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Hierarchical Matching Maximum Likelihood Estimation for Digital-Array Monopulse Tracking Radar

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ABSTRACT: Direction of arrival (DOA) estimation is an important issue for radar and communication applications. Monopulse is widely used to obtain the DOA result by the complex ratio from the sigma and delta beams of the antenna. In the case of digital array systems, various methods based on the covariance matrix of the received signal have been proposed to obtain DOA result. However, it is impractical for tracking radar scenarios, as the covariance matrix is not easy to obtain. Nevertheless, there is merely one target echo in the vicinity of the range cell as forecasted. Thus, the Maximum Likelihood Estimation (MLE) is a relatively good estimator for tracking radar, which has high accuracy and robustness. However, MLE is often very computationally resource-intensive, as it needs to search the whole steering vector set. In this paper, in order to utilize MLE effectively, we propose an algorithm to quickly search the steering vector set by binary tree hierarchical matching method, which can significantly reduce the computational cost. Furthermore, the computational complexity and accuracy performance have been studied from both theoretical analysis and simulation perspectives.

1. INTRODUCTION

Monopulse is a simultaneous lobbing technique for precisely determining the angle of arrival of echo signal [1–4]. Monopulse has been widely used in precision tracking radar to get accurate position of the targets, which is also applied in many kinds of modern search radar to improve the angle estimation accuracy. As to practical application, the antenna array is usually divided into four quadrants, and signals forming these quadrants are transmitted to the Σ-Δ network to get the sum signal, delta-azimuth signal, and delta-elevation signal, respectively [5–7].

In a digital array system, direction of arrival (DOA) algorithms are used, in general, to estimate the number of incident plane waves and their angles of incidence, which is also named as angle spectrum estimation. Many DOA estimation approaches have been designed, and their performances have been investigated, such as Capon method [8], maximum entropy method [9], Multiple Signal Classification (MUSIC) method [10], Minimum Norm technique [11], and the Estimation of Signal Parameters via Rotational Invariance Techniques (ESPRIT) [12]. The subspace approach makes use of a fundamental property possessed by those eigenvectors of the array spectral density matrix that are associated with eigenvalues that are larger than the noise level [13]. Wang et al. propose a coprime array DOA estimation algorithm that is based on single-snapshot direct data domain approach [14]. The various methods mentioned above are based on the sample covariance matrix (SCM) of received signal of the digital antenna array. However, under certain conditions, the SCM cannot be obtained for the target echo signal, such as in single-snapshot scenario [15]. Many compressed sensing DOA methods have been proposed [16] for single-snapshot.

Digital array tracking radar systems may employ monopulse angle estimation method through digital beamforming (DBF) [17]. In this case, multiple spatial channels are available for multifunctional array signal processing, including digital monopulse beamforming. There are some criteria for monopulse design such as achieving lower sidelobe level for the sum and difference beam [18]. The derivation of monopulse angle accuracy for digital array radar to achieve Cramer-Rao lower bound (CRLB) is presented by Takahashi et al. [19].

In this paper, we focus on a particular application scenario, the digital array radar that is employed for target tracking. Tracking radar measures the coordinates of a target and provides data that may be used to determine the target trajectory and to predict its future position, such as Millimeter-Wave Monopulse Radar for Space Debris Detection [20]. As for tracking radar, it is reasonable to assume that there is only one target's echo in a certain range cell. In many cases, even for digital arrays tracking radar, angle estimation still relies on monopulse. For tracking radars, with respect to the angular measurement of the target, the most concerning aspects are high accuracy and high robustness. Hence, the most fundamental approach, maximum likelihood estimation (MLE) method, performs more favorably. Additionally, under the constraint of a single target echo, compressed sensing-based algorithms, such as orthogonal matching pursuit (OMP) [21], sparse Bayesian learning (SBL) [22], and Least Absolute Shrinkage and Selection Operator (LASSO) [23] methods, are equivalent to MLE.

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MLE demands sampling within the angle domain. For a one-dimensional array, it is not a complex issue. Nevertheless, the actual tracking radar needs to undertake two-dimensional measurements, which are azimuth and elevation. Thus, the calculation consumption of MLE cannot be afforded by real-time processor. In order to utilize MLE for angle measurement, we have proposed a binary tree hierarchical matching method to quickly search for the target angle in the steering vector set. The application of recursive methods can significantly reduce the computational cost. The remainder of this letter is organized as follows. In Section 2, a signal model is introduced. In Section 3, binary tree hierarchical matching method is proposed in detail. Computational complexity and accuracy performance have been researched from both theoretical analysis and simulation perspectives in Sections 4 and 5. Finally, we give some conclusions in Section 6.

2. SIGNAL MODEL

For clarity sake, the uniform linear geometry of antenna array is considered. Although we use a uniform linear array as an example to describe the algorithm, the MLE is independent of the array topology. The received signal is processed by low-noise amplifiers, mixers, filters, analog-to-digital converters (ADC), and converted to baseband, as depicted in Figure 1. The output of digital array can be expressed by the received complex vector

$$\mathbf{x} = (x_0, x_1, \ldots, x_{N-1})^{\mathrm{T}} \tag{1}$$

in which, $()^{\mathrm{T}}$ denotes the transpose of matrix, and N represents the number of elements in the digital array antenna. As for tracking radar, in certain range cell, we assume only one target echo. Thus, the received complex vector can be denoted as

$$\mathbf{x} = \kappa\mathbf{a}(\theta) + \mathbf{n} \tag{2}$$

where $\mathbf{a}(\theta)$ is the steering vector, $\mathbf{n}$ the noise vector, κ the complex amplitude of signal, and θ the direction of arriving to the array.

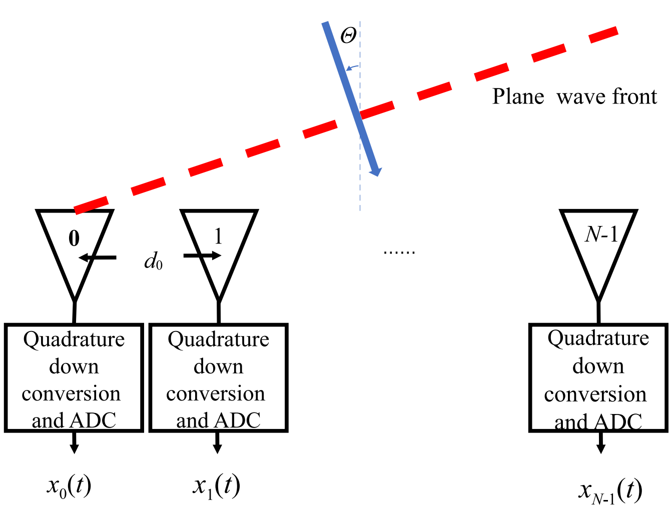

FIGURE 1. Illustration of a uniform linear antenna array.

In the uniform linear arrays, the steering vector can be described as

$$\mathbf{a}(\theta) = \left[1, \exp\left(\mathrm{j}\frac{2\pi}{\lambda}d_0 \sin\theta\right), \ldots, \exp\left(\mathrm{j}\frac{2\pi(N-1)}{\lambda}d_0 \sin\theta\right)\right]^{\mathrm{T}} \tag{3}$$

where λ is the wavelength of signal, and d_0 is the element spacing in uniform linear arrays.

There are two methods to achieve target angle measurement in digital array tracking radar. The first method is to obtain sigma-beam and delta-beam in monopulse, by specific beamforming filters. In this situation, the antenna elements should be positioned in a symmetric fashion around the antenna center. Specifically, the antenna aperture of the N-element linear array should be divided into two quadrants with $N/2$ elements each. The sigma-beam of monopulse is

$$\mathbf{w}_{\Sigma}(n) = \mathbf{a}(n)^*, \quad n = 0, 1, \ldots, N-1 \tag{4}$$

The delta-beam of monopulse is

$$\begin{aligned} \mathbf{w}_{\Delta}(n) &= \mathbf{a}(n)^*, \quad n = 0, 1, \ldots, N/2-1 \\ \mathbf{w}_{\Delta}(n) &= -\mathbf{a}(n)^*, \quad n = N/2, N/2+1, \ldots, N-1 \end{aligned} \tag{5}$$

$\mathbf{a}(n)$ is the coefficient of the spatial beamforming filter. Consequently, we can get complex ratio by the output of sigma-beam and delta-beam,

$$\eta = \frac{\mathbf{w}_{\Delta}^{\mathrm{H}} \cdot \mathbf{x}}{\mathbf{w}_{\Sigma}^{\mathrm{H}} \cdot \mathbf{x}} \tag{6}$$

where $()^{\mathrm{H}}$ denotes the conjugate transpose operator of matrix. The complex ratio can be used to obtain angle estimation result by monopulse.

The second method is to perform MLE for angle estimation. Compared to monopulse, MLE does not require symmetry of the array. We assume that $\mathbf{n}$ in (2) is zero-means complex white Gaussian random processes, correspondingly $\mathbf{n} \sim CN(0, \sigma^2\mathbf{I}_{N\times N})$. Thus, the likelihood function of the received complex vector

$$P(\mathbf{x}|\theta,\kappa) = \frac{1}{\pi^N\sigma^2}\exp\left(-\frac{(\mathbf{x}-\kappa\mathbf{a}(\theta))^{\mathrm{H}}(\mathbf{x}-\kappa\mathbf{a}(\theta))}{\sigma^2}\right) \tag{7}$$

The MLE of (7) is

$$\hat{\theta} = \arg\max_{\theta} P(\mathbf{x}|\theta,\kappa) = \arg\min \|\mathbf{x} - \kappa\mathbf{a}(\theta\|^2 \tag{8}$$

As for the uniform linear arrays, MLE is equivalent to discrete Fourier transform (DFT), for which fast Fourier transform (FFT) method is quite less complex even with the overcomplex grid search. However, under the condition of general array topology, DFT cannot be utilized, while MLE still works. From a practical perspective, the set of steering vectors in MLE can even be obtained through external calibration rather than theoretical derivation.

To solve the MLE problem mentioned above, the spatial domain can be discretized into the following sampling set

$$\Omega = \{\theta_0, \theta_1, \ldots, \theta_M\} \tag{9}$$

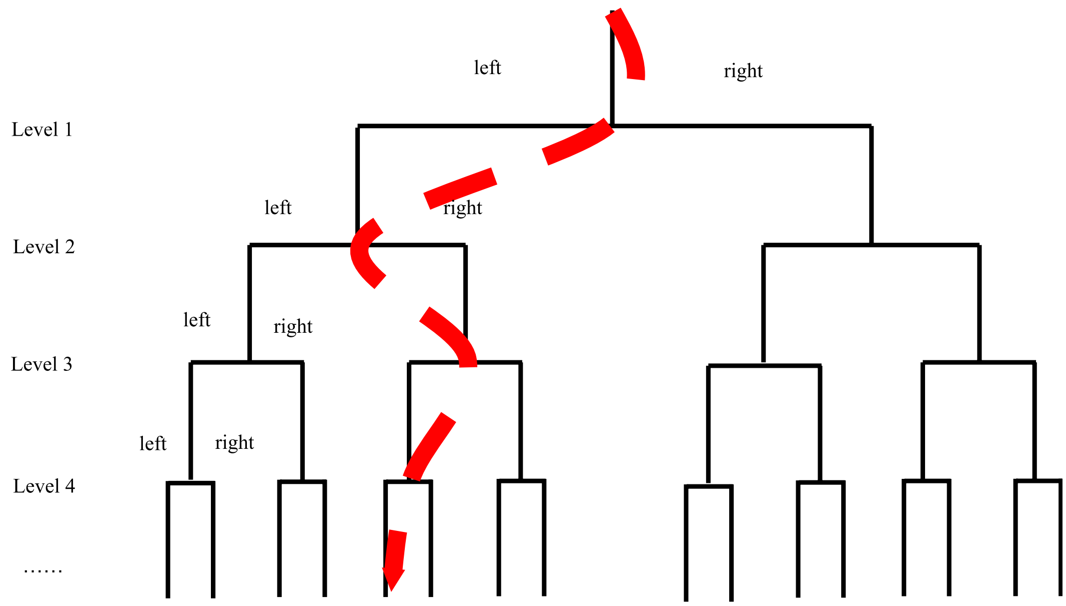

FIGURE 2. Path to find MLE.

The maximum sampling interval should satisfy

$$\max\left(|\theta_m - \theta_n|\right) < s\frac{\lambda}{Nd_0} \tag{10}$$

We can match the steering vectors corresponding to each angle in Ω with $\mathbf{x}$. Thus, MLE can be described as

$$\widehat{\theta} = \arg\max_{\theta}\left(\left|\left(\mathbf{a}(\theta)\right)^{\mathrm{H}} \cdot \mathbf{x}\right|\right), \quad \theta \in \Omega \tag{11}$$

In order to achieve better performance of accuracy, the sampling interval should be narrow enough, which leads to much more computational complexity by using (11). To solve (11), compressed sensing-based algorithms, such as OMP and SBL methods, are equivalent to MLE.

3. HIERARCHICAL MATCHING METHOD FOR MLE

The reason for the high computational complexity of (11) is that it requires matching each element of the steering vector set with the received data one by one. The essence of the binary tree hierarchical matching method is to achieve the process of finding the most matching element through hierarchy. Firstly, we need to determine whether the most matching element is on the left or right side by the following significant factors

$$F_{\text{left}} = \sum_{\theta\in\left(-\frac{\pi}{2},0\right)} \left|\left(\mathbf{a}(\theta_k)\right)^{\mathrm{H}} \cdot \mathbf{x}\right| \tag{12}$$

$$F_{\text{right}} = \sum_{\theta\in\left(0,\frac{\pi}{2}\right)} \left|\left(\mathbf{a}(\theta_k)\right)^{\mathrm{H}} \cdot \mathbf{x}\right| \tag{13}$$

If the significant factor (12) is larger than (13), then the next step is to focus on the left side of steering vector set; otherwise, we will choose the right one. Figure 2 denotes the path to find MLE, within binary tree hierarchical matching method. Consequently, the total number of levels in the hierarchy should be calculated in this way

$$L = \log_2 M \tag{14}$$

According to (12) and (13), the computational complexity has not decreased. On the contrary, the hierarchy structure increases more computational complexity. Therefore, we propose a modified significance factor expression

$$\overline{F}^{l}_{\text{left}} = \sum_{\theta\in LF(\Omega_l)} \left(\mathbf{x}\cdot\left(\mathbf{a}\left(\theta\right)\right)^{\mathrm{H}}\right)\left(\left(\mathbf{a}\left(\theta\right)\right)^{\mathrm{H}}\cdot\mathbf{x}\right) = \mathbf{x}^{\mathrm{H}}\mathbf{G}^{l}_{\text{left}}\mathbf{x} \tag{15}$$

$$\overline{F}^{l}_{\text{right}} = \sum_{\theta\in RT(\Omega_l)} \left(\mathbf{x}\cdot\left(\mathbf{a}\left(\theta\right)\right)^{\mathrm{H}}\right)\left(\mathbf{a}\left(\theta\right)^{\mathrm{H}}\cdot\mathbf{x}\right) = \mathbf{x}^{\mathrm{H}}\mathbf{G}^{l}_{\text{right}}\mathbf{x} \tag{16}$$

where,

$$\mathbf{G}^{l}_{\text{left}} = \sum_{\theta\in LF(\Omega_l)} \mathbf{a}\left(\theta\right)\mathbf{a}\left(\theta\right)^{\mathrm{H}} \tag{17}$$

$$\mathbf{G}^{l}_{\text{right}} = \sum_{\theta\in RT(\Omega_l)} \mathbf{a}\left(\theta\right)\mathbf{a}\left(\theta\right)^{\mathrm{H}} \tag{18}$$

in which, we consider using $LF(\cdot)$ and $RT(\cdot)$ to represent the 'left hand' and 'right hand' subsets of the set, respectively. Thus, $\mathbf{G}^{l}_{\text{left}}$ and $\mathbf{G}^{l}_{\text{right}}$ denote 'left hand' and 'right hand' characteristic matrix, respectively.

$\mathbf{G}^{l}_{\text{left}}$ and $\mathbf{G}^{l}_{\text{right}}$ are square matrices with dimensions $N \times N$. As the elements number of Ω_l decreases, the difference between the steering vectors also decreases, leading to the rank of $\mathbf{G}^{l}_{\text{left}}$ and $\mathbf{G}^{l}_{\text{right}}$ becoming smaller. It is easy to conclude that $\mathbf{G}^{l}_{\text{left}}$ and $\mathbf{G}^{l}_{\text{right}}$ satisfy conjugate symmetry. Consequently, both of them can be eigen-decomposed and arranged in descending order of eigenvalues

$$\mathbf{G}_{\text{left}}^{l} \approx \sum_{k \leq c_l} \lambda_{kl} \mathbf{u}_{kl} \mathbf{u}_{kl}^{\text{H}} \tag{19}$$

$$\mathbf{G}_{\text{right}}^{l} \approx \sum_{k \leq c_l} \chi_{kl} \mathbf{v}_{kl} \mathbf{v}_{kl}^{\text{H}} \tag{20}$$

where λ_{kl} is the k-th eigenvalue of $\mathbf{G}_{\text{left}}^{l}$ in l-th level and satisfies $\lambda_{kl} \geq \lambda_{(k+1)l}$. $\mathbf{u}_{kl}$ is the eigenvector corresponding to λ_{kl}.

Similarly, χ_{kl} is the k-th eigenvalue of $\mathbf{G}_{\text{right}}^{l}$ in l-th level and satisfies $\chi_{kl} \geq \chi_{(k+1)l}$. $\mathbf{v}_{kl}$ is the eigenvector corresponding to χ_{kl}. c_l is the cutdown number in l-th level. The eigenvalues λ_{kl} and χ_{kl}, as well as eigenvectors $\mathbf{u}_{kl}$ and $\mathbf{v}_{kl}$, can be pre-calculated and stored in real-time radar signal processor. Thus, (15)) and (16)) can be expressed by the formula

$$\overline{F}_{\text{left}}^{l} \approx \sum_{k \leq c_l} \lambda_{kl} |\mathbf{x}^{\text{H}} \mathbf{u}_{kl}|^2 \tag{21}$$

$$\overline{F}_{\text{right}}^{l} \approx \sum_{k \leq c_l} \chi_{kl} |\mathbf{x}^{\text{H}} \mathbf{v}_{kl}|^2 \tag{22}$$

As mentioned above, the rank of $\mathbf{G}_{\text{left}}^{l}$ and $\mathbf{G}_{\text{right}}^{l}$ decreases as the level update. Therefor, the cutdown number satisfies

$$c_1 > c_2 > \ldots > c_L \tag{23}$$

We apply the following descent strategy for c_l

$$c_{l+1} = \text{ceil}(\alpha c_l) \tag{24}$$

α is the decay factor that satisfies: $0 < \alpha < 1$, and ceil( ) is the upward rounding function. As the level-l approaches $\log_2 M$, c_l becomes less and less. And, at the bottom level ($L = \log_2 M$), c_L can be set as 1. Under these circumstances, $\mathbf{G}_{\text{left}}^{l}$ and $\mathbf{G}_{\text{right}}^{l}$ are single 'left steering vector' and 'right steering vector', respectively. Figure 3 illustrates the flowchart of the binary tree hierarchical matching method.

Algorithm Binary tree hierarchical matching method

Initialization:
Searching set: $\mathbf{A}_{\text{global}}$
Get left sub-set: $\mathbf{A}_{\text{left}}$ from $\mathbf{A}_{\text{global}}$
Get right sub-set: $\mathbf{A}_{\text{right}}$ from $\mathbf{A}_{\text{global}}$
$L = \log_2(M)$
$l = 1$
while ($l \neq L$) **do**
  Calculate $\overline{F}_{\text{left}}$ and $\overline{F}_{\text{right}}$ separately, based on $\mathbf{A}_{\text{left}}$ and $\mathbf{A}_{\text{right}}$, by pre-stored the eigenvalues λ_{kl}, χ_{kl} and the eigenvectors $\mathbf{u}_{kl}$ and $\mathbf{v}_{kl}$;
  if $\overline{F}_{\text{left}} > \overline{F}_{\text{right}}$ **then**
    $\mathbf{A}_{\text{global}} = \mathbf{A}_{\text{left}}$;
  else
    $\mathbf{A}_{\text{global}} = \mathbf{A}_{\text{right}}$;
  end if
  Update $\mathbf{A}_{\text{left}}$ and $\mathbf{A}_{\text{right}}$ based on $\mathbf{A}_{\text{global}}$;
  $l{+}{+}$;
end while

FIGURE 3. The flowchart of the algorithm.

TABLE 1. Comparison of computational complexity of two algorithms.

	direct matching	hierarchical matching
inner product	M	$\leq c_1 \log_2 M$
complex multiplication	NM	$\leq c_1 N \log_2 M$
complex addition	$(N-1)M$	$\leq c_1 (N-1) \log_2 M$
comparative operation	$M-1$	$\log_2 M$

4. ANALYSIS AND SIMULATION RESULTS OF COMPUTATIONAL COMPLEXITY

In this section, we compare and analyze the computational complexity of the direct matching MLE algorithm and the hierarchical matching MLE algorithm. In both algorithms, complex vector inner product and comparison operations are required, where complex vector inner product can be decomposed into complex multiplication and complex addition operations. Table 1 compares the computational complexity of two algorithms, which shows that the hierarchical matching MLE algorithm has significantly lower computational complexity. Compared to direct matching MLE, hierarchical matching MLE reduces the computational complexity from $O(M)$ to $O(\log M)$.

Taking a 64-unit uniform linear array as an example, we compare the calculation time of the direct MLE search algorithm and binary tree MLE search algorithm, where the element spacing of the uniform linear array is $\lambda/2$, and the signal to noise ratio (SNR) of single receiving antenna is 0 dB. The decay factor α is 0.618, which is golden section. Although the case of a uniform linear array is used in this paper to demonstrate the algorithm process, in fact, nonuniform arrays, which are often used in tracking radars, and digital array feed technologies are also applicable to this algorithm. Especially for those digital array feed systems where the steering vector cannot be analytically given, without an analytically expressed steering vector, it is impossible to use algorithms such as Newton's iterative method for parameter estimation, while direct matching method is still applicable.

The code development environment is octave on a 64 bit Windows 7 operating system, and the computer's CPU is Intel (R) Core (TM) i7-7700@3.6 GHz, installed with 32 GB of system memory. Table 2 illustrates the comparison of computation time consumption between the two algorithms. It is obvious that the time consumption of direct matching MLE algorithm

TABLE 2. Comparison of time consumptions of two algorithms.

M	direct matching (ms)	$\log_2 M$	hierarchical matching (ms)
1024	3.066	10	3.695
2048	5.555	11	4.884
4096	7.116	12	4.831
8192	11.73	13	3.435
16384	21.28	14	3.397
32768	41.46	15	3.511
65536	83.01	16	3.736

increases significantly with the increase of M. As the time required for algorithm function calls is significantly longer than the computation time, the time consumption of the hierarchical matching MLE algorithm remains.

5. ANALYSIS SIMULATION RESULTS OF ACCURACY PERFORMANCE

The signal received in uniform linear array can be denoted as follows:

$$x(n)=S(\theta,n)+w(n)=\kappa\exp\left(\mathrm{j}\frac{2\pi}{\lambda}nd_0\sin\theta\right)+w(n) \quad (25)$$

where n is the antenna index, and $n=0,1,\ldots,N-1$. $w(n)\sim CN(0,\sigma^2)$. According to statistical signal processing [24], the CRLB of parameter estimation under the condition of Gaussian noise is:

$$\operatorname{var}\left(\widehat{\theta}\right)\geq\frac{\sigma^2}{\sum_{n=0}^{N-1}\left(\frac{\partial S(\theta,n)}{\partial\theta}\right)\left(\frac{\partial S(\theta,n)}{\partial\theta}\right)^{\mathrm{H}}} \quad (26)$$

Substituting (25) into (26), the CRLB of angle of estimate can be expressed:

$$\operatorname{var}\left(\widehat{\theta}\right)\geq\frac{\sigma^2}{\left(\kappa\frac{2\pi d_0\cos\theta}{\lambda}\right)^2\sum_{n=0}^{N-1}n^2} \quad (27)$$

We have

$$\operatorname{var}\left(\widehat{\theta}\right)\geq\frac{3\lambda^2}{2\left(\pi d_0\cos\theta\right)^2\left(2N^3-3N^2+N\right)\mathrm{SNR}} \quad (28)$$

where SNR is

$$\mathrm{SNR}=\frac{\kappa^2}{\sigma^2} \quad (29)$$

Monte Carlo method was used to evaluate the accuracy performance of algorithms, comparing the direct matching MLE method with hierarchical matching MLE method. The SNR level of the unit antenna is set from -10 to 8 dB in the step of 1 dB. Simulate 1000 times at each SNR level and calculate the root mean square(RMS) of the calculation error.

The performance of computational accuracy is illustrated by Figure 4. Under the condition of high SNR level, the performance of both the direct matching algorithm and hierarchical matching algorithm can approach the CRLB. It is also obvious that the direct matching method can achieve the above approximation process faster than the hierarchical matching method, and the SNR gap between the two is approximately 4 dB. The reason is that the direct matching method matches all the spaces one by one, while the hierarchical matching algorithm is realized through multiple-subspace matching. In the case of low SNR, there is a certain probability of making mistakes in the above subspace matching. Although the performance of hierarchical matching method is somewhat degraded in the case of low SNR, its calculation speed is much faster than that of direct matching method.

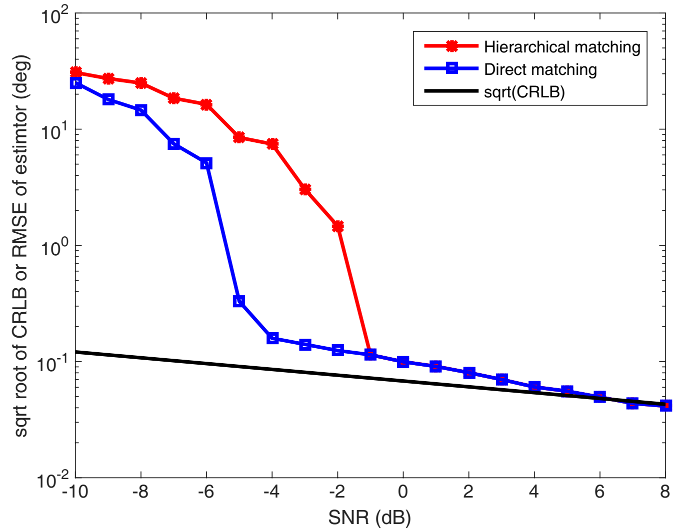

FIGURE 4. Performance of computational accuracy.

6. CONCLUSION

Hierarchical matching algorithm for digital tracking radar has been proposed in this paper, based on decomposition and approximate expression of left and right hand characteristics matrices. According to the theoretical analysis and simulation results, hierarchical matching method can utilize MLE with much less computational complexity with little accuracy performance loss. Compared to direct matching MLE, hierarchical matching MLE reduces the computational complexity from $O(M)$ to $O(\log M)$. The maximum SNR gap between two algorithms is approximately 4 dB. Due to the use of eigenvalues and eigenvectors from the feature matrix for approximate representation, this method can also be extended to other situations where MLE is employed.

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