vanishes from the problem.

If there is no magnetic field, $\nabla \times \mathbf{A} = \mathbf{0}$, and no electrostatic field, $-\nabla\phi = \mathbf{0}$, as is the case outside a long solenoid, (12) as well as (20) result in

$$m\frac{\mathrm{d}\dot{\mathbf{r}}}{\mathrm{d}t} + \frac{e}{c}\frac{\partial \mathbf{A}}{\partial t} = \mathbf{0}. \tag{21}$$

The Hamiltonian formalism does not change anything in this respect except that it facilitates the transition to quantum mechanics via canonical quantization, see below.

3.1. Time Dependence of the Vector Potential

As already pointed it is an intuitively obvious fact that the magnetic field of the moving charged particle will superpose on the magnetic field from the source and result in a changing magnetic energy, see Eq. (63) below. This is obviously in conflict with Eq. (17). Consequently the effect of this interaction cannot be described while assuming a time-independent $\mathbf{A}$. A force $e\mathbf{E}_A$ (see Eq. (5)) is well known to arise when the magnetic flux through a loop changes (Faraday induction). It seems reasonable to assume that the flux inside a stationary solenoid should remain essentially constant as an electron passes. When this problem instead is viewed from the equivalent system in which the solenoid moves past the electron a time dependent vector potential will obviously arise as has been pointed out by Boyer [19]. This indicates the subtlety of the problem.

Equation (21) shows that when the magnetic and electrostatic fields are zero, the particle will move as a free particle provided that $\partial\mathbf{A}/\partial t = \mathbf{0}$. This could be the case if the test particle approximation was valid; i.e., when the moving charge does not affect the source. As will be shown below this can never be exact; the system of charged particles is an interacting system. If the currents that produce $\mathbf{A}$ have fixed geometry a reasonable approximation is that

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{a}(\mathbf{r})\Phi(t). \tag{22}$$

Here $\mathbf{a}(\mathbf{r})$ describes how the vector potential varies in space while $\Phi(t)$ represents a magnetic flux and depends on the strength of the currents that flow in the fixed geometry of conductors that represent the source of the external field. The fact that the vector potential conveys physical information beyond that of the fields has been stressed by Reiss [17] and Comay [31] among others.

Use of (22) shows that equation (21), with $\mathbf{B} = \mathbf{0}$, can be written

$$m\frac{\mathrm{d}\dot{\mathbf{r}}}{\mathrm{d}t} + \frac{e}{c}\mathbf{a}(\mathbf{r})\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \mathbf{0}. \tag{23}$$

Integration with respect to time now gives

$$m\dot{\mathbf{r}}(t) + \frac{e}{c}\mathbf{a}(\mathbf{r})\Phi(t) = m\dot{\mathbf{r}}(t) + \frac{e}{c}\mathbf{A}(\mathbf{r}, t) = \mathbf{p} = \text{constant}. \tag{24}$$

The canonical (or generalized or conjugate) momentum (6) is thus constant when the magnetic and electrostatic fields are zero but the electrodynamic field due to the time dependence of the vector potential is of importance. The importance of the generalized momentum and the vector potential has been stressed by Konopinski [32, 33].

3.2. Hamiltonian Operator and Schrödinger's Equation

Canonical quantization of the Hamiltonian H is achieved by replacing $\mathbf{p}$ with the operator $\hat{\mathbf{p}} = -\mathrm{i}\hbar\,\partial/\partial\mathbf{r} \equiv -\mathrm{i}\hbar\nabla$. Putting this into (16) gives us the Hamiltonian operator

$$\hat{H} = \frac{1}{2m}\left[-\mathrm{i}\hbar\frac{\partial}{\partial\mathbf{r}} - \frac{e}{c}\mathbf{A}(\mathbf{r})\right]^2. \tag{25}$$

When this Hamiltonian operates on a wave function $\psi(\mathbf{r})$ one must take the non-commutation of the operators $\hat{\mathbf{p}}$ and $\mathbf{r}$ into account. We find that

$$\left[\frac{\partial}{\partial\mathbf{r}} \cdot \mathbf{A}(\mathbf{r})\right]\psi(\mathbf{r}) = \mathbf{A} \cdot \frac{\partial\psi}{\partial\mathbf{r}} + \psi\frac{\partial\mathbf{A}}{\partial\mathbf{r}}. \tag{26}$$

If the Coulomb gauge is imposed one has

$$\frac{\partial \mathbf{A}}{\partial \mathbf{r}} = \nabla \cdot \mathbf{A} = 0.$$

The Schrödinger equation becomes [5, 34]

$$\hat{H}\psi = \left[\frac{-\hbar^2}{2m}\nabla^2 + \mathrm{i}\hbar\frac{e}{mc}\mathbf{A}\cdot\nabla + \frac{e^2}{2mc^2}\mathbf{A}^2\right]\psi. \tag{27}$$

It is clear that a free particle wave function can not be a solution of this equation even if the magnetic field is zero.

4. THE DARWIN LAGRANGIAN — CONSERVATION LAWS AND EQUATIONS OF MOTION

In 1920 C. G. Darwin [21] derived a Lagrangian for a system of electrically and magnetically interacting charged particles by eliminating the independent field degrees of freedom. The Darwin Lagrangian accurately determines the dynamics of systems with electromagnetic interactions as long as radiation can be neglected and velocities are reasonably non-relativistic. The Darwin approach is briefly mentioned in many advanced textbooks, e.g., Schwinger et al. [35]. A review of the literature can be found in [26]. The main point here is that the Darwin approach allows one to go beyond the test particle approximation assumed in the treatment above.

4.1. The Lagrangian

The Lagrangian of a free particle is $L_{\mathrm{m}} = -mc^2\sqrt{1-\mathbf{v}^2/c^2} \approx -mc^2 + (m/2)\mathbf{v}^2 + (m/8)\mathbf{v}^4/c^2 + \ldots$. Below we assume that the particles are not moving with relativistic speeds and thus that $\frac{m}{2}\mathbf{v}^2$ is accurate enough for the free particle Lagrangian (kinetic energy). The Darwin Lagrangian is then

$$L(\mathbf{r}_a, \mathbf{v}_a) = \sum_{a=1}^{N}\left[\frac{1}{2}m_a\mathbf{v}_a^2 + \frac{e_a}{2c}\mathbf{v}_a\cdot\mathbf{A}_a(\mathbf{r}_a) - \frac{e_a}{2}\phi_a(\mathbf{r}_a)\right], \tag{28}$$

where,

$$\mathbf{A}_a(\mathbf{r}_a) = \sum_{b(\neq a)}^{N}\frac{e_b[\mathbf{v}_b + (\mathbf{v}_b\cdot\hat{\mathbf{e}}_{ab})\hat{\mathbf{e}}_{ab}]}{2c|\mathbf{r}_a - \mathbf{r}_b|}, \tag{29}$$

and,

$$\phi_a(\mathbf{r}_a) = \sum_{b(\neq a)}^{N}\frac{e_b}{|\mathbf{r}_a - \mathbf{r}_b|}. \tag{30}$$

Here the position and velocity vectors of the particles are $\mathbf{r}_a$ and $\mathbf{v}_a$ respectively, m_a and e_a their rest masses and charges respectively, and $\hat{\mathbf{e}}_{ba} = (\mathbf{r}_a - \mathbf{r}_b)/|\mathbf{r}_a - \mathbf{r}_b|$ is the unit vector pointing from b to a.

Note carefully that the introduction of the vector quantity $\mathbf{A}_a$ in (29) appears to introduce a vector potential in the formalism. In fact this quantity is *not* a field but simply a vector function of particle positions and velocities which simplifies the algebraic expression of the Darwin Lagrangian (even if one can recognize it as the Coulomb gauge vector potential from moving charges). Sharma [36] states that problems related to the Feynman's disk paradox [37] can not be treated without using a field formulation of electrodynamics. In fact this class of problems, which are closely related to the Aharonov-Bohm phenomenon [38], are eminently suited for the action-at-a-distance Darwin Lagrangian formulation.

4.2. Conservation Laws

The Darwin Lagrangian is invariant under translations $\mathbf{r}_a \to \mathbf{r}_a + \boldsymbol{\epsilon}$. It is also invariant under rotations. Finally there is no explicit time dependence. According to Noether's theorem this means that there is a conserved momentum, angular momentum and energy. Stettner [39] derived the same conserved quantities (to order $(v/c)^2$) from the energy momentum tensor.

One finds that the conserved total momentum is

$$\mathbf{p}_{\rm tot} = \sum_{a=1}^{N} \mathbf{p}_a, \tag{31}$$

where

$$\mathbf{p}_a = m_a \mathbf{v}_a + \frac{e_a}{c}\mathbf{A}_a(\mathbf{r}_a). \tag{32}$$

are the canonical momenta of the particles. The conserved angular momentum is found to be $\mathbf{L} = \sum_a \mathbf{r}_a \times \mathbf{p}_a$. Finally the conserved energy is

$$E = \sum_{a=1}^{N}\left[\frac{1}{2}m_a\mathbf{v}_a^2 + \frac{e_a}{2c}\mathbf{v}_a\cdot\mathbf{A}_a(\mathbf{r}_a) + \frac{e_a}{2}\phi_a(\mathbf{r}_a)\right] \tag{33}$$

and differs from L in (28) only in the sign of the last term, the Coulomb energy (as long as the non-relativistic kinetic energy is used).

It is well known that the momentum of a charged particle is given by Eq. (32); one can even find a derivation in the textbook by Kittel [40] (Appendix G). A more recent derivation and discussion is by Essén [41] where also the origin in the Darwin formalism is pointed out. Redinz [42] shows that the vector potential contribution to the energy (33) represents work done by velocity and acceleration dependent forces. A recent discussion on the nature of the electromagnetic momentum is by Singal [43]. The unease with vector potentials in classical electromagnetism, due to their gauge freedom, has impaired the reputation of this formula and prevented its use. Below we will argue that this is the crucial concept for understanding the dynamics of the Aharonov-Bohm effect.

5. CONSERVATION LAWS AND THE TEST PARTICLE APPROXIMATION

It is of interest to consider what the conservation laws say about the motion of a single particle (electron) of charge e at $\mathbf{r}_e$ outside a system of N_s particles $a = 1,\ldots,N_s = N-1$. The role of the Coulomb interaction is well known so we consider only the magnetic interaction. The energy (33) can then be written

$$E = \frac{1}{2}m_e\mathbf{v}_e^2 + \frac{e}{2c}\mathbf{v}_e\cdot\mathbf{A}_s(\mathbf{r}_e) + \tag{34}$$

$$\sum_{a=1}^{N_s}\left\{\frac{1}{2}m_a\mathbf{v}_a^2 + \frac{e_a}{2c}\mathbf{v}_a\cdot[\mathbf{A}_{sa}(\mathbf{r}_a) + \mathbf{A}_e(\mathbf{r}_a)]\right\}. \tag{35}$$

Here

$$\mathbf{A}_s(\mathbf{r}_e) = \sum_{b=1}^{N_s}\frac{e_b[\mathbf{v}_b + (\mathbf{v}_b\cdot\hat{\mathbf{e}}_{eb})\hat{\mathbf{e}}_{eb}]}{2c|\mathbf{r}_e - \mathbf{r}_b|} \tag{36}$$

$$\mathbf{A}_{sa}(\mathbf{r}_a) = \sum_{b(\neq a)}^{N_s}\frac{e_b[\mathbf{v}_b + (\mathbf{v}_b\cdot\hat{\mathbf{e}}_{ba})\hat{\mathbf{e}}_{ba}]}{2c|\mathbf{r}_a - \mathbf{r}_b|} \tag{37}$$

$$\mathbf{A}_e(\mathbf{r}_a) = \frac{e[\mathbf{v}_e + (\mathbf{v}_e\cdot\hat{\mathbf{e}}_{ea})\hat{\mathbf{e}}_{ea}]}{2c|\mathbf{r}_a - \mathbf{r}_e|} \tag{38}$$

where $\hat{\mathbf{e}}_{ea} = (\mathbf{r}_a - \mathbf{r}_e)/|\mathbf{r}_a - \mathbf{r}_e|$.

Here one notes that the two interaction terms, the second term of Eq. (34) and the last of the three sums in (35), are equal,

$$\frac{e}{2c}\mathbf{v}_e \cdot \mathbf{A}_s(\mathbf{r}_e) = \sum_{a=1}^{N_s} \frac{e_a}{2c}\mathbf{v}_a \cdot \mathbf{A}_e(\mathbf{r}_a). \tag{39}$$

Using this the energy (or Lagrangian) can be written as a sum of two terms, the energy of the electron E_e and the energy of the system E_s. So we have

$$E = E_e + E_s, \tag{40}$$

where

$$E_e = \frac{1}{2}m_e\mathbf{v}_e^2 + \frac{e}{c}\mathbf{v}_e \cdot \mathbf{A}_s(\mathbf{r}_e) \tag{41}$$

and,

$$E_s = \sum_{a=1}^{N_s}\left[\frac{1}{2}m_a\mathbf{v}_a^2 + \frac{e_a}{2c}\mathbf{v}_a \cdot \mathbf{A}_{sa}(\mathbf{r}_a)\right]. \tag{42}$$

Here E_e represents the kinetic plus (magnetic) interaction energy of the electron and the system respectively while E_s only refers to the particles of the system responsible for producing $\mathbf{A}_s$. As long as one can ignore the Coulomb electrostatic interactions this energy (Lagrangian) $E = L$ is a constant.

The conserved total momentum of Eq. (31) can be written

$$\mathbf{p}_{\text{tot}} = \mathbf{p}_e + \mathbf{p}_s \tag{43}$$

where $\mathbf{p}_e$ is

$$\mathbf{p}_e = m_e\mathbf{v}_e + \frac{e}{c}\mathbf{A}_s(\mathbf{r}_e) \tag{44}$$

and

$$\mathbf{p}_s = \sum_{a=1}^{N_s}\left[m_a\mathbf{v}_a + \frac{e_a}{c}\mathbf{A}_a(\mathbf{r}_a)\right]. \tag{45}$$

One notes that each of the terms $\mathbf{A}_a(\mathbf{r}_a) = \mathbf{A}_{sa}(\mathbf{r}_a) + \mathbf{A}_e(\mathbf{r}_a)$ in the last sum contains a contribution from the electron. From the constancy of $\mathbf{p}_{\text{tot}}$ one finds that if

$$\frac{\mathrm{d}\mathbf{p}_e}{\mathrm{d}t} = \mathbf{F}_e, \text{ then } \frac{\mathrm{d}\mathbf{p}_s}{\mathrm{d}t} = -\mathbf{F}_e \tag{46}$$

in accordance with the principle of action and reaction.

Using the test particle approximation in calculating the electron motion one assumes that the energy E_s of the system producing the field is constant. When combined with the total conservation laws this implies that E_e is conserved. In many cases it seems that this should at least be a good approximation (neglecting electrostatic interactions). If $\mathbf{F}_e = \mathbf{0}$ then one can assume that also $\mathbf{p}_e$ and $\mathbf{p}_s$ are both constant.

5.1. The Equations of Motion

Here we will derive the equations of motion,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \mathbf{v}_a} - \frac{\partial L}{\partial \mathbf{r}_a} = 0, \tag{47}$$

for the system from its Lagrangian (28). These equations can be found explicitly in a 1939 paper by Primakoff and Holstein [44], as well as in the books by Page and Adams [45] and by Podolsky and Kunz [46], but they have not received much attention in the last half century, see however Redinz [42] and Boyer [47]. One message of these equations is quite clear: there is no such thing as a particle moving in given external fields. The moving particle will always affect the motions of the particles producing the forces. The question then is when this effect can be ignored and when it cannot.

The canonical momentum is

$$\frac{\partial L}{\partial \mathbf{v}_a} = \mathbf{p}_a \tag{48}$$

and is given in Eq. (32). Its time derivative is then

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \mathbf{v}_a} = m_a \dot{\mathbf{v}}_a + \frac{e_a}{c}\frac{\partial \mathbf{A}_a(\mathbf{r}_a)}{\partial t} + \frac{e_a}{c}\mathbf{v}_a \cdot \frac{\partial \mathbf{A}_a(\mathbf{r}_a)}{\partial \mathbf{r}_a}. \tag{49}$$

According to the Euler-Lagrange equations (47) this should the be equal to the canonical force

$$\frac{\partial L}{\partial \mathbf{r}_a} = \frac{\partial}{\partial \mathbf{r}_a}\sum_{b=1}^{N}\left[\frac{e_b}{2c}\mathbf{v}_b \cdot \mathbf{A}_b(\mathbf{r}_b) - \frac{e_b}{2}\phi_b(\mathbf{r}_b)\right] \tag{50}$$

$$= \frac{e_a}{c}\mathbf{v}_a \cdot \frac{\partial}{\partial \mathbf{r}_a}\mathbf{A}_a(\mathbf{r}_a) + \frac{e_a}{c}\mathbf{v}_a \times \left(\frac{\partial}{\partial \mathbf{r}_a} \times \mathbf{A}_a(\mathbf{r}_a)\right) - e_a\frac{\partial \phi_a(\mathbf{r}_a)}{\partial \mathbf{r}_a}. \tag{51}$$

Here Eq. (10) has been used. Combining (49) and (51) we find

$$m_a \dot{\mathbf{v}}_a + \frac{e_a}{c}\frac{\partial \mathbf{A}_a(\mathbf{r}_a)}{\partial t} = \frac{e_a}{c}\mathbf{v}_a \times \left(\frac{\partial}{\partial \mathbf{r}_a} \times \mathbf{A}_a(\mathbf{r}_a)\right) - e_a\frac{\partial \phi_a(\mathbf{r}_a)}{\partial \mathbf{r}_a}. \tag{52}$$

This thus turns out to be essentially the same equation as Eq. (12).

The above calculation shows that the time-dependence of $\mathbf{A}_a(\mathbf{r}_a)$ due to the motion of particle a, i.e., the time dependence of $\mathbf{r}_a(t)$, vanishes from the equation of motion. In the derivation of Section 2 this also happened when the terms $(e/c)(\dot{\mathbf{r}} \cdot \nabla)\mathbf{A}$ appeared on both sides of the equation. In the Darwin Lagrangian formalism one can investigate the second term on the left hand side explicitly. According to Eq. (29) we can write $\mathbf{A}_a(\mathbf{r}_a)$ explicitly in the form

$$\mathbf{A}_a(\mathbf{r}_a) = \mathbf{A}_a(\mathbf{r}_a;\ \mathbf{r}_b(t), \mathbf{v}_b(t)) \equiv \mathbf{A}_a(\mathbf{r}_a, t). \tag{53}$$

This means that

$$\frac{\partial \mathbf{A}_a(\mathbf{r}_a)}{\partial t} = \frac{\partial \mathbf{A}_a(\mathbf{r}_a, t)}{\partial t} \tag{54}$$

and that the partial (or explicit) time derivative of $\mathbf{A}_a(\mathbf{r}_a)$ is due to the time-dependence caused by the motion of all particles of the system except particle a itself. Using this and Eq. (29) we find the explicit equation,

$$\frac{e_a}{c}\frac{\partial \mathbf{A}_a(\mathbf{r}_a, t)}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{b(\neq a)}^{N}\frac{e_a e_b[\mathbf{v}_b(t) + (\mathbf{v}_b(t) \cdot \hat{\mathbf{e}}_{ab}(t))\hat{\mathbf{e}}_{ab}(t)]}{2c^2|\mathbf{r}_a - \mathbf{r}_b(t)|}\right) = \tag{55}$$

$$\sum_{b(\neq a)}^{N}\left[\left(\frac{\dot{\mathbf{v}}_b + (\dot{\mathbf{v}}_b \cdot \hat{\mathbf{e}}_{ba})\hat{\mathbf{e}}_{ba}}{2c^2}\right)\frac{e_a e_b}{r_{ba}} - \left(\frac{\mathbf{v}_b^2 - 3(\mathbf{v}_b \cdot \hat{\mathbf{e}}_{ba})^2}{2c^2}\right)\frac{e_a e_b \hat{\mathbf{e}}_{ba}}{r_{ba}^2}\right], \tag{56}$$

for this term. When moved to the right hand side of Eq. (52) this expression represents the often forgotten velocity and acceleration dependent "electric" forces from Eq. (5). When all particles, except particle a, move in conductors of fixed geometry it is not unreasonable to approximate this expression by,

$$\frac{e_a}{c}\frac{\partial \mathbf{A}_a(\mathbf{r}_a, t)}{\partial t} = \frac{e_a}{c}\mathbf{a}_a(\mathbf{r}_a)\frac{\mathrm{d}\Phi_a(t)}{\mathrm{d}t}, \tag{57}$$

as was done in Eq. (22).

The equations of motion found above differ in important aspects from what most physicists are used to. In electrostatics and Newtonian gravity of point particles the acceleration of a particle in the system is determined by the positions of the other particles. Eqs. (52) and (55)–(56) show that the acceleration of a particles requires knowledge of the positions, velocities, and accelerations of the other particles! If nothing else this shows that the classical dynamics of charged particles is quite subtle.

5.2. The Status of the Canonical Momentum

Using Eqs. (52) and (54) we can write the equation of motion for particle a

$$\frac{\mathrm{d}}{\mathrm{d}t} m_a \mathbf{v}_a + \frac{\partial}{\partial t} \frac{e_a}{c} \mathbf{A}_a(\mathbf{r}_a, t) = \frac{e_a}{c} \mathbf{v}_a \times \mathbf{B}_a(\mathbf{r}_a) + e_a \mathbf{E}_{Ca}(\mathbf{r}_a). \tag{58}$$

Here $\mathbf{B}_a$ and $\mathbf{E}_{Ca}$ are the magnetic and electrostatic (Coulomb) fields produced by the other particle of the system. We see that this is in agreement with Eq. (13) where the fields were assumed to be given external fields. The difference here is that one can find the explicit result (55)–(56) for the partial time derivative of the $\mathbf{A}$-field.

We find that the fields $\mathbf{B}_a$ and $\mathbf{E}_{Ca}$ produce forces that cause the canonical momentum to depend on time. In particular this leads to the important conclusion that if the system does not produce magnetic and electrostatic forces ($\mathbf{B}_a = \mathbf{E}_{Ca} = \mathbf{0}$) at particle a, then

$$\frac{\mathrm{d}}{\mathrm{d}t} m_a \mathbf{v}_a + \frac{\partial}{\partial t} \frac{e_a}{c} \mathbf{A}_a(\mathbf{r}_a, t) = \mathbf{0}. \tag{59}$$

Integrating with respect to time, while keeping $\mathbf{r}_a$ constant, gives

$$\mathbf{p}_a = m_a \mathbf{v}_a + \frac{e_a}{c} \mathbf{A}_a(\mathbf{r}_a, t) = \text{const.} \tag{60}$$

so, for this case, the canonical momentum of particle a is a conserved quantity. The kinetic momentum $m_a \mathbf{v}_a$ is not predicted to be constant.

6. ELECTRON MOTION OUTSIDE A LONG SOLENOID

We now assume that particle a is a particle of charge e and mass m_e and that the rest of the system constitutes a long solenoid. In the ideal case this means that $\mathbf{B}_a = \mathbf{E}_{Ca} = \mathbf{0}$ so we should have that the canonical momentum (60) is conserved. We also assume that the particle moves in a plane perpendicular to the solenoid, see Fig. 1. We thus have only two degrees of freedom for the electron in the plane. From this and the conservation of canonical momentum the motion of the electron can be deduced, as will be done below.

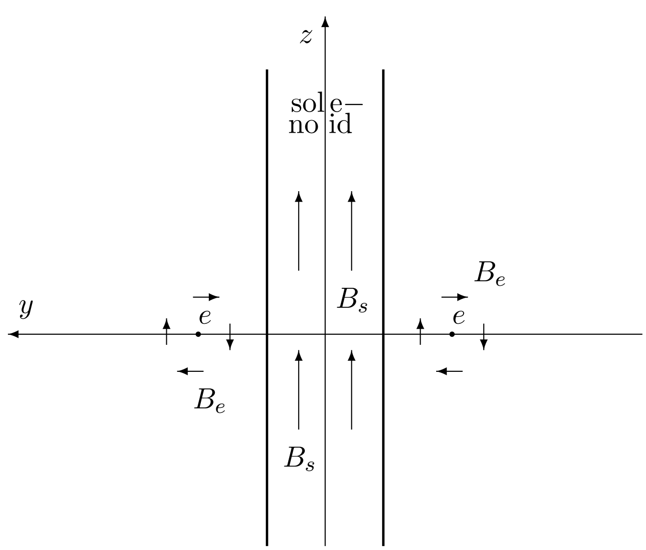

Figure 1. This figure shows a cross section of the solenoid with interior magnetic field B_s along the z-axis in a right-handed coordinate system with y-axis to the left and x-axis into the figure. A charged particle with charge $e > 0$ moves along the positive x-axis (into figure). When the particle trajectory has positive y-coordinate its magnetic field B_e will subtract from the solenoid field. For negative y-coordinate (to the left of the solenoid) its field will add to the interior field.

6.1. Vector Potential of the Solenoid and the Magnetic Interaction Energy

We model the long solenoid as a long circular cylinder of radius ρ_s and height Z_s on which current circulates. The current density is such that it can be described as the rigid rotation of a surface current density

$$\sigma = \frac{N_s e}{2\pi \rho_s Z_s} \tag{61}$$

with some angular velocity ω. In the limit $N_s, Z_s \to \infty$ and $N_s/Z_s \to \lambda$ the resulting (Coulomb gauge) vector potential is well known to be,

$$\mathbf{A}_s(\mathbf{r}) = \frac{1}{2} B_s \rho_s^2 \frac{\hat{\mathbf{e}}_\varphi(\varphi)}{\rho} = \frac{\Phi_B}{2\pi} \frac{\hat{\mathbf{e}}_\varphi(\varphi)}{\rho} \tag{62}$$

where ρ, φ are cylinder coordinates ($\mathbf{r} = \rho\, \hat{\mathbf{e}}_\rho(\varphi) + z\hat{\mathbf{e}}_z$), and $B_s = (4\pi/c)\sigma\rho_s\omega$ is the interior magnetic field. Φ_B is the magnetic flux, i.e., the cross section area of the cylinder times the magnetic field $\Phi_B = B_s \pi \rho_s^2$. The expressions (62) are valid for $\rho \geq \rho_s$ i.e., outside and on the cylinder. The rotation of this vector field is zero, so the corresponding (exterior) magnetic field $\mathbf{B} = \nabla \times \mathbf{A}_s$ vanishes.

The vector potential (62) leads to the magnetic interaction energy,

$$U_i = \frac{1}{4\pi} \int_V \mathbf{B}_s \cdot \mathbf{B}_e \mathrm{d}V = \frac{e}{c} \mathbf{v} \cdot \mathbf{A}_s(\mathbf{r}) = \frac{e\,\Phi_B}{c\,2\pi} \dot{\varphi}, \tag{63}$$

between a passing electron and the solenoid, as elucidated by Essén and Stén [10]. Here $\mathbf{B}_e$ is the magnetic field from the moving electron and $\mathbf{v} = \dot{\rho}\, \hat{\mathbf{e}}_\rho(\varphi) + \rho\dot{\varphi}\, \hat{\mathbf{e}}_\varphi(\varphi)$ its velocity. This energy expression is based on standard expressions for electromagnetic energy [27] and is in good agreement with Eq. (41). It also agrees with the intuitive notion that the energy goes down when the magnetic field of the passing particle subtracts from the solenoid field and up when it adds to it, see Fig. 1.

One notes that this energy is simply the time-derivative of the electron azimuthal angle φ. This means that in the usual Lagrangian formalism Eq. (1) becomes

$$L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2} m\dot{\mathbf{r}}^2 + \frac{e\Phi_B}{2\pi c} \dot{\varphi} - e\phi(\mathbf{r}, t). \tag{64}$$

The solenoid thus only contributes a total time derivative $\dot{\varphi}$ to the Lagrangian. Usually such a total time derivative will not influence the dynamics, as follows from the variational principle. One may note, however, that the azimuthal angle φ is multivalued and not well defined at the origin. This indicates that the standard result may not apply.

When electrostatic effects can be neglected Eq. (64) can be regarded as an energy arising from Eq. (1) since according to the Darwin Lagrangian approach the second term is in fact quadratic in velocities, not linear, even though this fact is not evident in the test particle approach where the angular velocity of the charge producing the external field is regarded as constant. This indicates that one should consider the energy to be

$$E = \frac{1}{2} m\dot{\mathbf{r}}^2 + \frac{e\Phi_B}{2\pi c} \dot{\varphi}. \tag{65}$$

That is, the sum of the kinetic energy of the passing electron and its magnetic interaction energy with the solenoid. An energy expression, however, does not lead to an equation of motion, so the actual motion of the electron remains unclear. An exception to this state of affairs is the case of one-dimensional motion (e.g., along a line). In this case a conserved energy can be used to analyze the motion. When this is done one finds essentially the same result as found below using canonical momentum conservation.

6.2. Vector Potential of the Solenoid and the Canonical Momentum

Let us return to Eq. (12) with the electrodynamic force

$$e\mathbf{E}_A = -\frac{e}{c} \frac{\partial \mathbf{A}}{\partial t}. \tag{66}$$

Using the solenoid vector potential (62) here the simple test particle approximation tells us that the partial time derivative is zero, since the position of the electron ρ, φ is held constant, and the solenoid is assumed completely unaffected by the electron. As noted above when finding the equation of motion from the Darwin Lagrangian Eqs. (52)–(55) this can not be correct. The moving electron will cause changes in the flux Φ_B inside the solenoid, so we should consider the expression

$$\mathbf{A}_s(\mathbf{r},t) = \frac{\Phi_B(t)}{2\pi}\,\frac{\hat{\mathbf{e}}_\varphi(\varphi)}{\rho}. \tag{67}$$

The electrodynamic force is thus

$$e\mathbf{E}_A = -\frac{e}{c}\frac{\partial \mathbf{A}_s(\mathbf{r},t)}{\partial t} = -\frac{e}{2\pi c}\frac{\hat{\mathbf{e}}_\varphi(\varphi)}{\rho}\frac{\mathrm{d}\Phi_B(t)}{\mathrm{d}t}. \tag{68}$$

This is essentially Faraday's law: a change in flux induces an emf [48]. Eq. (59) is now

$$m_e\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \frac{e}{2\pi c}\frac{\hat{\mathbf{e}}_\varphi(\varphi)}{\rho}\frac{\mathrm{d}\Phi_B(t)}{\mathrm{d}t} = \mathbf{0}. \tag{69}$$

Integration with respect to time keeping m_e and $\hat{\mathbf{e}}_\varphi(\varphi)/\rho$ constant gives

$$m_e\mathbf{v}(t) + \frac{e\Phi_B(t)}{2\pi c}\frac{\hat{\mathbf{e}}_\varphi(\varphi)}{\rho} = m_e\mathbf{v}_\infty \tag{70}$$

where $\mathbf{v}_\infty$ is the velocity of the particle far from the solenoid. The treatment of the partial time derivative and its integration above seems unfamiliar, but this is what one gets by following the mathematics.

6.3. Motion of the Passing Electron Assuming Canonical Momentum Conservation

We now investigate the motion of the passing electron under the assumption that the conserved momentum is given by Eq. (60) and the vector potential by (62). We also assume that the inertia of the solenoid is many orders of magnitude larger that the inertia of the electron so that we can consider $\Phi_B(t)$ as constant compared to $\mathbf{v}(t)$. The calculation will produce the experimentally verified shift.

Assuming motion in the xy-plane and that far from the solenoid the electron momentum is $\mathbf{p} = m_e v_0 \hat{\mathbf{e}}_x$ with y-coordinates $y_\infty = \pm\ell$. The two signs determine on which side of the solenoid the electron passes, see Fig. 1. We find that (32) or (60) gives

$$m_e v_0 \hat{\mathbf{e}}_x = m_e\mathbf{v} + \frac{e\Phi_B}{2\pi c}\,\frac{\hat{\mathbf{e}}_\varphi(\varphi)}{\rho} \tag{71}$$

using (62). We rewrite this in the form

$$\mathbf{v} = v_0\hat{\mathbf{e}}_x - \frac{e\Phi_B}{2m_e\pi c}\,\frac{-\hat{\mathbf{e}}_x\sin\varphi + \hat{\mathbf{e}}_y\cos\varphi}{\sqrt{x^2+y^2}}. \tag{72}$$

Introducing the notation

$$\delta v \equiv \frac{e\Phi_B}{2m_e\pi c\ell} \tag{73}$$

and using $\sin\varphi = y/\sqrt{x^2+y^2}$ and $\cos\varphi = x/\sqrt{x^2+y^2}$ we find

$$\dot{x}\hat{\mathbf{e}}_x + \dot{y}\hat{\mathbf{e}}_y = v_0\hat{\mathbf{e}}_x - \delta v\frac{-\ell y\hat{\mathbf{e}}_x + \ell x\hat{\mathbf{e}}_y}{x^2+y^2}. \tag{74}$$

The two components of electron velocity are then

$$\dot{x} = v_0 + \delta v\,\frac{\ell y}{x^2+y^2}, \tag{75}$$

$$\dot{y} = -\delta v\,\frac{\ell x}{x^2+y^2}. \tag{76}$$

We now investigate these assuming that δv is small compared to v_0.

We first note that $\dot{y} = 0$ at $x = 0, \pm\infty$. Since δv is small it is thus reasonable to assume that the trajectory is close to a straight line with constant $y = \pm\ell$. At $x = 0$, when the electron is closest to the solenoid, Eq. (75) then gives

$$\dot{x} = v_0 \pm \delta v, \tag{77}$$

$$\dot{y} = 0, \tag{78}$$

so there is clearly a difference in speeds depending on which side of the solenoid the electron passes.

From (75) our straight line approximation gives the equation of motion

$$\dot{x}(t) = v_0 \pm \delta v \frac{\ell^2}{x^2(t) + \ell^2} \tag{79}$$

for the electron parallel to the x-axis. This is a separable differential equation which we now solve.

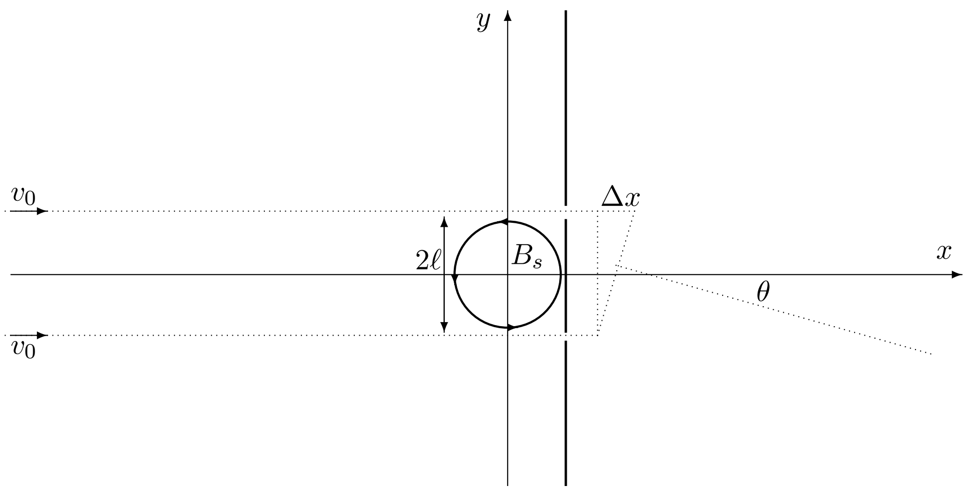

Figure 2. This figure shows a cross section of the solenoid with interior magnetic field B_s in the positive z-direction. The righthanded coordinate system used is shown as well as the double-slit screen with a distance 2ℓ between the slits. Charged particles of charge $e > 0$ and initial speed v_0 will get different speeds depending on the sign of the y-coordinate of the slit. Particles with positive y-coordinate will be faster and Δx ahead after passing the solenoid. The wave-mechanical deflection angle is θ. For electrons with $e < 0$ the deflection is in the opposite direction.

7. THE DIFFERENCE IN PASSAGE TIME AND DISTANCE TRAVELLED

Separating the differential equation (79) gives

$$\mathrm{d}t = \frac{\mathrm{d}x}{v_0 \pm \delta v \dfrac{\ell^2}{x^2 + \ell^2}}. \tag{80}$$

By integration we can find the time the electrons take to go from $x = -L$ to $x = L$

$$T_\pm(L) = \int_{-L}^{L} \mathrm{d}t = \int_{-L}^{L} \frac{\mathrm{d}x}{v_0 \pm \delta v \dfrac{\ell^2}{x^2 + \ell^2}} \tag{81}$$

The integral gives

$$T_\pm(L) = \frac{2L}{v_0} \mp \frac{2\ell\, \delta v \arctan\left(\dfrac{L}{\ell}\sqrt{\dfrac{v_0}{v_0 + \delta v}}\right)}{v_0\sqrt{v_0(v_0 + \delta v)}}. \tag{82}$$

Assuming small δv we find

$$T_{\pm}(L) \approx \frac{2L}{v_0} \mp \frac{2\ell\delta v}{v_0^2} \arctan\left(\frac{L}{\ell}\right) \tag{83}$$

The time difference for $L \to \infty$ is then

$$\Delta T = \frac{2\pi\ell\delta v}{v_0^2}, \tag{84}$$

using that $\arctan(\infty) = \pi/2$. We finally find that the difference in distance travelled along the x-axis is

$$\Delta x = v_0 \Delta T = \frac{2\pi\ell\delta v}{v_0}, \tag{85}$$

to first order in δv.

To verify this classically one must carefully measure these small differences in time or distance travelled for the charged particles passing on either side of the solenoid. Quantum interference effects arising from wave-mechanical behaviour of electrons should of course be much more sensitive to the differences. Two beams a distance 2ℓ apart will have wave fronts that are shifted by the distance Δx, see Fig. 2. This means that the interference pattern should be deflected by an angle given by

$$\tan\theta \approx \theta = \frac{\Delta x}{2\ell} = \frac{\pi\delta v}{v_0}. \tag{86}$$

Using the expression (73) for δv we find

$$\theta = \frac{\pi}{v_0} \frac{e\Phi_B}{2m_e\pi c\ell} = \frac{e\Phi_B}{p_0 2\ell c}, \tag{87}$$

where $p_0 = m_e v_0$ is the initial and final momentum of the electrons, and 2ℓ is the distance between the slits. Note that we have used gaussian units throughout. Changing to SI-units for $e\Phi_B$ the speed of light c in the denominator goes away. This angle is then seen to be the same as that found in the quantum mechanical treatment of the Aharonov-Bohm effect, see Boyer [12].

8. CONCLUSIONS

We started from the usual Lagrangian for a charged particle in given external fields and derived the usual equation of motion with the Lorentz force. We pointed out that the significance of the term $-c^{-1}\partial\mathbf{A}/\partial t$ in the electric field remains unclear in this formalism. To find out what it means we started from the Darwin Lagrangian and derived the equation motion for one particle in an N-particle system. This gives an explicit expression for this term and leads to the conclusion that the vector potential, being part of the canonical momentum, will affect the motion of a particle even if there is no electrostatic or magnetic field. In fact one may regard the experimental confirmation of the Aharonov-Bohm shift as a confirmation of this subtle classical effect.

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