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Lensing by a Single Interface: Perfect Focus Point Rather Than a Drain Point

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ABSTRACT: An exact analytical solution is obtained for the problem of finding the field of a linear electric current located near the interface between half-spaces filled with ordinary and perfectly matched double-negative media. To achieve this, a novel approach is introduced that, for the first time, enabled the correct analytic continuation of the function describing the field into the entire half-space filled with a double-negative medium. The analysis of this solution shows that the current source field, upon reaching the point of perfect focusing, passes through it and then moves off to infinity, rather than disappearing at that point, as claimed in earlier works.

1. INTRODUCTION

Progress in the experimental realization of double-negative (DNG) materials reveals new perspectives on many electromagnetic phenomena [1, 2]. This is most evident in the study of the field behavior near the interface between ordinary and DNG media. Layered media that include DNG materials exhibit unique electrodynamic properties [3], such as the effect of perfect focusing. To obtain an expression for the field of a current source in such a planar layered medium, spectral decomposition method is typically used. Thus, the interaction between a linear current source and a DNG half-space or DNG slab is modeled using different Sommerfeld integrals in each layer of such a medium [4, 5]. However, when transitioning from a DNG medium with $\varepsilon_{DNG} < 0$ and $\mu_{DNG} < 0$ to the case of a perfectly matched planar lens with $\varepsilon_{DNG} = -\varepsilon_0$ and $\mu_{DNG} = -\mu_0$, spatial regions are divided into two types [6–8]. In regions of the first type, the corresponding Sommerfeld integrals are calculated explicitly, while in regions of the second type, these integrals become divergent. A challenge arises in deriving expressions for the field in regions of the second type. In [6, 8, 9], additional assumptions regarding the field's possible behavior near the focal plane, which separates regions of the first and second type, are made. Based on these assumptions, the field is continued from regions of the first type into regions of the second type. The variants of such continuation proposed in [6] and [8] lead to the conclusion that the field's singularity point in a perfectly matched DNG medium (slab in [6] and half-space in [8]) is a drain point of electromagnetic energy rather than a focus, which implies a non-physical disappearance of energy at this point from the real space. This results in a violation of the radiation condition: the Poynting vector flux in DNG half-space appears coming from infinity (Fig. 2 in [8]). This is a consequence of the assumption that the analytic continuation through the focal plane should be carried out continuously, while ignoring the fact that the original function has a branch point on this plane. On the other hand, the additional assumption used in [9] results in an infinitely large jump of the field at the focus point.

In this work, we restrict our analysis to the field of a linear electric current near a DNG half-space. Without any additional assumptions, we demonstrate that the analytic continuation, which accounts for the presence of a branch point of the Hankel function, automatically results in the singularity in a perfectly matched DNG half-space being a point of perfect focusing rather than an energy drain point. In this case, the field has a finite discontinuity in the focal plane.

2. PROBLEM FORMULATION AND SOLUTION

We are looking for the field $E_y = E_y(x, z)$ of a linear electric current $\vec{j} = \{0, j_y, 0\}$,

$$j_y = I_0 \delta(x) \delta(z - z_0) e^{-i\omega t} \quad (z_0 > 0) \tag{1}$$

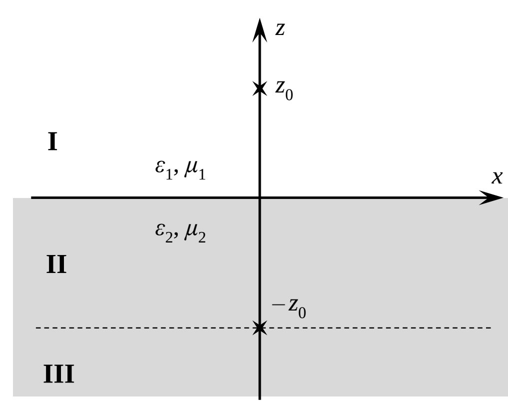

FIGURE 1. Geometry of the boundary value problem.

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in a two-layered medium (Fig. 1). This field is E-polarized with components

$$H_x = -\frac{1}{i\omega\mu}\frac{\partial E_y}{\partial z}, \quad H_z = \frac{1}{i\omega\mu}\frac{\partial E_y}{\partial x}, \qquad E_x = E_z = H_y = 0. \tag{2}$$

To find it, we solve the following boundary value problem [10]

$$\begin{cases} \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \omega^2 \varepsilon_s \mu_s\right] E_y^{(s)}(x,z) = -g_0 \delta(x)\,\delta(z - z_0), \\ \quad -\infty < \ x, z \ < \infty, \\ E_y^{(1)}\big|_{z=0} = E_y^{(2)}\big|_{z=0}, \quad \frac{1}{\mu_1}\frac{\partial E_y^{(1)}}{\partial z}\Big|_{z=0} = \frac{1}{\mu_2}\frac{\partial E_y^{(2)}}{\partial z}\Big|_{z=0}, \\ \lim\limits_{r\to\infty} \sqrt{r}\left(\frac{\partial E_y^{(s)}}{\partial r} - ik_s E_y^{(s)}\right) = 0. \end{cases} \tag{3}$$

Here, the first line is the Helmholtz equation; the third line represents the continuity condition of the field at the interface $z = 0$; the fourth line is the Sommerfeld radiation condition [11]; the function $E_y^{(s)}(x,z)$ and permittivity ε_s and permeability μ_s are defined in the upper ($s = 1$, $z > 0$) and lower ($s = 2$, $z < 0$) half-spaces. Besides that, $r = \sqrt{x^2 + z^2}$, $k_s = \sqrt{\varepsilon_s \mu_s}\,\omega$, $g_0 = i\omega\mu_1 I_0/2\pi$.

The solution to the problem (3) is known, which is obtained using the method of spectral decomposition [10–12]. In this study, we employ the field representation derived in [10]:

$$E_y^{(1)} = \frac{ig_0}{4\pi}\left\{\int_{-\infty}^{\infty} \frac{\exp\{i\xi x + i\kappa_1 |z - z_0|\}}{\kappa_1} d\xi + \int_{-\infty}^{\infty} \frac{\exp\{i\xi x + i\kappa_1 (z + z_0)\}}{\kappa_1} R(\xi)\, d\xi\right\}, \tag{4}$$

$$E_y^{(2)} = \frac{ig_0}{4\pi}\int_{-\infty}^{\infty} \frac{\exp\{i\xi x - i\kappa_2 z + i\kappa_1 z_0\}}{\kappa_1} T(\xi)\, d\xi. \tag{5}$$

Here,

$$R(\xi) = \frac{\mu_2\kappa_1 - \mu_1\kappa_2}{\mu_2\kappa_1 + \mu_1\kappa_2} \quad \text{and} \quad T(\xi) = \frac{2\mu_2\kappa_1}{\mu_2\kappa_1 + \mu_1\kappa_2} \tag{6}$$

are reflection and transmission coefficients, and $\kappa_s(\xi) = \sqrt{k_s^2 - \xi^2}\ (s = 1, 2)$.

Next, we consider the case of perfectly impedance matched media: $\varepsilon_2 = -\varepsilon_1$, $\mu_2 = -\mu_1$. As known from [8], in this scenario, the reflection coefficient from the boundary of such media is $R = 0$; the transmission coefficient is $T = 1$; and $\kappa_2(\xi) = -\kappa_1(\xi)$. Therefore, from (4) and (5) follows

$$E_y^{(1)} = \frac{ig_0}{4\pi}\int_{-\infty}^{\infty} \frac{\exp\{i\xi x + i\kappa_1 |z - z_0|\}}{\kappa_1} d\xi = \frac{ig_0}{4} H_0^{(1)}\left(k_1\sqrt{x^2 + (z - z_0)^2}\right) \quad (0 < z < \infty), \tag{7}$$

$$E_y^{(2)} = \frac{ig_0}{4\pi}\int_{-\infty}^{\infty} \frac{\exp\{i\xi x + i\kappa_1 (z + z_0)\}}{\kappa_1} d\xi \quad (-\infty < z < 0). \tag{8}$$

Here, $H_0^{(1)}(\cdots)$ is the Hankel function of the first kind [13]. The convergence of the integral in (8) depends on the sign of the second term in the exponent's argument. In region **II** ($-z_0 < z < 0$, see Fig. 1), from (8) follows

$$E_y^{(2)} = \frac{ig_0}{4\pi}\int_{-\infty}^{\infty} \frac{\exp\{i\xi x + i\kappa_1 |z + z_0|\}}{\kappa_1} d\xi = \frac{ig_0}{4} H_0^{(1)}\left(k_1\sqrt{x^2 + (z + z_0)^2}\right) \quad (-z_0 < z < 0), \tag{9}$$

The representations (7) and (9) are fully consistent with the corresponding formulas (8a) and (8b) obtained in [8], taking into account the difference in sign in the time dependence between those expressions and the present study.

In region **III** ($-\infty < z < -z_0$), the integral in (8) diverges. Therefore, to obtain the expression for the field $E_y^{(2)}$ in this region, it is necessary to perform an analytic continuation of the function $H_0^{(1)}(k_1\rho)$ from (9). The point $\rho = \sqrt{x^2 + (z + z_0)^2} = 0$ is a branch point of this function. It is known [13] that in the course of analytic continuation of the Hankel function, as the branch point is encircled, the following transformation occurs:

$$H_0^{(1)}\left(e^{in\pi}\, k_1\rho\right) = H_0^{(1)}(k_1\rho) - 2nJ_0(k_1\rho), \tag{10}$$

where $J_0(\ldots)$ is the Bessel function and $n = 0, 1, 2, \ldots$. The criterion that unambiguously determines the physically correct result of such continuation is the Sommerfeld radiation condition. For DNG half-space, this requirement translates to [5]

$$\lim_{\rho\to\infty} \sqrt{\rho}\left(\frac{\partial E_y^{(2)}}{\partial \rho} + ik_1 E_y^{(2)}\right) = 0. \tag{11}$$

For $n = 1$ (single encirclement around the singularity of the function $H_0^{(1)}(k_1\rho)$), from (10) we obtain $H_0^{(1)}\left(e^{i\pi}k_1\rho\right) = -H_0^{(2)}(k_1\rho)$. Therefore, the analytic continuation of the Hankel function (9) into region **III** yields

$$E_y^{(2)} = -\frac{ig_0}{4\pi}\int_{-\infty}^{\infty} \frac{\exp\{-i\beta x - i\kappa_1 |z + z_0|\}}{\kappa_1} d\beta = -\frac{ig_0}{4} H_0^{(2)}(k_1\rho) \quad (-\infty < z < -z_0), \tag{12}$$

where $\kappa_1(\beta) = \sqrt{k_1^2 - \beta^2}$, Im $\kappa_1(\beta) \le 0$ [11]. Taking into account the asymptotic behavior of the function $H_0^{(2)}(k_1\rho)$ for $k_1\rho \gg 1$, it is easy to see that the function (12) satisfies the radiation condition (11). For any $n \ne 1$, this condition is not fulfilled.

As can be seen from (2), (9), and (12), the field components E_y and H_z have a discontinuity crossing the focal plane $z = -z_0$. It can be shown that this discontinuity is a direct consequence of the effect of perfect focusing.

Let us now examine the distribution of the obtained field's Poynting vector

$$\vec{S} = \text{Re } \vec{E} \times \vec{H}^* = \text{Re } \{\vec{x}_0 \cdot E_y H_z^* - \vec{z}_0 \cdot E_y H_x^*\} \tag{13}$$

in each of the three regions. Here, * denotes the complex conjugate. Given that $H_0^{(1,2)}(\chi) = J_0(\chi) \pm iN_0(\chi)$ ($J_0(\chi)$ and $N_0(\chi)$ are the Bessel and Neumann functions), and the Wronskian $\text{W}\,(J_0(\chi), N_0(\chi)) = 2/\pi\chi$ [13], from (2) and (13) we deduce

$$\vec{S} = \sigma \begin{cases} \vec{\rho}_1/\rho_1^2, & 0 < z < \infty, \\ -\vec{\rho}_2/\rho_2^2, & -z_0 < z < 0, \\ \vec{\rho}_2/\rho_2^2, & -\infty < z < -z_0, \end{cases} \tag{14}$$

where $\vec{\rho}_1 = \vec{e}_x \cdot x + \vec{e}_z \cdot (z - z_0)$, $\vec{\rho}_2 = \vec{e}_x \cdot x + \vec{e}_z \cdot (z + z_0)$, and $\sigma = \dfrac{\omega\mu_1}{32\pi^3} I_0^2$.

As can be seen from (14), the energy flux from the source at the point $(0, z_0)$ propagates in radial directions. Upon crossing the interface $z = 0$, negative refraction occurs, resulting in all the electromagnetic energy passing into the DNG medium's layer $-z_0 < z < 0$ being directed to the point $(0, -z_0)$. From this point, as indicated by the last formula in (14), the energy spreads radially in the half-space $-\infty < z < -z_0$ going towards infinity (Fig. 2). Thus, the point $(0, -z_0)$ is a point of **perfect focusing**, not an energy drain as claimed in [8]. Additionally, from (14), it follows that the non-zero energy fluxes, which are adjacent to the focal plane on both sides, have opposite signs, thus indicating a discontinuity of the field in the focal plane.

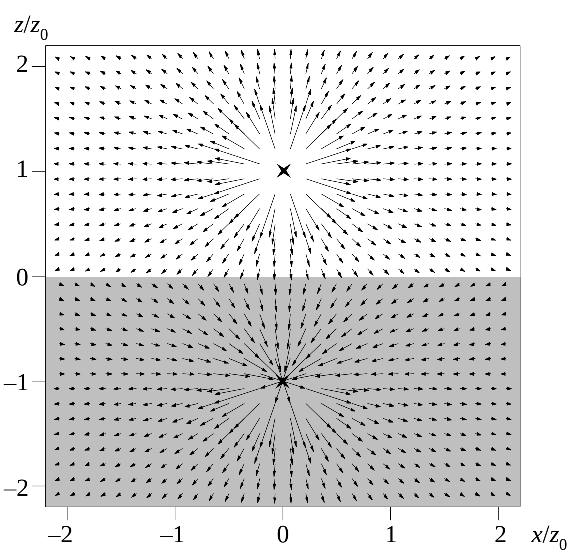

FIGURE 2. Poynting vector of the field of a linear current source near the interface between ordinary and DNG media. The source is at $(x, z) = (0, z_0)$, the focus point is $(x, z) = (0, -z_0)$.

From the integral representations (7), (9), (12), it can be concluded that in region **I** ($z > 0$), the phase front of the cylindrical wave propagates from the source point; in region **II** ($-z_0 < z < 0$), it propagates from the focus point; and in region **III** ($-\infty < z < -z_0$), it propagates towards the focus point.

Evanescent spatial modes decay exponentially in the ordinary medium (region **I**) as they propagate away from the source (see (7)). After crossing the interface, in region **II**, their amplitudes increase (see (9)), reaching the focus point $(0, -z_0)$ the same values that they had when leaving the source point $(0, z_0)$. After passing through the focus point, in region **III**, their amplitudes decay exponentially again (see (12)).

3. CONCLUSION

An exact analytical solution has been obtained for the two-dimensional problem of finding the field of a linear electric current located near the interface between ordinary and perfectly matched DNG media. This solution shows that the point of perfect focusing is not a drain for the field's energy, as claimed in [6, 8]. The approach proposed in our work allows the mandatory radiation condition in the DNG half-space to be satisfied, thereby eliminating the physically incorrect arrival of waves from infinity, which occurs in the solution obtained in [8].

It has also been shown that in the considered case of perfectly matched ordinary and DNG media, the physically intuitive property of boundedness and continuity of the electromagnetic field, typical for ordinary homogeneous media, is violated in the DNG medium not only at the focus point but throughout the focal plane. This is a consequence of the idealized nature of the model under consideration, where the matching conditions of ordinary and DNG media are strictly satisfied, and no losses are present. Violation of any of these conditions eliminates both the field's discontinuity and the singularity at the focus point.

The proposed method for determining the field of a linear current near a perfectly matched DNG half-space can be applied to solve a similar problem for a perfectly matched DNG planar layer. Moreover, it can also be extended to the case of three-dimensional problem that describes the field of a point current source.

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