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An Addition to Binomial Array Antenna Theory

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ABSTRACT: This research extends the theory of binomial array antennas. A closed-form expression for the half-power beamwidth of the array factor of an array antenna is presented. The expression is correct for the ratios of element spacing to wavelength greater than or equal to one quarter. Also, exact equations for determining the half-power beamwidth for main-beam steering are derived. A comparison with an approximate formula for the half-power beamwidth of binomial array antennas is carried out.

1. INTRODUCTION

Array antennas are powerful structures in communications. They are used in many technical devices in areas, such as mobile communications, satellite communications, radars for civil and military use, and astronomy, among others. Array antennas come in different geometric shapes to achieve directivity diagrams according to specific design requirements [1]. However, one of the most common design problems with array antennas is the synthesis of uniform linear array (ULA) antennas. The main problem is obtaining a ULA with a minimum number of elements N and a strong directivity — a narrow main lobe and the smallest possible side lobes. In historical terms, the array antenna theory is constantly evolving.

Various approaches are used for array antenna designs. The classical methods of synthesis, such as Fourier series, Schelkunoff, Woodward-Lawson, Dolph-Chebyshev, Riblet, and Taylor, can be found in the literature [2–8]. Window functions have been used to improve the array factor (AF) directivity [9, 10]. Various iterative algorithms for optimizing the array antenna patterns are described in the literature [11–15]. New mathematical functions for array antenna synthesis have also been introduced [16–18]. The theory of antenna arrays has been developed in a number of publications which include spectral factorization, elements sparsity, mutual coupling, and mounting-platform effects [19–21].

One of the design methods is that of obtaining an AF pattern without side lobes. Such an array antenna is described in [22–26]. Historically, the first array antenna with AF without side lobes is a binomial array antenna. The idea for array antennas with binomial coefficients was introduced nearly a century ago by Stone [27]. An important feature of binomial array antennas is that their array factor has no side lobes when the elements are spaced at less than or equal to half wavelength, that is, $d \leq \lambda/2$, with d being the distance between the elements and λ being the wavelength. Otherwise, side lobes appear, as shown in Fig. 1.

The theory of binomial array antennas was extended in [28], in which approximate closed-form expressions for the half-power beamwidth (HPBW) of the array factor and maximum
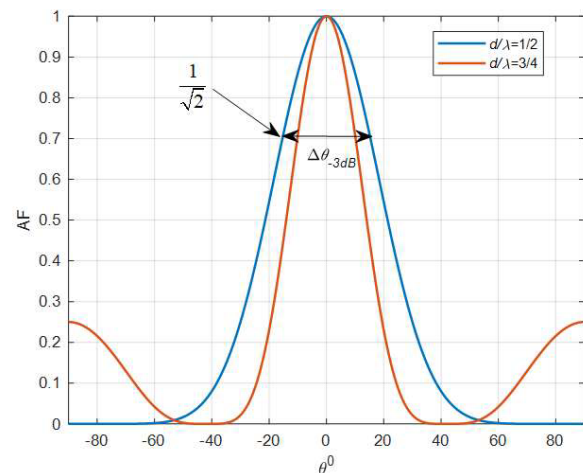
FIGURE 1. Array factors of five-element binomial array antenna, for $d/\lambda = 1/2$ and $d/\lambda = 3/4$.

directivity for the half-wavelength distance spacing are described. HPBW is usually denoted as $\Delta\theta_{-3\,\text{dB}}$, with θ being the azimuthal angle in the horizontal plane containing the axis with the elements of the ULA and $-3\,\text{dB}$ subscript referring to the fact that halving the power corresponds to a reduction by 3 dB of that power when being expressed in decibels. Over the past decade, the theory has been supplemented with a compact expression for the directivity of the binomial array antenna for an arbitrary d/λ ratio [29]. In [30], a study of a 10-element binomial array antenna for Global System for Mobile Communications applications is described. The authors propose an optimal interelement distance equal to 0.82λ.

In this research, an exact closed-form expression for the HPBW of the AF of the binomial array antenna is proposed. The expression is valid for ratios $d/\lambda \geq 0.25$. Moreover, exact expressions for determining the HPBW for main-beam steering are derived. This study represents an addition to the binomial array antenna theory. The array antenna elements are assumed to be isotropic point radiators.

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2. THEORY AND MAIN RESULTS

The theory of binomial arrays antennas is included in the most widely used textbooks and lecture notes on antennas [2–6]. In [28], an approximate expression for the HPBW of a binomial array antenna is described

$$\Delta\theta \approx \frac{1.06}{\sqrt{N-1}}, \tag{1}$$

where N is the number of array elements. This approximate formula is valid only for half-wavelength interelement distance.

In [4], an expression for the HPBW of a binomial array antenna in the ψ-space is derived. Here, ψ refers to the phase shift between two adjacent elements, equal to $\psi = kd\sin\theta$, with $k = 2\pi/\lambda$ being the wavenumber and θ being the azimuthal angle in the $[-90°, 90°]$ range

$$\Delta\psi_{-3\,\text{dB}} = 4\arccos\left(2^{-\frac{0.5}{N-1}}\right). \tag{2}$$

Again in [4], for beamwidth mapping from the ψ-space to the θ-space, a map linearization about the main lobe direction θ_0 is performed using the derivative of ψ with respect to θ, evaluated at θ_0. As a result, the following expression is obtained

$$\Delta\theta_{-3\,\text{dB}} = \frac{\Delta\psi_{-3\,\text{dB}}}{kd\cos\theta_0}. \tag{3}$$

This approach gives approximate values of HPBW for array antennas with a low number of radiators and wide HPBWs, because the derivative is equal to

$$\frac{d\psi}{d\theta} = \lim_{\Delta\theta\to 0}\left(\frac{\Delta\psi}{\Delta\theta}\right). \tag{4}$$

When $\cos\theta_0 = 0$, an approximate expression is used

$$\Delta\theta_{-3\,\text{dB}} = 2\sqrt{\frac{\Delta\psi_{-3\,\text{dB}}}{kd}}. \tag{5}$$

So far, the described analytical expressions (1) and (3) for the HPBW as functions of the azimuthal angle θ have certain inaccuracies. That is why, in the exposition below, a solution for HPBW for the array factor of the binomial array antenna is presented.

The normalized complex array factor of the binomial array antenna has the form

$$\text{AF} = \frac{1}{2^{N-1}}\left|\sum_{m=0}^{N-1} a_m \exp\left(jm\psi\right)\right|, \tag{6}$$

where a_m denotes the binomial coefficients given as

$$a_m = \frac{(N-1)!}{m!\,(N-1-m)!}, \quad m = 0, 1, ..., N-1. \tag{7}$$

To obtain the HPBW of the AF, it is convenient to use an expression for AF equivalent to Eq. (5), given in [5] without binomial coefficients

$$\text{AF} = \sqrt{\frac{(1+\cos\psi)^{N-1}}{2^{N-1}}} = (0.5 + 0.5\cos\psi)^{\frac{N-1}{2}}. \tag{8}$$

Thus, the following equation needs to be solved (see Fig. 1)

$$\begin{aligned}\text{AF} &= \sqrt{\frac{(1+\cos\psi_{-3\,\text{dB}})^{N-1}}{2^{N-1}}} \\ &= \sqrt{\frac{(1+\cos(kd\sin\theta_{-3\,\text{dB}}))^{N-1}}{2^{N-1}}} = \frac{1}{\sqrt{2}}.\end{aligned} \tag{9}$$

As outlined in Appendix A, from Eq. (8), we determine

$$\theta_{-3\,\text{dB}} = \arcsin\left(\frac{1}{kd}\arccos\left(2^{\frac{N-2}{N-1}} - 1\right)\right). \tag{10}$$

Because of the symmetry of the AF with respect to $\theta = 0°$ (see Fig. 1), the angle obtained in (9) needs to be doubled

$$\Delta\theta_{-3\,\text{dB}} = 2\arcsin\left(\frac{1}{kd}\arccos\left(2^{\frac{N-2}{N-1}} - 1\right)\right), \quad N \geq 2. \tag{11}$$

3. MAIN BEAM STEERING OF THE BINOMIAL ARRAY ANTENNA

The described theory of the binomial array antenna is valid in cases when the AF determines a radiation pattern perpendicular to the array axis. For main beam steering, the angle of the main beam direction θ_0 is included in (7) to obtain

$$\text{AF} = (0.5 + 0.5\cos(kd\sin\theta - kd\sin\theta_0))^{\frac{N-1}{2}}. \tag{12}$$

The main beam steering leads to a certain asymmetry of the main lobe and increases the HPBW and side lobes, as shown in Fig. 2. To obtain the HPBW, an approach that differs from the one described in [4] is proposed. In this case, it is necessary to calculate two expressions accounting for the two different slopes of the two sides of the main lobe, using (9)

$$\theta_1 = \arcsin\left(\sin\theta_0 - \frac{1}{kd}\arccos\left(2^{\frac{N-2}{N-1}} - 1\right)\right); \tag{13}$$

$$\theta_2 = \arcsin\left(\sin\theta_0 + \frac{1}{kd}\arccos\left(2^{\frac{N-2}{N-1}} - 1\right)\right). \tag{14}$$

HPBW is obtained from the difference of θ_2 and θ_1

$$\Delta\theta_{-3\,\text{dB}} = \theta_2 - \theta_1. \tag{15}$$

Figure 2 shows AF steering of a 7-element binomial array antenna with an interelement distance equal to $d = \lambda/2$ for $\theta_0 = 0°$ (red) and $\theta_0 = 30°$ (blue). The calculated HPBWs are $\Delta\theta_{-3\,\text{dB}} = 24.7475°$, and $\Delta\theta_{-3\,\text{dB}} = 28.9856°$, respectively.
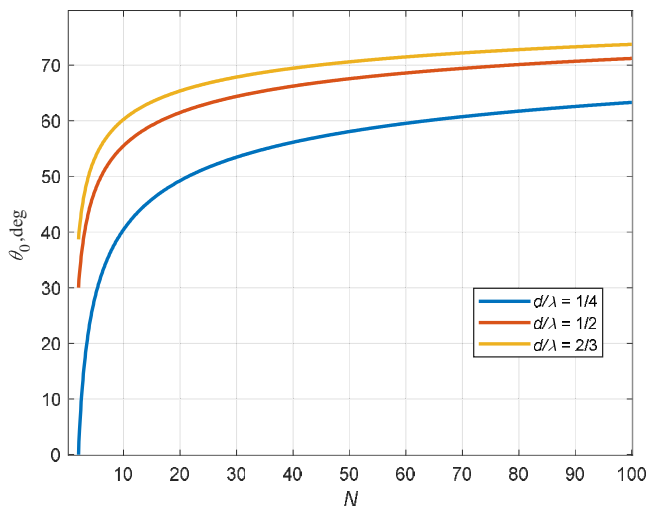
4. DISCUSSION

The maximum value of $\Delta\theta_{-3\,\text{dB}}$ is $180°$. This is satisfied when the minimal value of the AF is equal to $1/\sqrt{2}$. If the minimal value is greater than $1/\sqrt{2}$, Equation (10) does not provides a real solution because of the following.

First, we explore the case when the main lobe is perpendicular to the array axis, i.e., $\theta_0 = 0°$. Since the argument of the
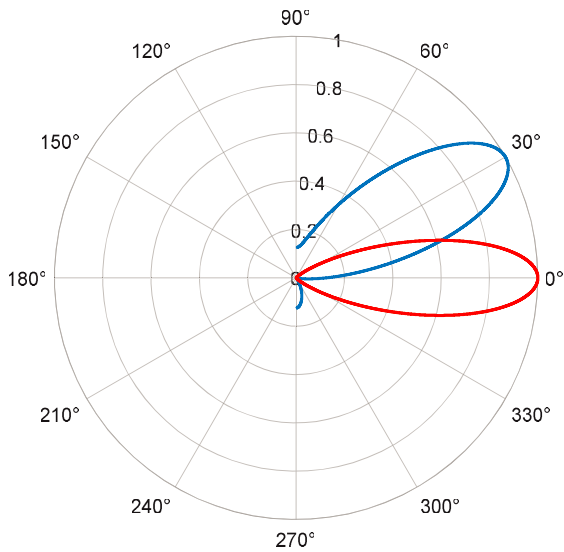
FIGURE 2. Steered AF of a binomial array antenna, $N = 7$; $d = \lambda/2$; $\theta_0 = 30°$ (blue), $\Delta\theta_{-3\,\text{dB}} = 28.9856°$; $\theta_0 = 0°$ (red), $\Delta\theta_{-3\,\text{dB}} = 24.7475°$.

FIGURE 3. Applicability limit of Eq. (10), $N \geq 2$.

FIGURE 4. Maximal steer angle, for $N \geq 2$, $d/\lambda = 1/4$, $d/\lambda = 1/2$, and $d/\lambda = 2/3$.

FIGURE 5. Absolute error of the approximate expression in Eq. (1), for $N \geq 2$, $d/\lambda = 1/2$.
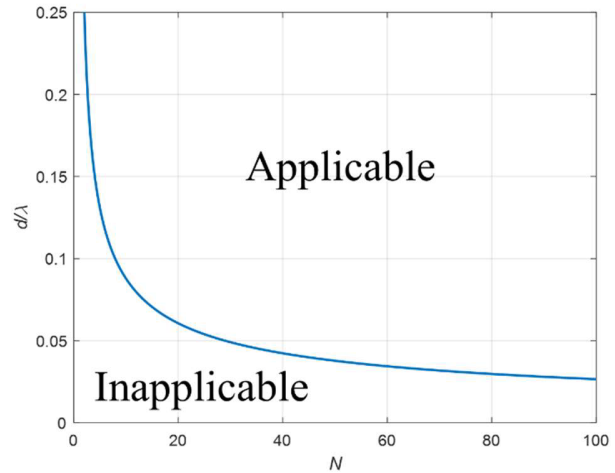
arcsine in Eq. (10) varies in the $[-1, 1]$ interval, the restriction for obtaining a real-valued solution can be formulated as

$$\frac{1}{kd} \arccos\left(2^{\frac{N-2}{N-1}} - 1\right) \leq 1. \tag{16}$$
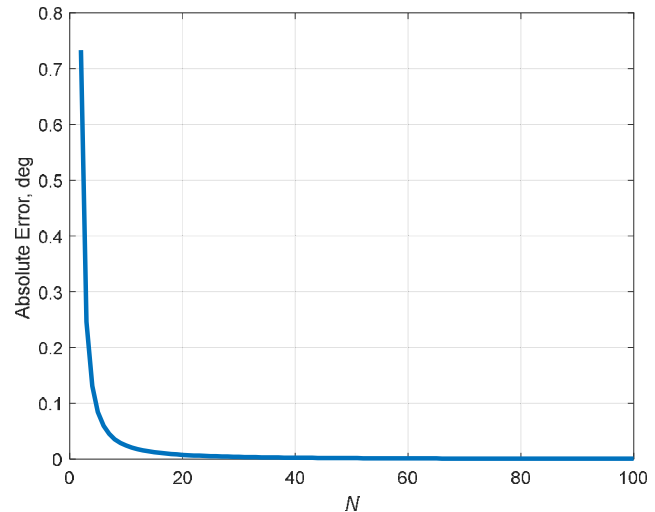
For a two-element array ($N = 2$), the left hand of the inequality (15) is a positive number equal to $\lambda/(4d)$. Substituting this expression in (15), we obtain $d/\lambda \geq 0.25$. For $N = 2$ and $d/\lambda = 0.25$, $\Delta\theta_{-3\,\text{dB}} = 180°$. Evidently, the applicability limit of Eq. (10) is defined when the argument of the arcsine is equal to 1. From this condition, it follows that

$$\frac{d}{\lambda} = \frac{1}{2\pi} \arccos\left(2^{\frac{N-2}{N-1}} - 1\right). \tag{17}$$

Figure 3 depicts the relation from Eq. (17). The curve shows the values of N and d/λ, for which the HPBW equals $180°$.

In the array antenna synthesis, an interelement distance shorter than $\lambda/4$ is not recommended due to the mutual coupling between the adjacent radiators. From this reasoning, the study assumes that the interelement distance is greater than or equal to $\lambda/4$. Following the same rules, when the main lobe is steered, equating (13) to $1/\sqrt{2}$ we can derive the maximal steer angle

$$\theta_0 = \arcsin\left(1 - \frac{1}{kd} \arccos\left(2^{\frac{N-2}{N-1}} - 1\right)\right). \tag{18}$$

Figure 4 shows the dependence of the maximal steer angle on the number of array antenna elements for 3 different interelement distances. It is seen that two-element $\lambda/4$-spaced binomial array antenna cannot be steered.

The derived exact formula for the HPBW allows for checking of the accuracy of the approximate formula given in Eq. (1).
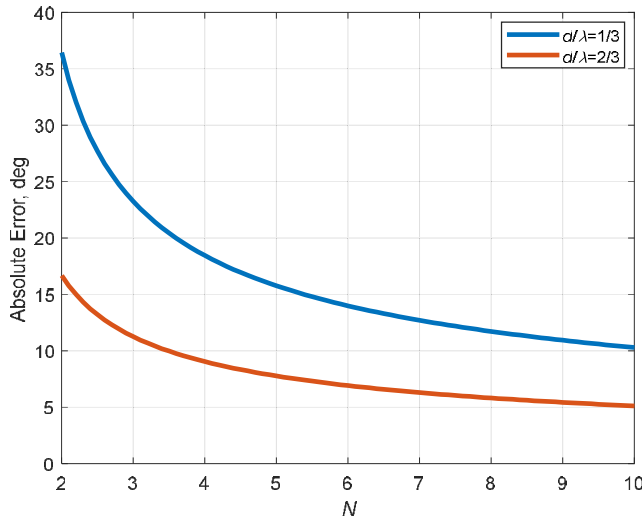
FIGURE 6. Absolute error of the approximate expression in Eq. (1), for $N \geq 2$, $d/\lambda = 1/3$, and $d/\lambda = 2/3$.

FIGURE 7. Absolute error for Eq. (3), $N \geq 2$.

Fig. 5 shows the absolute error of the approximate expression in degrees for $d/\lambda = 1/2$. The absolute error is defined as the absolute value of the difference between the exact value and the approximate value obtained by the proposed method.

The absolute error for the half-wavelength distance between the elements is smaller than $0.8°$. For other interelement distances, the absolute error significantly increases, as shown in Fig. 6.

It can be concluded that the approximate expression is easy to use for $d/\lambda = 0.5$ due to the lower computational complexity.

The same test for Eq. (3) is carried out, with the assumption that $\cos \theta_0 = 1$. Fig. 7 shows the absolute error for Eq. (3) for $d/\lambda = 0.45$, $d/\lambda = 0.5$, and $d/\lambda = 0.55$. The absolute error is shown to decrease when the number of elements N and the d/λ ratio increase.

This nonsignificant inaccuracy is due to the approximation in the map linearization. To eliminate the error, when $\cos \theta_0 = 1$, Eq. (3) becomes
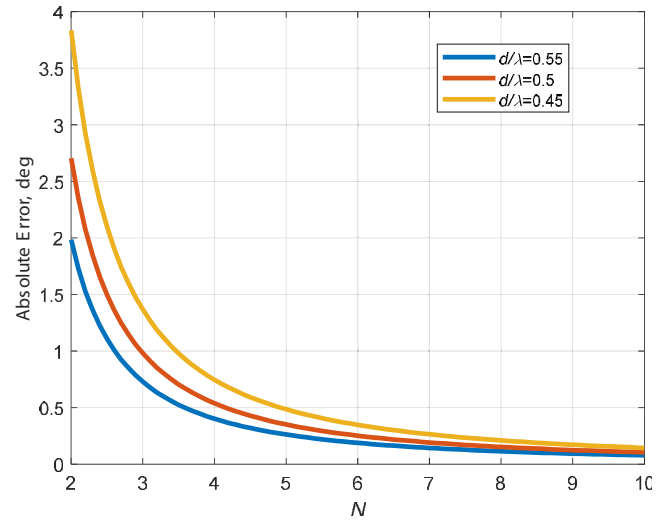
$$\Delta \theta_{-3\,\text{dB}} = 2 \left(\arcsin \frac{\Delta \psi_{-3\,\text{dB}}}{2kd} \right); \quad \theta \in [-90°, 90°]. \quad (19)$$

5. CONCLUSION

From the presented theory, the following conclusions can be drawn. An exact formula for HPBW of the AF of a binomial array antenna has been derived in (10). Exact expressions for HPBW for the steered main beam are derived in (12)–(14).

The applicability limits of inequality (15) and the maximal steer angle given in Eq. (17), depending on the number of elements and the d/λ ratio, have been defined. Due to the elements' mutual coupling, the most frequently used interelement distances in practice are in the 0.5λ–0.8λ range. Therefore, the proposed theory can be useful in the binomial array antenna design.

The absolute error for HPBW of the approximate expression in Eq. (1) has been explored and illustrated in Fig. 5. For

$d/\lambda = 0.5$, the absolute error for the HPBW in the approximate expression is less than $0.8°$, which makes it suitable for use because of the simplicity.

The proposed study complements the theory of the binomial array antennas. In the paper, the array antenna elements are assumed to be isotropic point radiators. The figures depict array factors. Practical implementation involves issues related to all the effects that occur when constructing an antenna array: mutual coupling between the neighbor elements, inaccurate distance between the elements, inaccuracies in the construction of the elements, insertion losses, etc. The described addition to the theory gives guidance for reducing these negative effects. In a future work, the proposed theory will be extended for two-dimensional binomial array antennas.

APPENDIX A. DERIVATION OF EQ. (9). STARTING WITH EQ. (8)

$$\text{AF} = \sqrt{\frac{(1 + \cos \psi_{-3\,\text{dB}})^{N-1}}{2^{N-1}}} = \frac{1}{\sqrt{2}}, \quad (A1)$$

and raising both sides to the power of 2, we obtain

$$\frac{(1 + \cos \psi_{-3\,\text{dB}})^{N-1}}{2^{N-1}} = \frac{1}{2} = 2^{-1}; \quad (A2)$$

$$(1 + \cos \psi_{-3\,\text{dB}})^{N-1} = 2^{N-1} 2^{-1} = 2^{N-2}. \quad (A3)$$

Raising both sides to the power of $1/(N-1)$, we obtain

$$1 + \cos \psi_{-3\,\text{dB}} = 2^{\frac{N-2}{N-1}}. \quad (A4)$$

Rearranging the last equality, we obtain

$$\psi_{-3\,\text{dB}} = \arccos \left(2^{\frac{N-2}{N-1}} - 1 \right). \quad (A5)$$

Noting that $\psi_{-3\,\text{dB}} = kd \sin \theta_{-3\,\text{dB}}$, we obtain

$$kd \sin \theta_{-3\,\text{dB}} = \arccos \left(2^{\frac{N-2}{N-1}} - 1 \right), \quad \text{or} \quad (A6)$$

$$\sin\theta_{-3\,\text{dB}} = \frac{1}{kd}\arccos\left(2^{\frac{N-2}{N-1}} - 1\right), \qquad \text{(A7)}$$

and finally

$$\theta_{-3\,\text{dB}} = \arcsin\left(\frac{1}{kd}\arccos\left(2^{\frac{N-2}{N-1}} - 1\right)\right). \qquad \text{(A8)}$$

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