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Admittance and Impedance Relations at Moving Boundaries

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ABSTRACT: Admittance and impedance (Leontovich) matching conditions at the boundary of a good conductor find widespread usage in the formulation and (numerical) solution of electromagnetic problems. Starting with the known relationships at a stationary interface, we derive manifestly covariant admittance and impedance relations in a flat space-time for a conducting body which moves with uniform velocity in free space. Explicit formulas (in the ordinary space, that is) are given for both isotropic and anisotropic conductors. Under the same hypotheses, we also derive, at the conducting interface, the surface density of four-force by means of the normal component of the relevant energy-momentum tensor. The low-velocity limit of the formulas is also presented because it is of particular interest for practical applications. Moreover, since the covariant admittance and impedance relations as well as the matching condition of the energy-momentum tensor require the unitary four-vector perpendicular to a surface in motion, we outline, in the appendices, the derivation of unitary four-vectors tangential to a hyper-line and perpendicular to a hyper-surface in the Lorentz space.

1. INTRODUCTION

Good conductors (i.e., materials characterized macroscopically by a large bulk conductivity) are amenable to an approximate, though accurate, description in terms of equivalent matching conditions which relate the electromagnetic entities on the boundary by means of a (possibly dyadic) surface admittance or impedance [1–3, 4, Section 3.4.2]. Such an approach is feasible because, owing to the high value of conductivity, the electromagnetic field in the conductor is exponentially damped and hence, remains essentially non-null only right under the boundary in a thin layer whose thickness is the order of the skin depth. In fact, the approximation amounts not so much to neglecting the exponential tail of the field in the bulk of the material as to assuming a (possibly infinite) planar boundary for the derivation of the equivalent surface admittance and impedance [5, Example 7.2, 6, Section 9.1.3, 7, Section 2.9]. Consequently, the model may be less accurate in the presence of and close to sharp bends, corners, and tips of the actual boundary.

This limitation notwithstanding, there is abundant evidence in the literature that admittance and impedance boundary conditions have been and can be profitably used for the formulation of electromagnetic problems, including wave radiation and scattering, waveguides, transmission lines, and transformers, in tandem with mainstream solution strategies such as the finite-difference time-domain method [8], boundary element method [9, 10], integral equations and the Method of Moments [11, Section 13.2.6, 12], finite element method [13], and even combination thereof [14].

Whatever the strategy of choice, the main advantage of a surface description (over the statement of the Ohm law in the bulk) is that the spatial region occupied by the conducting body can be excluded from the computational domain. As a result, the solution time is reduced accordingly, and the delicate calculation of the comparatively small fields inside the conductor is avoided altogether. While these features can be beneficial also for the numerical solution of electromagnetic problems which involve good conductors in motion, still, admittance and impedance relations at a moving interface do not appear to have been derived. Further, a recently published resource [15, Sections 2.3–2.4] seems to indicate that the more-than-a-century-old electrodynamics of moving media is far from being obsolete.

For all these reasons, we undertake the task of filling the gap, and by applying Minkowski's powerful frame-hopping technique and the principle of Lorentz covariance (e.g., [16, Chapter 8]), we derive *manifestly covariant* admittance and impedance relationships (Section 2) on the boundary of a conducting body in uniform motion in free space, under the assumption that the surface admittance and impedance of the body at rest are known and independent of time. We consider both isotropic and anisotropic conductors described, respectively, by scalar and dyadic surface admittances and impedances. Under the same hypotheses, in Section 3 we determine the power lost in the conductor and the Lorentz force on the boundary per unit area by means of the matching conditions obeyed by the relevant energy-momentum tensor. Also, we give the low-velocity approximation of the formulas, which should come in handy for the numerical solution of practical engineering problems, e.g., plane-wave scattering from moving metallic targets by means of a modified electric-field integral equation.

As is the case with bulk constitutive relationships (e.g., [15, Section 2.4, 16, Section 8.3, 17, Section 4.2]), it is found that the motion has a substantial impact on the macroscopic description of the conducting interface in a few key aspects, namely,

(1) if the conductor is isotropic, the scalar surface admittance and impedance pass over into non-symmetric dyadics;

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(2) the equivalent surface currents (electric and magnetic) end up being driven by fields ($\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$) *and* flux densities ($\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{D}}$);
(3) effective charge densities appear on the surface in addition to the true ones already present when the body is stationary;
(4) both true and effective surface charge densities undergo distortion owing to the Lorentz contraction of the body.

As a matter of necessity, the formulas rely heavily on the notion of (ordinary) unit vectors tangential and normal to the material interface. These vectors, however, are distorted by the motion in a way that is far from obvious. Therefore, as a bonus, Appendix A introduces a unitary four-vector tangential to a regular hyper-line (and hence, to the hyper-surface whereon the line lies) whereas Appendix B reviews and extends the concept of unitary four-vector perpendicular to a hyper-surface. Lastly, in Appendix C, a decomposition of the metric tensor on a hyper-surface is given that is akin to the representation of the identity dyadic on a surface in the ordinary space.

Throughout the paper, the underlying manifold for the electromagnetic phenomena is the pseudo-Euclidean Lorentz space $\mathbb{L}^4$ (Minkowski's flat space-time) equipped with metric tensor $\mathbf{g}$ (signature $\{+,-,-,-\}$) and Levi-Civita pseudo-tensor $\boldsymbol{\epsilon}$, with the specific choice [18]

$$\epsilon^{\alpha\beta\eta\nu} := \begin{cases} 1/\sqrt{-g} & \alpha,\beta,\eta,\nu \text{ an even permutation of } 0,1,2,3 \\ -1/\sqrt{-g} & \alpha,\beta,\eta,\nu \text{ an odd permutation of } 0,1,2,3 \\ 0 & \text{any two indices coincide} \end{cases} \quad (1)$$

where $g := \det[g_{\alpha\beta}] < 0$ is the determinant of the matrix associated with $\mathbf{g}$.

Concerning notation, boldface Roman and cursive letters (e.g., $\mathbf{v}$, $\hat{\mathbf{n}}$, $\boldsymbol{\mathcal{E}}$) indicate three-dimensional vectors in the ordinary space; boldface letters with a bar (e.g., $\overline{\boldsymbol{\mathcal{I}}}$, $\overline{\boldsymbol{\mathcal{T}}}_{em}$, $\overline{\mathbf{Y}}$) denote dyadics; and boldface Sans Serif letters (e.g., $\mathbf{g}$, $\boldsymbol{\epsilon}$, M) denote tensors in $\mathbb{L}^4$. Einstein's summation convention is adopted, Greek indices range in the set $\{0,1,2,3\}$ with zero indicating the time component, and Latin indices range in the set $\{1,2,3\}$. Finally, $\mathbf{v}$ is the velocity of the body, c_0 the speed of light in vacuum, $\boldsymbol{\varsigma} := \mathbf{v}/c_0$ the vector speed parameter, and $\gamma := 1/(1-\varsigma^2)^{1/2}$ the Lorentz factor.

2. FIELDS AND FLUX DENSITIES

2.1. Isotropic Conductors

In an inertial reference frame K', where an isotropic conducting body is at rest and occupies a domain $V \subset \mathbb{R}^3$ (Figure 1), the admittance and impedance relationships on $S := \partial V$ read

$$\hat{\mathbf{n}}' \times \boldsymbol{\mathcal{H}}' = -Y_S \hat{\mathbf{n}}' \times (\hat{\mathbf{n}}' \times \boldsymbol{\mathcal{E}}') = Y_S(\overline{\boldsymbol{\mathcal{I}}} - \hat{\mathbf{n}}'\hat{\mathbf{n}}') \cdot \boldsymbol{\mathcal{E}}' \quad (2)$$

$$\boldsymbol{\mathcal{E}}' \times \hat{\mathbf{n}}' = -Z_S \hat{\mathbf{n}}' \times (\hat{\mathbf{n}}' \times \boldsymbol{\mathcal{H}}') = Z_S(\overline{\boldsymbol{\mathcal{I}}} - \hat{\mathbf{n}}'\hat{\mathbf{n}}') \cdot \boldsymbol{\mathcal{H}}' \quad (3)$$

where $Y_S > 0$ and $Z_S := 1/Y_S$ (physical dimension: $1/\Omega$ and Ω) are the scalar surface admittance and impedance, and $\overline{\boldsymbol{\mathcal{I}}}$ denotes the identity dyadic. We take the unit normal $\hat{\mathbf{n}}'$ on S positively directed outwards V (Figure 1) whereby the flux of the Poynting vector $\boldsymbol{\mathcal{S}}' := \boldsymbol{\mathcal{E}}' \times \boldsymbol{\mathcal{H}}'$ through S is negative, and

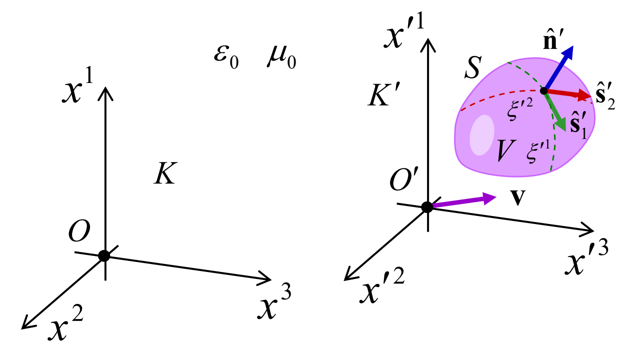

FIGURE 1. Inertial reference frames K and K' for the derivation of the admittance/impedance matching conditions at the material boundary S of a conducting body in uniform motion.

this occurrence signals that electromagnetic energy flows into the conductor and is lost to Joule heating.

In order to extend (2) and (3) to the case of a moving body, we formulate them in covariant form. In this regard, we observe that (2) has the structure of a *surface* Ohm's law of sorts (cf. [19–21]) where $\hat{\mathbf{n}}' \times \boldsymbol{\mathcal{H}}'$ plays the role of a surface current density which is related to the tangential electric field through a dyadic, even if the conductor is isotropic. Thus, we can expect $\hat{\mathbf{n}}' \times \boldsymbol{\mathcal{H}}'$ to be the spatial part of an equivalent surface four-current (cf. [17, Section 3.4])

$$J_S'^{\eta} := n'_\nu M'^{\nu\eta} \quad (4)$$

where

$$[n'_\nu] = \begin{pmatrix} 0 \\ \hat{\mathbf{n}}' \end{pmatrix} \quad (5)$$

is the (matrix representation of the) 'unitary' normal four-vector in K' (Appendix B) and

$$[M'^{\nu\eta}] := \begin{pmatrix} 0 & -c_0\boldsymbol{\mathcal{D}}' \\ c_0\boldsymbol{\mathcal{D}}' & \overline{\boldsymbol{\mathcal{I}}} \times \boldsymbol{\mathcal{H}}' \end{pmatrix} \quad (6)$$

is the (matrix representation of the) Maxwell tensor M'. Moreover, $J_S'^{0} := \hat{\mathbf{n}}' \cdot \boldsymbol{\mathcal{D}}' c_0$ is proportional to charge (at rest) which is distributed on S with surface density ϱ'_S. As a result, we find

$$n'_\nu M'^{\nu\eta} = \frac{1}{c_0} Y_S U'^{\eta\alpha} N'_{\alpha\beta} V'^{\beta} + \varrho'_S V'^{\eta} \quad (7)$$

where

$$[N'_{\alpha\beta}] := \begin{pmatrix} 0 & \boldsymbol{\mathcal{E}}' \\ -\boldsymbol{\mathcal{E}}' & c_0\overline{\boldsymbol{\mathcal{I}}} \times \boldsymbol{\mathcal{B}}' \end{pmatrix} \quad (8)$$

is the (matrix representation of the) Faraday tensor N', and V'^{α} is the four-velocity in K'. The symmetric tensor field

$$U'^{\eta\alpha} := g^{\eta\alpha} + n'^{\eta} n'^{\alpha} - \frac{1}{c_0^2} V'^{\eta} V'^{\alpha} \quad (9)$$

with matrix representation

$$[U'^{\eta\alpha}] = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{n}}'\hat{\mathbf{n}}' - \overline{\boldsymbol{\mathcal{I}}} \end{pmatrix} \quad (10)$$

is charged with the task of extracting the part of the four-vector $N'_{\alpha\beta}V'^{\beta}$ that is tangential to the hyper-surface $(\mathbb{R}\times S)\subset\mathbb{L}^4$.

Actually, on the grounds of identity (C1), we can conclude that $U'^{\eta\alpha}$ behaves as the restriction of the metric $g^{\eta\alpha}$ for tensors on $\mathbb{R}\times S$. The following interesting properties of $U'^{\eta\alpha}$

$$\begin{aligned} &n'_\eta U'^{\eta\alpha}=\frac{1}{c_0}V'_\eta U'^{\eta\alpha}=0, \quad U'^{\eta\alpha}U'^{\cdot\beta}_{\alpha}=U'^{\eta\beta} \\ &U'^{\eta\alpha}g_{\eta\alpha}=2, \qquad\qquad U'^{\eta\alpha}U'_{\eta\alpha}=2 \end{aligned} \tag{11}$$

are proved by inspection on account of (5). The first one, in particular, implies that n'_η and V'_η are eigenvectors associated with two null eigenvalues of $U'^{\eta\alpha}$, whereby the matrix $[U'^{\eta\alpha}]$ does not possess an ordinary inverse. The other two eigenvectors, associated with a double unitary eigenvalue, are any two linearly independent four-vectors tangential to $\mathbb{R}\times S$ or, equivalently, perpendicular to both n'^η and V'^η (Appendices A and B).

By virtue of the principle of Lorentz covariance, we get

$$\begin{aligned} n_\nu M^{\nu\eta} \;=\; &\frac{Y_S}{c_0}\left(g^{\eta\alpha}+n^\eta n^\alpha-\frac{V^\eta V^\alpha}{c_0^2}\right)N_{\alpha\beta}V^\beta \\ &+\varrho'_S V^\eta \end{aligned} \tag{12}$$

the desired admittance boundary conditions in the frame K. Of the four scalar equations represented by (7) and (12) only three are independent because

$$\begin{aligned} n_\eta n_\nu M^{\nu\eta} \;=\; &\frac{Y_S}{c_0}n_\eta\left(g^{\eta\alpha}+n^\eta n^\alpha-\frac{V^\eta V^\alpha}{c_0^2}\right)N_{\alpha\beta}V^\beta \\ &+\varrho'_S n_\eta V^\eta=0 \end{aligned} \tag{13}$$

in view of the antisymmetry of $M^{\nu\eta}$, the first of properties (11), and definition (B7) of the unit normal four-vector on the boundary of a surface in motion with velocity $\mathbf{v}$. The matrix representation of the surface identity tensor $U^{\alpha\beta}$ in K reads

$$[U^{\alpha\beta}] = \begin{pmatrix} \dfrac{1}{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}-\gamma^2 & \dfrac{\hat{\mathbf{n}}\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma}}{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}-\gamma^2\boldsymbol{\varsigma} \\ \dfrac{\hat{\mathbf{n}}\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma}}{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}-\gamma^2\boldsymbol{\varsigma} & \dfrac{\hat{\mathbf{n}}\hat{\mathbf{n}}}{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}-\overline{\boldsymbol{\mathcal{I}}}-\gamma^2\boldsymbol{\varsigma}\boldsymbol{\varsigma} \end{pmatrix} \tag{14}$$

whence we obtain the asymptotic expansion

$$\begin{aligned} [U^{\alpha\beta}] = &\begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{n}}\hat{\mathbf{n}}-\overline{\boldsymbol{\mathcal{I}}} \end{pmatrix} + \begin{pmatrix} 0 & \hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\boldsymbol{\varsigma}) \\ \hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\boldsymbol{\varsigma}) & \overline{\boldsymbol{\mathcal{O}}} \end{pmatrix} \\ &+\begin{pmatrix} |\hat{\mathbf{n}}\times\boldsymbol{\varsigma}|^2 & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{n}}\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2-\boldsymbol{\varsigma}\boldsymbol{\varsigma} \end{pmatrix} + o(\varsigma^2), \quad \varsigma\to 0 \end{aligned} \tag{15}$$

which shows that the relativistic effects due to the motion of S are first order in $\hat{\mathbf{n}}\times\boldsymbol{\varsigma}$.

From (12), we derive the explicit form of the admittance relationships at the boundary of a conductor in motion as

$$\begin{aligned} \hat{\mathbf{n}}\cdot\boldsymbol{\mathcal{D}} \;=\; &\gamma Y_S\mathbf{v}\cdot\frac{\overline{\boldsymbol{\mathcal{I}}}-\hat{\mathbf{n}}\hat{\mathbf{n}}}{c_0^2\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}}\cdot(\boldsymbol{\mathcal{E}}+\mathbf{v}\times\boldsymbol{\mathcal{B}}) \\ &+\gamma\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\,\varrho'_S, \quad \mathbf{r}\in S \end{aligned} \tag{16}$$

$$\begin{aligned} \hat{\mathbf{n}}\cdot\mathbf{v}\boldsymbol{\mathcal{D}}+\hat{\mathbf{n}}\times\boldsymbol{\mathcal{H}} \;=\; &\gamma Y_S\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2} \\ &\left[\overline{\boldsymbol{\mathcal{I}}}-\hat{\mathbf{n}}\hat{\mathbf{n}}\cdot\frac{\overline{\boldsymbol{\mathcal{I}}}-\boldsymbol{\varsigma}\boldsymbol{\varsigma}}{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\right] \\ &\cdot(\boldsymbol{\mathcal{E}}+\mathbf{v}\times\boldsymbol{\mathcal{B}}) \\ &+\gamma\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\,\varrho'_S\mathbf{v}, \quad \mathbf{r}\in S \end{aligned} \tag{17}$$

where the extra factor $\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}$ that multiplies ϱ'_S accounts for the distortion of the charge density that is caused by the Lorentz contraction of S in the direction of $\mathbf{v}$. Besides, the motion is responsible for the appearance of an effective surface charge density in the right member of (16).

The surface dyadic field

$$\overline{\boldsymbol{\mathcal{U}}}(\mathbf{r}):=\overline{\boldsymbol{\mathcal{I}}}-\hat{\mathbf{n}}\hat{\mathbf{n}}\cdot\frac{\overline{\boldsymbol{\mathcal{I}}}-\boldsymbol{\varsigma}\boldsymbol{\varsigma}}{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}, \quad \mathbf{r}\in S \tag{18}$$

has the properties

$$\begin{aligned} &(\hat{\mathbf{n}}-\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma}\boldsymbol{\varsigma})\cdot\overline{\boldsymbol{\mathcal{U}}}=0, \quad \overline{\boldsymbol{\mathcal{U}}}\cdot\hat{\mathbf{n}}=0 \\ &\overline{\boldsymbol{\mathcal{U}}}:\overline{\boldsymbol{\mathcal{I}}}=2, \quad \overline{\boldsymbol{\mathcal{U}}}:\overline{\boldsymbol{\mathcal{U}}}=2+\frac{(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2|\hat{\mathbf{n}}\times\boldsymbol{\varsigma}|^2}{[1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2]^2} \end{aligned} \tag{19}$$

which imply that $\hat{\mathbf{n}}-\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma}\boldsymbol{\varsigma}$ and $\hat{\mathbf{n}}$ are left and right eigenvectors associated with a null eigenvalue. The other left and right eigenvectors of $\overline{\boldsymbol{\mathcal{U}}}$ are $\hat{\mathbf{n}}\times\boldsymbol{\varsigma}$, $\hat{\mathbf{n}}\times(\hat{\mathbf{n}}\times\boldsymbol{\varsigma})$, and $\hat{\mathbf{n}}\times\boldsymbol{\varsigma}$, $\hat{\mathbf{n}}\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma}-\gamma^2[1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2]\boldsymbol{\varsigma}$, respectively.

With a little algebra, we obtain the alternative expression

$$\begin{aligned} \hat{\mathbf{n}}\times(\boldsymbol{\mathcal{H}}+\boldsymbol{\mathcal{D}}\times\mathbf{v}) \;=\; &\gamma Y_S\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\left(\overline{\boldsymbol{\mathcal{I}}}-\boldsymbol{\varsigma}\boldsymbol{\varsigma}\right) \\ &\cdot\left[\overline{\boldsymbol{\mathcal{I}}}-\hat{\mathbf{n}}\hat{\mathbf{n}}\cdot\frac{\overline{\boldsymbol{\mathcal{I}}}-\boldsymbol{\varsigma}\boldsymbol{\varsigma}}{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\right] \\ &\cdot(\boldsymbol{\mathcal{E}}+\mathbf{v}\times\boldsymbol{\mathcal{B}}), \quad \mathbf{r}\in S \end{aligned} \tag{20}$$

by combining (16) and (17) to eliminate ϱ'_S. Compared to (2), (17) relates an effective electric field to an effective surface electric current through a dyadic field which, as anticipated, is not symmetric because $\overline{\boldsymbol{\mathcal{U}}}(\mathbf{r})\neq[\overline{\boldsymbol{\mathcal{U}}}(\mathbf{r})]^T$, although $\overline{\boldsymbol{\mathcal{I}}}-\boldsymbol{\varsigma}\boldsymbol{\varsigma}$ and $\overline{\boldsymbol{\mathcal{U}}}$ commute. By contrast, it can be checked that the dyadic field in the right member of (20) is symmetric. Finally, in most practical applications, $|\mathbf{v}|$ happens to be much smaller than c_0, and one can conveniently use

$$\hat{\mathbf{n}}\cdot\boldsymbol{\mathcal{D}}=\varrho'_S \tag{21}$$

$$\hat{\mathbf{n}}\cdot\mathbf{v}\boldsymbol{\mathcal{D}}+\hat{\mathbf{n}}\times\boldsymbol{\mathcal{H}}=Y_S\left(\overline{\boldsymbol{\mathcal{I}}}-\hat{\mathbf{n}}\hat{\mathbf{n}}\right)\cdot(\boldsymbol{\mathcal{E}}+\mathbf{v}\times\boldsymbol{\mathcal{B}})+\varrho'_S\mathbf{v} \tag{22}$$

$$\hat{\mathbf{n}}\times(\boldsymbol{\mathcal{H}}+\boldsymbol{\mathcal{D}}\times\mathbf{v})=Y_S\left(\overline{\boldsymbol{\mathcal{I}}}-\hat{\mathbf{n}}\hat{\mathbf{n}}\right)\cdot(\boldsymbol{\mathcal{E}}+\mathbf{v}\times\boldsymbol{\mathcal{B}}) \tag{23}$$

to the first order in $\mathbf{v}/c_0$.

The principle of duality [5, Section 6.7] — applied to (16) and (17) with the specific substitution $Y_S\to -Z_S$ — allows writing the corresponding impedance relationships right away, though it is instructive to start over with (3). To find our way around, we may regard $\boldsymbol{\mathcal{E}}'\times\hat{\mathbf{n}}'$ as a surface magnetic current density related to the tangential magnetic field by means of a dyadic. Therefore, $\boldsymbol{\mathcal{E}}'\times\hat{\mathbf{n}}'$ should be the spatial part of an equivalent magnetic surface four-current, say,

$$\star J'^{\eta}_{MS} := n'_\nu \star N'^{\nu\eta} = \frac{1}{2}\epsilon^{\alpha\beta\nu\eta} n'_\nu N'_{\alpha\beta} \tag{24}$$

where $\star N'^{\nu\eta}$ denotes the Hodge dual of the Faraday tensor in (8), and the normal four-vector in K' is given by (5). Further, $\star J'^{0}_{MS} := \hat{\mathbf{n}}'\cdot\boldsymbol{\mathcal{B}}' c_0$ is proportional to magnetic charge (at rest) which is concentrated on S with surface density ϱ'_{MS}. Accordingly, we find

$$n'_\nu \star N'^{\nu\eta} = \frac{Z_S}{c_0}\left(g^{\eta\alpha} + n'^{\eta} n'^{\alpha} - \frac{V'^{\eta}V'^{\alpha}}{c_0^2}\right)\star M'_{\alpha\beta}V'^{\beta} + \varrho'_{MS}V'^{\eta} \tag{25}$$

where

$$\star M'_{\alpha\beta} := \frac{1}{2}\epsilon_{\eta\nu\alpha\beta}M'^{\eta\nu} \tag{26}$$

is the Hodge dual of the Maxwell tensor given in (6).

Then, by virtue of the principle of covariance, we get

$$n_\nu \star N^{\nu\eta} = \frac{Z_S}{c_0}\left(g^{\eta\alpha} + n^{\eta} n^{\alpha} - \frac{V^{\eta}V^{\alpha}}{c_0^2}\right)\star M_{\alpha\beta}V^{\beta} + \varrho'_{MS}V^{\eta} \tag{27}$$

the impedance relationship in the frame K. Also in (27), only three scalar equations out of four are independent. At this stage, we are able to recognize that the duality transformations

$$\begin{aligned} M^{\nu\eta} &\longrightarrow -(\star N^{\nu\eta}), \quad & N_{\alpha\beta} &\longrightarrow \star M_{\alpha\beta} \\ \varrho'_S &\longrightarrow -\varrho'_{MS}, & Y_S &\longrightarrow -Z_S \end{aligned} \tag{28}$$

turn (12) into (27) and vice-versa.

From (27), we obtain the explicit form of the impedance relationships at the boundary of a conductor in motion as

$$\hat{\mathbf{n}}\cdot\boldsymbol{\mathcal{B}} = \gamma Z_S \mathbf{v}\cdot\frac{\overline{\boldsymbol{\mathcal{I}}} - \hat{\mathbf{n}}\hat{\mathbf{n}}}{c_0^2\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}}\cdot(\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{D}}\times\mathbf{v}) + \gamma\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\,\varrho'_{MS}, \quad \mathbf{r}\in S \tag{29}$$

$$\hat{\mathbf{n}}\cdot\mathbf{v}\boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{E}}\times\hat{\mathbf{n}} = \gamma Z_S\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\left[\overline{\boldsymbol{\mathcal{I}}} - \hat{\mathbf{n}}\hat{\mathbf{n}}\cdot\frac{\overline{\boldsymbol{\mathcal{I}}} - \boldsymbol{\varsigma}\boldsymbol{\varsigma}}{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\right]\cdot(\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{D}}\times\mathbf{v}) + \gamma\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\,\varrho'_{MS}\mathbf{v}, \quad \mathbf{r}\in S \tag{30}$$

with the alternative expression

$$(\boldsymbol{\mathcal{E}} + \mathbf{v}\times\boldsymbol{\mathcal{B}})\times\hat{\mathbf{n}} = \gamma Z_S\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\left(\overline{\boldsymbol{\mathcal{I}}} - \boldsymbol{\varsigma}\boldsymbol{\varsigma}\right)\cdot\left[\overline{\boldsymbol{\mathcal{I}}} - \hat{\mathbf{n}}\hat{\mathbf{n}}\cdot\frac{\overline{\boldsymbol{\mathcal{I}}} - \boldsymbol{\varsigma}\boldsymbol{\varsigma}}{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\right]\cdot(\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{D}}\times\mathbf{v}), \quad \mathbf{r}\in S \tag{31}$$

derived from (29) and (30) by eliminating the 'true' magnetic charge density. Asymptotically, we have

$$\hat{\mathbf{n}}\cdot\boldsymbol{\mathcal{B}} = \varrho'_{MS} \tag{32}$$

$$\hat{\mathbf{n}}\cdot\mathbf{v}\boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{E}}\times\hat{\mathbf{n}} = \varrho'_{MS}\mathbf{v} + Z_S\left(\overline{\boldsymbol{\mathcal{I}}} - \hat{\mathbf{n}}\hat{\mathbf{n}}\right)\cdot(\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{D}}\times\mathbf{v}) \tag{33}$$

$$(\boldsymbol{\mathcal{E}} + \mathbf{v}\times\boldsymbol{\mathcal{B}})\times\hat{\mathbf{n}} = Z_S\left(\overline{\boldsymbol{\mathcal{I}}} - \hat{\mathbf{n}}\hat{\mathbf{n}}\right)\cdot(\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{D}}\times\mathbf{v}) \tag{34}$$

to the first order in $\mathbf{v}/c_0$.

At the cost of a little algebra, we can formulate (27) alternatively in terms of the tensors **N** and **M** instead of the dual entities given in (24) and (26). By starting with

$$\frac{1}{2}\epsilon^{\iota\kappa\nu\eta}n_\nu N_{\iota\kappa} = \frac{1}{2}\epsilon^{\nu\iota\kappa\eta}n_{[\nu}N_{\iota\kappa]} = \varrho'_{MS}V^{\eta} + \frac{Z_S}{2c_0}\left(g^{\eta\alpha} + n^{\eta}n^{\alpha} - \frac{V^{\eta}V^{\alpha}}{c_0^2}\right)\epsilon_{\iota\kappa\alpha\beta}M^{\iota\kappa}V^{\beta} \tag{35}$$

contracting throughout with $-\epsilon_{\lambda\xi\zeta\eta}$, and using the identity

$$-\epsilon_{\eta\lambda\xi\zeta}\epsilon^{\eta\nu\iota\kappa} = \begin{vmatrix}\delta^\nu_\lambda & \delta^\iota_\lambda & \delta^\kappa_\lambda \\ \delta^\nu_\xi & \delta^\iota_\xi & \delta^\kappa_\xi \\ \delta^\nu_\zeta & \delta^\iota_\zeta & \delta^\kappa_\zeta\end{vmatrix}, \quad \delta^\nu_\lambda := \begin{cases} 1, & \nu=\lambda \\ 0, & \nu\neq\lambda\end{cases} \tag{36}$$

we get

$$-\frac{1}{2}\epsilon_{\lambda\xi\zeta\eta}\epsilon^{\nu\iota\kappa\eta}n_{[\nu}N_{\iota\kappa]} = 3n_{[\lambda}N_{\xi\zeta]} = -\varrho'_{MS}\epsilon_{\lambda\xi\zeta\eta}V^{\eta} - \frac{Z_S}{2c_0}\epsilon_{\lambda\xi\zeta\eta}\left(g^{\eta\alpha} + n^{\eta}n^{\alpha} - \frac{V^{\eta}V^{\alpha}}{c_0^2}\right)\epsilon_{\alpha\iota\kappa\beta}M^{\iota\kappa}V^{\beta} \tag{37}$$

and finally

$$n_{[\lambda}N_{\xi\zeta]} = \frac{1}{3}\varrho'_{MS}\epsilon_{\eta\lambda\xi\zeta}V^{\eta} + \frac{Z_S}{3!\,c_0}\epsilon_{\eta\lambda\xi\zeta}\epsilon_{\alpha\iota\kappa\beta}\left(g^{\eta\alpha} + n^{\eta}n^{\alpha} - \frac{V^{\eta}V^{\alpha}}{c_0^2}\right)M^{\iota\kappa}V^{\beta} \tag{38}$$

where the rank-3 antisymmetric tensors in the right-hand side represent effective surface magnetic currents that arise from convection and conduction. Compared to (27), (38) is redundant because 40 scalar equations (out of the 64 ones it represents) are trivial identities, whereas the remaining 24 ones are instances of four equations, each being repeated six times.

2.2. Anisotropic Conductors

To extend the previous results to dyadic surface admittance and impedance for an anisotropic conducting boundary, we rewrite (2) and (3) as

$$\hat{\mathbf{n}}'\times\boldsymbol{\mathcal{H}}' = \overline{\boldsymbol{\mathcal{Y}}}_S\cdot\boldsymbol{\mathcal{E}}', \qquad \hat{\mathbf{n}}'\cdot\overline{\boldsymbol{\mathcal{Y}}}_S = \overline{\boldsymbol{\mathcal{Y}}}_S\cdot\hat{\mathbf{n}}' = 0 \tag{39}$$

$$\boldsymbol{\mathcal{E}}'\times\hat{\mathbf{n}}' = \overline{\boldsymbol{\mathcal{Z}}}_S\cdot\boldsymbol{\mathcal{H}}', \qquad \hat{\mathbf{n}}'\cdot\overline{\boldsymbol{\mathcal{Z}}}_S = \overline{\boldsymbol{\mathcal{Z}}}_S\cdot\hat{\mathbf{n}}' = 0 \tag{40}$$

where

$$\overline{\boldsymbol{\mathcal{Y}}}_S := \mathcal{Y}^{11}\hat{\mathbf{s}}'_1\hat{\mathbf{s}}'_1 + \mathcal{Y}^{12}\hat{\mathbf{s}}'_1\hat{\mathbf{s}}'_2 + \mathcal{Y}^{21}\hat{\mathbf{s}}'_2\hat{\mathbf{s}}'_1 + \mathcal{Y}^{22}\hat{\mathbf{s}}'_2\hat{\mathbf{s}}'_2 \tag{41}$$

$$\overline{\boldsymbol{\mathcal{Z}}}_S := \mathcal{Z}^{11}\hat{\mathbf{s}}_1'\hat{\mathbf{s}}_1' + \mathcal{Z}^{12}\hat{\mathbf{s}}_1'\hat{\mathbf{s}}_2' + \mathcal{Z}^{21}\hat{\mathbf{s}}_2'\hat{\mathbf{s}}_1' + \mathcal{Z}^{22}\hat{\mathbf{s}}_2'\hat{\mathbf{s}}_2' \quad (42)$$

and $\hat{\mathbf{s}}_1'$, $\hat{\mathbf{s}}_2' := \hat{\mathbf{n}}' \times \hat{\mathbf{s}}_1'$ are two orthogonal unit vectors tangential to S (Figure 1). Therefore, $\hat{\mathbf{s}}_1'$, $\hat{\mathbf{s}}_2'$, and $\hat{\mathbf{n}}' = \hat{\mathbf{s}}_1' \times \hat{\mathbf{s}}_2'$ form an orthonormal right-handed triple on S, and

$$\overline{\boldsymbol{\mathcal{I}}} = \hat{\mathbf{s}}_1'\hat{\mathbf{s}}_1' + \hat{\mathbf{s}}_2'\hat{\mathbf{s}}_2' + \hat{\mathbf{n}}'\hat{\mathbf{n}}' \quad (43)$$

whereby (41) and (42) consistently pass over into

$$\overline{\boldsymbol{\mathcal{Y}}}_S = Y_S(\overline{\boldsymbol{\mathcal{I}}} - \hat{\mathbf{n}}'\hat{\mathbf{n}}'), \quad \overline{\boldsymbol{\mathcal{Z}}}_S = Z_S(\overline{\boldsymbol{\mathcal{I}}} - \hat{\mathbf{n}}'\hat{\mathbf{n}}') \quad (44)$$

for an isotropic boundary. Further, we recognize that $\hat{\mathbf{s}}_1'$, $\hat{\mathbf{s}}_2'$, and $-\hat{\mathbf{n}}'$ are the spatial parts of three four-vectors $s_1'^{\alpha}$, $s_2'^{\alpha}$, and n'^{α}, in accordance with definitions (A8) and (B7) with $\boldsymbol{\varsigma} = 0$.

Next, on the basis of (41) and (42), we introduce admittance and impedance tensors as[1]

$$\begin{aligned} Y'^{\eta\alpha} := -\big(&\mathcal{Y}^{11}s_1'^{\eta}s_1'^{\alpha} + \mathcal{Y}^{12}s_1'^{\eta}s_2'^{\alpha} \\ &+\mathcal{Y}^{21}s_2'^{\eta}s_1'^{\alpha} + \mathcal{Y}^{22}s_2'^{\eta}s_2'^{\alpha}\big) \end{aligned} \quad (45)$$

$$\begin{aligned} Z'^{\eta\alpha} := -\big(&\mathcal{Z}^{11}s_1'^{\eta}s_1'^{\alpha} + \mathcal{Z}^{12}s_1'^{\eta}s_2'^{\alpha} \\ &+\mathcal{Z}^{21}s_2'^{\eta}s_1'^{\alpha} + \mathcal{Z}^{22}s_2'^{\eta}s_2'^{\alpha}\big) \end{aligned} \quad (46)$$

which have the matrix form

$$[Y'^{\eta\alpha}] = \begin{pmatrix} 0 & 0 \\ 0 & -\overline{\boldsymbol{\mathcal{Y}}}_S \end{pmatrix}, \quad [Z'^{\eta\alpha}] = \begin{pmatrix} 0 & 0 \\ 0 & -\overline{\boldsymbol{\mathcal{Z}}}_S \end{pmatrix} \quad (47)$$

and obey the conditions

$$\begin{aligned} n_\eta' Y'^{\eta\alpha} = 0 = V_\eta' Y'^{\eta\alpha}, \quad Y'^{\eta\alpha} n_\alpha' = 0 = Y'^{\eta\alpha} V_\alpha' \\ n_\eta' Z'^{\eta\alpha} = 0 = V_\eta' Z'^{\eta\alpha}, \quad Z'^{\eta\alpha} n_\alpha' = 0 = Z'^{\eta\alpha} V_\alpha' \end{aligned} \quad (48)$$

because n_η' and V'^{η} are orthogonal to $s_1'^{\eta}$ and $s_2'^{\eta}$ (Appendices B and C). In the special case of isotropic boundary, (45) and (46) are reduced to $Y_S U'^{\eta\alpha}$ and $Z_S U'^{\eta\alpha}$ by force of identity (C1). Although (45) and (46) are not elegant, they carry over immediately to the reference frame K where the body is in motion, while accounting for the Lorentz contraction of the surface S.

Thanks to these positions, it is straightforward to write (39) and (40) in covariant form as

$$n_\nu' M'^{\nu\eta} = \frac{1}{c_0} Y'^{\eta\alpha} N_{\alpha\beta}' V'^{\beta} + \varrho_S' V'^{\eta} \quad (49)$$

$$n_\nu' \star N'^{\nu\eta} = \frac{1}{c_0} Z'^{\eta\alpha} \star M_{\alpha\beta}' V'^{\beta} + \varrho_{MS}' V'^{\eta} \quad (50)$$

and, by the principle of covariance, we get

$$n_\nu M^{\nu\eta} = \frac{1}{c_0} Y^{\eta\alpha} N_{\alpha\beta} V^{\beta} + \varrho_S' V^{\eta} \quad (51)$$

$$n_\nu \star N^{\nu\eta} = \frac{1}{c_0} Z^{\eta\alpha} \star M_{\alpha\beta} V^{\beta} + \varrho_{MS}' V^{\eta} \quad (52)$$

in the reference frame K. Apparently, these relationships are dual to one another in view of (28) updated with the specific substitution $Y^{\eta\alpha} \to -Z^{\eta\alpha}$. Each one of (51) and (52) contains only three independent scalar equations in light of (48) and the antisymmetry of the electromagnetic tensors.

The explicit representation of the entries of $Y^{\eta\alpha}$ and $Z^{\eta\alpha}$ in K is unwieldy, e.g.,

$$[Y^{\eta\alpha}] = \begin{pmatrix} Y^{00} & \mathbf{Y}^r \\ \mathbf{Y}^l & \overline{\mathbf{Y}} \end{pmatrix} \quad (53)$$

where

$$\begin{aligned} Y^{00} = &-\mathcal{Y}^{11}\frac{\gamma^4(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1)^2}{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1)^2} - \mathcal{Y}^{22}\frac{\gamma^4(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2)^2}{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2)^2} \\ &-\frac{(\mathcal{Y}^{12}+\mathcal{Y}^{21})\gamma^4(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1)(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2)}{\sqrt{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1)^2}\sqrt{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2)^2}} \end{aligned} \quad (54)$$

$$\begin{aligned} \mathbf{Y}^r = &-\mathcal{Y}^{11}\frac{\gamma^2\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1(\hat{\mathbf{s}}_1+\gamma^2\boldsymbol{\varsigma}\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1)}{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1)^2} \\ &-\mathcal{Y}^{12}\frac{\gamma^2\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1(\hat{\mathbf{s}}_2+\gamma^2\boldsymbol{\varsigma}\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2)}{\sqrt{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1)^2}\sqrt{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2)^2}} \\ &-\mathcal{Y}^{21}\frac{\gamma^2\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2(\hat{\mathbf{s}}_1+\gamma^2\boldsymbol{\varsigma}\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1)}{\sqrt{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2)^2}\sqrt{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1)^2}} \\ &-\mathcal{Y}^{22}\frac{\gamma^2\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2(\hat{\mathbf{s}}_2+\gamma^2\boldsymbol{\varsigma}\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2)}{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2)^2} \end{aligned} \quad (55)$$

$$\begin{aligned} \mathbf{Y}^l = &-\mathcal{Y}^{11}\frac{(\hat{\mathbf{s}}_1+\gamma^2\boldsymbol{\varsigma}\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1)\gamma^2\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1}{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1)^2} \\ &-\mathcal{Y}^{12}\frac{(\hat{\mathbf{s}}_1+\gamma^2\boldsymbol{\varsigma}\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1)\gamma^2\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2}{\sqrt{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1)^2}\sqrt{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2)^2}} \\ &-\mathcal{Y}^{21}\frac{(\hat{\mathbf{s}}_2+\gamma^2\boldsymbol{\varsigma}\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2)\gamma^2\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1}{\sqrt{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2)^2}\sqrt{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1)^2}} \\ &-\mathcal{Y}^{22}\frac{(\hat{\mathbf{s}}_2+\gamma^2\boldsymbol{\varsigma}\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2)\gamma^2\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2}{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2)^2} \end{aligned} \quad (56)$$

$$\begin{aligned} \overline{\mathbf{Y}} = &-\mathcal{Y}^{11}\frac{(\hat{\mathbf{s}}_1+\gamma^2\boldsymbol{\varsigma}\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1)(\hat{\mathbf{s}}_1+\gamma^2\boldsymbol{\varsigma}\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1)}{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1)^2} \\ &-\mathcal{Y}^{12}\frac{(\hat{\mathbf{s}}_1+\gamma^2\boldsymbol{\varsigma}\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1)(\hat{\mathbf{s}}_2+\gamma^2\boldsymbol{\varsigma}\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2)}{\sqrt{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1)^2}\sqrt{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2)^2}} \\ &-\mathcal{Y}^{21}\frac{(\hat{\mathbf{s}}_2+\gamma^2\boldsymbol{\varsigma}\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2)(\hat{\mathbf{s}}_1+\gamma^2\boldsymbol{\varsigma}\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1)}{\sqrt{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2)^2}\sqrt{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1)^2}} \\ &-\mathcal{Y}^{22}\frac{(\hat{\mathbf{s}}_2+\gamma^2\boldsymbol{\varsigma}\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2)(\hat{\mathbf{s}}_2+\gamma^2\boldsymbol{\varsigma}\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2)}{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2)^2} \end{aligned} \quad (57)$$

though identity (C1) allows confirming that $Y^{\eta\alpha}$ and $Z^{\eta\alpha}$ do pass over into $Y_S U^{\eta\alpha}$ and $Z_S U^{\eta\alpha}$ if the conducting boundary is isotropic. Further, it can be checked by inspection that $Y^{\eta\alpha}$ and $Z^{\eta\alpha}$ are symmetric if so are $\overline{\boldsymbol{\mathcal{Y}}}_S$ and $\overline{\boldsymbol{\mathcal{Z}}}_S$ in (41) and (42).

[1]The leading minus sign here is a matter of choice and can be avoided by writing $N_{\beta\alpha}V^{\beta}$ and $\star M_{\beta\alpha}V^{\beta}$ in the subsequent (51) and (52).

A partially explicit expression of (51) is

$$\hat{\mathbf{n}} \cdot \boldsymbol{\mathcal{D}} = \frac{\gamma}{c_0}\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\left(Y^{00}\boldsymbol{\varsigma} - \mathbf{Y}^r\right) \cdot(\boldsymbol{\mathcal{E}} + \mathbf{v}\times\boldsymbol{\mathcal{B}}) + \gamma\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\,\varrho'_S, \quad \mathbf{r}\in S \quad (58)$$

$$\hat{\mathbf{n}}\cdot\mathbf{v}\boldsymbol{\mathcal{D}} + \hat{\mathbf{n}}\times\boldsymbol{\mathcal{H}} = \gamma\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\,\varrho'_S\mathbf{v} + \gamma\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\left(\mathbf{Y}^l\boldsymbol{\varsigma} - \overline{\mathbf{Y}}\right) \cdot(\boldsymbol{\mathcal{E}} + \mathbf{v}\times\boldsymbol{\mathcal{B}}), \quad \mathbf{r}\in S \quad (59)$$

whence we obtain the alternative relation

$$\hat{\mathbf{n}}\times(\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{D}}\times\mathbf{v}) = -\gamma\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2} \left(Y^{00}\boldsymbol{\varsigma}\boldsymbol{\varsigma} - \boldsymbol{\varsigma}\mathbf{Y}^r - \mathbf{Y}^l\boldsymbol{\varsigma} + \overline{\mathbf{Y}}\right) \cdot(\boldsymbol{\mathcal{E}} + \mathbf{v}\times\boldsymbol{\mathcal{B}}), \quad \mathbf{r}\in S \quad (60)$$

by eliminating ϱ'_S. For $|\mathbf{v}| \ll c_0$, one can use (21) and

$$\hat{\mathbf{n}}\cdot\mathbf{v}\boldsymbol{\mathcal{D}} + \hat{\mathbf{n}}\times\boldsymbol{\mathcal{H}} = \overline{\boldsymbol{\mathcal{Y}}}_S\cdot(\boldsymbol{\mathcal{E}} + \mathbf{v}\times\boldsymbol{\mathcal{B}}) + \varrho'_S\mathbf{v} \quad (61)$$

$$\hat{\mathbf{n}}\times(\boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{D}}\times\mathbf{v}) = \overline{\boldsymbol{\mathcal{Y}}}_S\cdot(\boldsymbol{\mathcal{E}} + \mathbf{v}\times\boldsymbol{\mathcal{B}}) \quad (62)$$

to the first order in $\mathbf{v}/c_0$. The corresponding formulas in terms of impedance follow also by duality [5, Section 6.7].

Since we have

$$\boldsymbol{\varsigma}\cdot\left(\mathbf{Y}^l\boldsymbol{\varsigma} - \overline{\mathbf{Y}}\right) = \boldsymbol{\varsigma}\cdot\mathbf{Y}^l\boldsymbol{\varsigma} - \boldsymbol{\varsigma}\cdot\overline{\mathbf{Y}} = Y^{00}\boldsymbol{\varsigma} - \mathbf{Y}^r \quad (63)$$

on account of (48), we can conclude that the effective charge appearing in (58) vanishes when $\mathbf{v} = 0$, without looking into the specific form of $Y^{\eta\alpha}$. Also in (59), effective surface electric current and effective electric field are related by a non-symmetric dyadic (even if $\mathcal{Y}^{12} = \mathcal{Y}^{21}$) because $\mathbf{Y}^l\boldsymbol{\varsigma} \neq \boldsymbol{\varsigma}\mathbf{Y}^l$. On the contrary, the dyadic field in the right-hand side of (60) is symmetric, if so is $\overline{\boldsymbol{\mathcal{Y}}}_S$, in which case $\mathbf{Y}^l = \mathbf{Y}^r$ and $\overline{\mathbf{Y}} = \overline{\mathbf{Y}}^T$.

Finally, the alternative expression

$$n_{[\lambda}N_{\xi\zeta]} = \frac{M^{\iota\kappa}V^\beta}{3!\,c_0}\epsilon_{\eta\lambda\xi\zeta}Z^{\eta\alpha}\epsilon_{\iota\kappa\alpha\beta} + \frac{1}{3}\varrho'_{MS}\epsilon_{\eta\lambda\xi\zeta}V^\eta \quad (64)$$

is obtained from (52) with the same steps that led us to (38).

3. RATE OF WORK AND FORCE DENSITY

In accordance with the admittance and impedance relations derived in Section 2, the electromagnetic field is assumed to be null inside the conductor, and hence, the boundary conditions for the energy-momentum tensor T [18, §33, 22, Chapter 12, 23, Section 5.9] at the interface of a good conductor are

$$n_\eta T^{\eta\nu} = -f^\nu_S, \quad \mathsf{X}\in\mathbb{R}\times S \quad (65)$$

where f^ν_S is the surface density of four-force, and X is the radius vector in the Lorentz space. In matrix form, f^ν_S reads

$$[f^\nu_S] = \begin{pmatrix} \dfrac{\boldsymbol{\mathcal{J}}_S\cdot\boldsymbol{\mathcal{E}}_{\text{ave}}/c_0}{\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}} \\ \dfrac{\varrho_S\boldsymbol{\mathcal{E}}_{\text{ave}} + \boldsymbol{\mathcal{J}}_S\times\boldsymbol{\mathcal{B}}_{\text{ave}}}{\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}} \end{pmatrix} = \begin{pmatrix} \dfrac{\boldsymbol{\mathcal{J}}_S\cdot\boldsymbol{\mathcal{E}}_{\text{ave}}/c_0}{\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}} \\ \dfrac{\mathbf{f}_S}{\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}} \end{pmatrix} \quad (66)$$

where ϱ_S, $\boldsymbol{\mathcal{J}}_S$, and $\mathbf{f}_S$ are the surface density of charge, current, and Lorentz force on S, and the subscript 'ave' serves to indicate the average (i.e., the arithmetic mean) of the fields on either side of S (cf. [17, Section 6.1]). Although in (66) we simply have $\boldsymbol{\mathcal{E}}_{\text{ave}} = \boldsymbol{\mathcal{E}}/2$ and $\boldsymbol{\mathcal{B}}_{\text{ave}} = \boldsymbol{\mathcal{B}}/2$ (where $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ are evaluated right on the positive side of S at time t) still, the calculation of the four-force with (66) is challenging, whereas (65) provides a far easier alternative.

To find a covariant expression of $n_\eta T^{\eta\nu}$ in terms of admittance and impedance tensors, it is expedient to rewrite (64) as

$$n_\eta N_{\alpha\beta} + n_\alpha N_{\beta\eta} + n_\beta N_{\eta\alpha} = \varrho'_{MS}\epsilon_{\xi\eta\alpha\beta}V^\xi + \frac{1}{2c_0}\epsilon_{\xi\eta\alpha\beta}Z^{\xi\lambda}\epsilon_{\iota\kappa\lambda\upsilon}M^{\iota\kappa}V^\upsilon \quad (67)$$

by exploiting the inherent antisymmetry of the Faraday tensor.

Then, we compute

$$\begin{aligned} n_\eta T^{\eta\nu} &= n_\eta\left(\frac{1}{c_0}M^{\eta\alpha}N_\alpha^{\cdot\nu} + \frac{1}{4c_0}g^{\eta\nu}M^{\alpha\beta}N_{\alpha\beta}\right) \\ &= \frac{1}{c_0}n_\eta M^{\eta\alpha}N_\alpha^{\cdot\nu} + \frac{1}{4c_0}g^{\eta\nu}M^{\alpha\beta} \left(\frac{1}{2c_0}\epsilon_{\xi\eta\alpha\beta}Z^{\xi\lambda}\epsilon_{\iota\kappa\lambda\upsilon}M^{\iota\kappa}V^\upsilon + \varrho'_{MS}\epsilon_{\xi\eta\alpha\beta}V^\xi - n_\alpha N_{\beta\eta} - n_\beta N_{\eta\alpha}\right) \\ &= \frac{1}{c_0}n_\eta M^{\eta\alpha}N_\alpha^{\cdot\nu} + \frac{g^{\eta\nu}}{4c_0}M^{\alpha\beta}\left(\varrho'_{MS}\epsilon_{\xi\eta\alpha\beta}V^\xi + \frac{1}{2c_0}\epsilon_{\xi\eta\alpha\beta}Z^{\xi\lambda}\epsilon_{\iota\kappa\lambda\upsilon}M^{\iota\kappa}V^\upsilon\right) - \frac{g^{\eta\nu}}{2c_0}n_\alpha M^{\alpha\beta}N_{\beta\eta} \\ &= \frac{1}{2c_0}n_\eta M^{\eta\alpha}N_\alpha^{\cdot\nu} + \frac{g^{\eta\nu}}{2c_0}\left(\frac{1}{2}M^{\alpha\beta}\varrho'_{MS}\epsilon_{\xi\eta\alpha\beta}V^\xi + \frac{1}{4c_0}M^{\alpha\beta}\epsilon_{\xi\eta\alpha\beta}Z^{\xi\lambda}\epsilon_{\iota\kappa\lambda\upsilon}M^{\iota\kappa}V^\upsilon\right) \\ &= \frac{1}{2c_0^2}\left(N_\alpha^{\cdot\nu}Y^{\alpha\iota}N_{\iota\kappa} + \star M_\alpha^{\cdot\nu}Z^{\alpha\iota}\star M_{\iota\kappa}\right)V^\kappa + \frac{1}{2c_0}\left(\varrho'_S N_\alpha^{\cdot\nu} + \varrho'_{MS}\star M_\alpha^{\cdot\nu}\right)V^\alpha \end{aligned} \quad (68)$$

where, in the last line, we have used (51) and introduced the Hodge dual of M as in (26). In this form, (68) holds for an anisotropic boundary with general (possibly non-symmetric) admittance and impedance tensors given by (45) and (46) rephrased in K. Also, in the rightmost member of (68), the terms involving the Faraday tensor and those involving the Maxwell tensor are the negative dual of one another, on the grounds of (28) updated with the substitution $Y^{\alpha\iota} \to -Z^{\alpha\iota}$.

As an example, from (65) and (68), we obtain the explicit expression of the surface density of four-force on an isotropic conductor in uniform motion by starting with

$$f^\nu_S = -\frac{1}{2c_0^2}\left(Y_S N_\alpha^{\cdot\nu}U^{\alpha\iota}N_{\iota\kappa} + Z_S\star M_\alpha^{\cdot\nu}U^{\alpha\iota}\star M_{\iota\kappa}\right)V^\kappa$$

$$-\frac{1}{2c_0}\left(\varrho'_S N_\alpha^{\cdot\nu} + \varrho'_{MS} \star M_\alpha^{\cdot\nu}\right) V^\alpha$$

$$= \frac{Y_S}{2c_0^2}(N^{\nu\iota} N_{\iota\kappa} V^\kappa + N^\nu_{\cdot\alpha} n^\alpha n^\iota N_{\iota\kappa} V^\kappa)$$

$$+\frac{Z_S}{2c_0^2}\left(\star M^{\nu\iota} \star M_{\iota\kappa} V^\kappa + \star M^\nu_{\cdot\alpha} n^\alpha n^\iota \star M_{\iota\kappa} V^\kappa\right)$$

$$-\frac{\varrho'_S}{2c_0} N_\alpha^{\cdot\nu} V^\alpha - \frac{\varrho'_{MS}}{2c_0} \star M_\alpha^{\cdot\nu} V^\alpha \tag{69}$$

having used (9) stated in the frame K. In light of (66) and (B7), we find the rate of work done by the field on the surface currents (or the power lost to Joule heating in S) per unit area as

$$\begin{aligned}\boldsymbol{\mathcal{J}}_S \cdot \boldsymbol{\mathcal{E}}_{\text{ave}} &= c_0\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\, f_S^0 \\ &= \frac{\gamma}{2} Y_S \sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\, \boldsymbol{\mathcal{E}}\cdot\overline{\boldsymbol{\mathcal{U}}}(\mathbf{r})\cdot(\boldsymbol{\mathcal{E}}+\mathbf{v}\times\boldsymbol{\mathcal{B}}) \\ &\quad+\frac{\gamma}{2} Z_S \sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\boldsymbol{\mathcal{H}}\cdot\overline{\boldsymbol{\mathcal{U}}}(\mathbf{r})\cdot(\boldsymbol{\mathcal{H}}+\boldsymbol{\mathcal{D}}\times\mathbf{v}) \\ &\quad+\frac{\gamma}{2}\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\,\varrho'_S \mathbf{v}\cdot\boldsymbol{\mathcal{E}} \\ &\quad+\frac{\gamma}{2}\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\,\varrho'_{MS}\mathbf{v}\cdot\boldsymbol{\mathcal{H}}, \quad \mathbf{r}\in S\end{aligned} \tag{70}$$

and the surface density of Lorentz force as

$$\begin{aligned}\mathbf{f}_S &= \frac{\gamma}{2} Y_S \sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\,(\boldsymbol{\mathcal{E}}+\mathbf{v}\times\boldsymbol{\mathcal{B}})\cdot\overline{\boldsymbol{\mathcal{U}}}(\mathbf{r})\times\boldsymbol{\mathcal{B}} \\ &\quad+\frac{\gamma}{2} Y_S \sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\,(\boldsymbol{\mathcal{E}}+\mathbf{v}\times\boldsymbol{\mathcal{B}})\cdot\overline{\boldsymbol{\mathcal{U}}}(\mathbf{r})\cdot\frac{\mathbf{v}}{c_0^2}\boldsymbol{\mathcal{E}} \\ &\quad-\frac{\gamma}{2} Z_S \sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\,(\boldsymbol{\mathcal{H}}+\boldsymbol{\mathcal{D}}\times\mathbf{v})\cdot\overline{\boldsymbol{\mathcal{U}}}(\mathbf{r})\times\boldsymbol{\mathcal{D}} \\ &\quad+\frac{\gamma}{2} Z_S \sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}(\boldsymbol{\mathcal{H}}+\boldsymbol{\mathcal{D}}\times\mathbf{v})\cdot\overline{\boldsymbol{\mathcal{U}}}(\mathbf{r})\cdot\frac{\mathbf{v}}{c_0^2}\boldsymbol{\mathcal{H}} \\ &\quad+\frac{\gamma}{2}\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\varrho'_S(\boldsymbol{\mathcal{E}}+\mathbf{v}\times\boldsymbol{\mathcal{B}}) \\ &\quad+\frac{\gamma}{2}\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\varrho'_{MS}(\boldsymbol{\mathcal{H}}+\boldsymbol{\mathcal{D}}\times\mathbf{v}), \quad \mathbf{r}\in S\end{aligned} \tag{71}$$

with the dyadic field $\overline{\boldsymbol{\mathcal{U}}}(\mathbf{r})$ defined in (18). Asymptotically ($|\mathbf{v}| \ll c_0$) we have

$$\begin{aligned}\boldsymbol{\mathcal{J}}_S\cdot\boldsymbol{\mathcal{E}}_{\text{ave}} &= \frac{Y_S}{2}\boldsymbol{\mathcal{E}}\cdot\left(\overline{\boldsymbol{\mathcal{I}}}-\hat{\mathbf{n}}\hat{\mathbf{n}}\right)\cdot(\boldsymbol{\mathcal{E}}+\mathbf{v}\times\boldsymbol{\mathcal{B}}) + \frac{1}{2}\varrho'_S\mathbf{v}\cdot\boldsymbol{\mathcal{E}} \\ &\quad+\frac{Z_S}{2}\boldsymbol{\mathcal{H}}\cdot\left(\overline{\boldsymbol{\mathcal{I}}}-\hat{\mathbf{n}}\hat{\mathbf{n}}\right)\cdot(\boldsymbol{\mathcal{H}}+\boldsymbol{\mathcal{D}}\times\mathbf{v})+\frac{1}{2}\varrho'_{MS}\mathbf{v}\cdot\boldsymbol{\mathcal{H}}\end{aligned} \tag{72}$$

$$\begin{aligned}\mathbf{f}_S &= \frac{Y_S}{2}(\boldsymbol{\mathcal{E}}+\mathbf{v}\times\boldsymbol{\mathcal{B}})\cdot\left(\overline{\boldsymbol{\mathcal{I}}}-\hat{\mathbf{n}}\hat{\mathbf{n}}\right)\times\boldsymbol{\mathcal{B}} \\ &\quad+\frac{Y_S}{2}(\boldsymbol{\mathcal{E}}+\mathbf{v}\times\boldsymbol{\mathcal{B}})\cdot\left(\overline{\boldsymbol{\mathcal{I}}}-\hat{\mathbf{n}}\hat{\mathbf{n}}\right)\cdot\frac{\mathbf{v}}{c_0^2}\boldsymbol{\mathcal{E}} \\ &\quad-\frac{Z_S}{2}(\boldsymbol{\mathcal{H}}+\boldsymbol{\mathcal{D}}\times\mathbf{v})\cdot\left(\overline{\boldsymbol{\mathcal{I}}}-\hat{\mathbf{n}}\hat{\mathbf{n}}\right)\times\boldsymbol{\mathcal{D}} \\ &\quad+\frac{Z_S}{2}(\boldsymbol{\mathcal{H}}+\boldsymbol{\mathcal{D}}\times\mathbf{v})\cdot\left(\overline{\boldsymbol{\mathcal{I}}}-\hat{\mathbf{n}}\hat{\mathbf{n}}\right)\cdot\frac{\mathbf{v}}{c_0^2}\boldsymbol{\mathcal{H}} \\ &\quad+\frac{1}{2}\varrho'_S(\boldsymbol{\mathcal{E}}+\mathbf{v}\times\boldsymbol{\mathcal{B}})+\frac{1}{2}\varrho'_{MS}(\boldsymbol{\mathcal{H}}+\boldsymbol{\mathcal{D}}\times\mathbf{v})\end{aligned} \tag{73}$$

to the first order in $\mathbf{v}/c_0$.

In (70) and (72), it is straightforward to identify the rate of work done by the average field $\boldsymbol{\mathcal{E}}_{\text{ave}} = \boldsymbol{\mathcal{E}}/2$ on the surface densities of conduction and convection electric currents as well as the rate of work done by the average field $\boldsymbol{\mathcal{H}}_{\text{ave}} = \boldsymbol{\mathcal{H}}/2$ on the surface densities of conduction and convection magnetic currents flowing on S. Analogous remarks apply to (71) and (73), where the various contributions can be construed as the Lorentz force on electric and magnetic sources.

For a stationary object, thanks to (2) and (3), formulas (70) and (71) pass over into

$$\begin{aligned}c_0 f_S^0 &= \frac{1}{2}Y_S\boldsymbol{\mathcal{E}}\cdot\left(\overline{\boldsymbol{\mathcal{I}}}-\hat{\mathbf{n}}\hat{\mathbf{n}}\right)\cdot\boldsymbol{\mathcal{E}}+\frac{1}{2}Z_S\boldsymbol{\mathcal{H}}\cdot\left(\overline{\boldsymbol{\mathcal{I}}}-\hat{\mathbf{n}}\hat{\mathbf{n}}\right)\cdot\boldsymbol{\mathcal{H}} \\ &= -\hat{\mathbf{n}}\cdot\boldsymbol{\mathcal{E}}\times\boldsymbol{\mathcal{H}} = -\hat{\mathbf{n}}\cdot\boldsymbol{\mathcal{S}}, \quad \mathbf{r}\in S\end{aligned} \tag{74}$$

$$\begin{aligned}\mathbf{f}_S &= \frac{1}{2}Y_S\boldsymbol{\mathcal{E}}\cdot\left(\overline{\boldsymbol{\mathcal{I}}}-\hat{\mathbf{n}}\hat{\mathbf{n}}\right)\cdot\overline{\boldsymbol{\mathcal{I}}}\times\boldsymbol{\mathcal{B}}+\frac{1}{2}\varrho'_S\boldsymbol{\mathcal{E}} \\ &\quad-\frac{1}{2}Z_S\boldsymbol{\mathcal{H}}\cdot\left(\overline{\boldsymbol{\mathcal{I}}}-\hat{\mathbf{n}}\hat{\mathbf{n}}\right)\cdot\overline{\boldsymbol{\mathcal{I}}}\times\boldsymbol{\mathcal{D}}+\frac{1}{2}\varrho'_{MS}\boldsymbol{\mathcal{H}} \\ &= \frac{1}{2}(\hat{\mathbf{n}}\times\boldsymbol{\mathcal{H}})\times\boldsymbol{\mathcal{B}}-\frac{1}{2}(\boldsymbol{\mathcal{E}}\times\hat{\mathbf{n}})\times\boldsymbol{\mathcal{D}}+\frac{\varrho'_S}{2}\boldsymbol{\mathcal{E}}+\frac{\varrho'_{MS}}{2}\boldsymbol{\mathcal{H}} \\ &= -\frac{1}{2}\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\mathcal{H}}\hat{\mathbf{n}}-\frac{1}{2}\boldsymbol{\mathcal{D}}\cdot\boldsymbol{\mathcal{E}}\hat{\mathbf{n}}+\hat{\mathbf{n}}\cdot\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{E}}+\hat{\mathbf{n}}\cdot\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{H}} \\ &= -\hat{\mathbf{n}}\cdot\overline{\boldsymbol{\mathcal{T}}}_{em}, \quad \mathbf{r}\in S\end{aligned} \tag{75}$$

which are the well-known results for the matching conditions of the Poynting vector and the Maxwell stress dyadic [5, Section 1.10].

4. CONCLUSIONS

For a good conductor in uniform motion in free space, we have derived manifestly covariant admittance and impedance relationships as well as an expression for the surface density of four-force on the boundary of the body. Whether the conductor be isotropic or anisotropic, (17), (30), and (59) show that the surface admittance and impedance pass over into non-symmetric dyadics, with the symmetry (when it is present in the rest frame) being broken by the motion.

Similarly to what happens with bulk constitutive relationships, the admittance and impedance relations combine the projections of $\boldsymbol{\mathcal{E}}+\mathbf{v}\times\boldsymbol{\mathcal{B}}$ and $\boldsymbol{\mathcal{H}}+\boldsymbol{\mathcal{D}}\times\mathbf{v}$ onto the conducting boundary in motion. According to (16), (29), and (58), the normal component of the flux densities is determined by the charges possibly present on the interface plus effective charges which depend on the surface admittance and impedance of the body at rest. Both true and effective surface charge densities are distorted because the body is Lorentz-contracted along $\mathbf{v}$.

APPENDIX A. TANGENTIAL FOUR-VECTORS

We look for a geometrical four-dimensional entity that generalizes the vector tangential to a regular line C, in particular when the latter is in motion. All observers agree that, in the rest frame K' of C, the vector

$$\mathbf{S}'(\mathbf{r}') := \frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}\xi'}, \quad S'^i = \frac{\mathrm{d}x'^i}{\mathrm{d}\xi'}, \quad \mathbf{r}'(\xi') \in C \tag{A1}$$

is tangential to C for any choice of the parameter ξ'. However, if we take ξ' as the *proper length* along the line, then the four-dimensional entity S' with contravariant components

$$[S'^\alpha] := \begin{pmatrix} 0 \\ \mathbf{S}' \end{pmatrix}, \quad S'^\alpha := \frac{\mathrm{d}X'^\alpha}{\mathrm{d}\xi'}, \quad \mathsf{X}' \in \mathbb{R} \times C \tag{A2}$$

is a *space-like* four-vector, and the squared pseudo-norm

$$S'_\alpha S'^\alpha = -|\mathbf{S}'|^2 < 0 \tag{A3}$$

is an invariant of the arc.

To find out the general structure of the four-vector S in a reference frame K where C moves with velocity $\mathbf{v}$, we apply an inverse Lorentz transformation (without rotations) to S' [16, 17]. By appending subscripts $\|$ and $\perp$ to indicate the parts of an ordinary vector that are parallel and perpendicular to $\mathbf{v}$, respectively, we have

$$\begin{aligned}[S^\alpha] &= \begin{pmatrix} \gamma & \gamma\boldsymbol{\varsigma} \\ \gamma\boldsymbol{\varsigma} & \underline{\mathcal{I}} + (\gamma-1)\hat{\mathbf{v}}\hat{\mathbf{v}} \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{S}' \end{pmatrix} = \begin{pmatrix} \gamma\boldsymbol{\varsigma}\cdot\mathbf{S}' \\ \gamma\mathbf{S}'_\| + \mathbf{S}'_\perp \end{pmatrix} \\ &= \begin{pmatrix} \gamma^2\boldsymbol{\varsigma}\cdot\mathbf{S} \\ \mathbf{S} + \gamma^2\boldsymbol{\varsigma}\boldsymbol{\varsigma}\cdot\mathbf{S} \end{pmatrix} = \gamma^2 \begin{pmatrix} \boldsymbol{\varsigma}\cdot\mathbf{S} \\ \mathbf{S} + \boldsymbol{\varsigma}\times(\boldsymbol{\varsigma}\times\mathbf{S}) \end{pmatrix} \end{aligned} \tag{A4}$$

where we have noticed that

$$\begin{aligned}\gamma\mathbf{S}'_\| + \mathbf{S}'_\perp &= \gamma^2\mathbf{S}_\| + \mathbf{S}_\perp = \gamma^2\mathbf{S} + (1-\gamma^2)\mathbf{S}_\perp \\ &= \mathbf{S} + \gamma^2\boldsymbol{\varsigma}\boldsymbol{\varsigma}\cdot\mathbf{S} = \gamma^2\mathbf{S} + \gamma^2\boldsymbol{\varsigma}\times(\boldsymbol{\varsigma}\times\mathbf{S}) \end{aligned} \tag{A5}$$

with $\mathbf{S}$ being the tangential vector from the viewpoint of an observer in K. With the help of the first part of (A5) we can also derive the relationships

$$|\mathbf{S}'| = \sqrt{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}})^2}\,|\mathbf{S}| = \gamma\sqrt{1-|\hat{\mathbf{s}}\times\boldsymbol{\varsigma}|^2}\,|\mathbf{S}| \tag{A6}$$

$$\hat{\mathbf{s}}' = \frac{\hat{\mathbf{s}}_\perp\sqrt{1-\varsigma^2} + \hat{\mathbf{s}}_\|}{\sqrt{1-\varsigma^2+(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}})^2}} = \frac{\hat{\mathbf{s}}_\perp\sqrt{1-\varsigma^2} + \hat{\mathbf{s}}_\|}{\sqrt{1-|\hat{\mathbf{s}}\times\boldsymbol{\varsigma}|^2}} \tag{A7}$$

where $\hat{\mathbf{s}}' := \mathbf{S}'/|\mathbf{S}'|$ and $\hat{\mathbf{s}} := \mathbf{S}/|\mathbf{S}|$ are the unit tangents to C in K' and K. Finally, by normalizing S in (A4) with $|\mathbf{S}'|$ in (A6), we obtain

$$[s^\alpha] = \begin{pmatrix} \dfrac{\gamma^2\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}}{\sqrt{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}})^2}} \\ \dfrac{\hat{\mathbf{s}} + \gamma^2\boldsymbol{\varsigma}\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}}{\sqrt{1+\gamma^2(\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}})^2}} \end{pmatrix} = \begin{pmatrix} \dfrac{\gamma\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}}{\sqrt{1-|\hat{\mathbf{s}}\times\boldsymbol{\varsigma}|^2}} \\ \gamma\dfrac{\hat{\mathbf{s}} + \boldsymbol{\varsigma}\times(\boldsymbol{\varsigma}\times\hat{\mathbf{s}})}{\sqrt{1-|\hat{\mathbf{s}}\times\boldsymbol{\varsigma}|^2}} \end{pmatrix} \tag{A8}$$

i.e., a 'unitary' space-time vector tangential to the hyper-line $\mathbb{R}\times C$.

APPENDIX B. NORMAL FOUR-VECTORS

There is also much use for a geometrical four-dimensional entity which extends the notion of (unit) normal vector on a $\mathcal{C}^2$-smooth surface S. We know that the unit normal looks different to an observer in relative motion with respect to S. Therefore, we start off in K', the rest frame of S, where all observers concur that a vector perpendicular to S is given by[2]

$$N'_i = \epsilon_{ijk}\frac{\partial x'^j}{\partial \xi'^1}\frac{\partial x'^k}{\partial \xi'^2}, \quad \mathbf{r}'(\xi'^1, \xi'^2) \in S \tag{B1}$$

with $\epsilon_{ijk} \in \{\pm 1, 0\}$ denoting the covariant holor of the Levi-Civita pseudo-tensor in $\mathbb{R}^3$, and ξ'^1 and ξ'^2 being two local coordinates on S (Figure 1). However, if we take ξ'^1 and ξ'^2 as *proper lengths* on S, and notice that $\epsilon_{ijk} = -\epsilon_{0ijk}$, then the four-dimensional entity N' with covariant components

$$N'_\alpha := -\epsilon_{0\alpha\beta\eta}\frac{\partial X'^\beta}{\partial \xi'^1}\frac{\partial X'^\eta}{\partial \xi'^2}, \quad [N'_\alpha] := \begin{pmatrix} 0 \\ \mathbf{N}' \end{pmatrix} \tag{B2}$$

is a *space-like* four-vector, and the squared pseudo-norm

$$N'_\alpha N'^\alpha = -|\mathbf{N}'|^2 < 0 \tag{B3}$$

is an invariant of the surface. Clearly, N' is also orthogonal to any four-vector with temporal component only, e.g., the four-velocity $V'^\alpha = c_0\delta_0^\alpha$.

To find out the general structure of the four-vector N in a frame K where S moves with velocity $\mathbf{v}$, we transform N'_α with an inverse Lorentz transformation[3] (without rotations) [17]

$$\begin{aligned}[N_\alpha] &= \begin{pmatrix} \gamma & -\gamma\boldsymbol{\varsigma} \\ -\gamma\boldsymbol{\varsigma} & \underline{\mathcal{I}} + (\gamma-1)\hat{\mathbf{v}}\hat{\mathbf{v}} \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{N}' \end{pmatrix} = \begin{pmatrix} -\gamma\boldsymbol{\varsigma}\cdot\mathbf{N}'_\| \\ \mathbf{N}'_\perp + \gamma\mathbf{N}'_\| \end{pmatrix} \\ &= \begin{pmatrix} -\gamma\boldsymbol{\varsigma}\cdot\mathbf{N}_\| \\ \gamma\mathbf{N}_\perp + \gamma\mathbf{N}_\| \end{pmatrix} = \begin{pmatrix} -\gamma\boldsymbol{\varsigma}\cdot\mathbf{N} \\ \gamma\mathbf{N} \end{pmatrix} \end{aligned} \tag{B4}$$

where we have noticed that

$$\gamma\mathbf{N} = \gamma\mathbf{N}'_\| + \mathbf{N}'_\perp \tag{B5}$$

with $\mathbf{N}$ being the vector perpendicular to S from the viewpoint of an observer in K. By means of (B5), it can be shown that the magnitudes of the normal vectors in K' and K are related as

$$|\mathbf{N}'| = \gamma\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}\,|\mathbf{N}| \tag{B6}$$

whereby we can normalize N in (B4) to get

$$[n^\alpha] := \frac{[N^\alpha]}{|\mathbf{N}'|} = -\frac{1}{\sqrt{1-(\boldsymbol{\varsigma}\cdot\hat{\mathbf{n}})^2}}\begin{pmatrix} \boldsymbol{\varsigma}\cdot\hat{\mathbf{n}} \\ \hat{\mathbf{n}} \end{pmatrix} \tag{B7}$$

[2] We are acutely aware that the same symbols N', N are being used for the Faraday tensor in Sections 2 and 3 (adhering to Van Bladel's choice [17]) and for the normal four-vector in this appendix. There should be little risk of confusion, though, because, in the index notation, these objects carry a different number of indices, and, more importantly, in this paper, they do not appear together in the same expressions, which, admittedly, would be awkward.

[3] The transformation matrix in (B4) is different from the one appearing in (A4) because here the four-vector being transformed is represented through its covariant components.

a 'unitary' space-time vector perpendicular to the hyper-surface $\mathbb{R} \times S$ [17, Problem 1.30, 24].

If S is a closed surface, boundary of a three-dimensional domain V in the ordinary space (as in Figure 1), we can think of $\hat{\mathbf{n}}'$ as being oriented positively *outwards* V, and definition (B7) preserves the positive orientation of $\hat{\mathbf{n}}$ in K. To avoid the appearance of the cumbersome leading minus sign, we may alternatively define the holor n^α as the negative of (B7), which is tantamount to taking $\hat{\mathbf{n}}$ (and $\hat{\mathbf{n}}'$) as positively oriented *inwards* V. If one makes this choice, though, in applying the four-dimensional Gauss theorem, one must keep in mind that the orientation of $\hat{\mathbf{n}}$ is reversed.

Since the first part of (B2) strongly suggests that a contraction was performed with a unit four-vector having temporal component only, we can further write

$$\begin{aligned} N'_\alpha &:= \frac{1}{c_0}\epsilon_{\alpha\nu\beta\eta}V'^\nu \frac{\partial X'^\beta}{\partial \xi'^1}\frac{\partial X'^\eta}{\partial \xi'^2} \\ &= \frac{1}{c_0}\epsilon_{\alpha\nu\beta\eta}V'^\nu \underset{1}{s}'^\beta \underset{2}{s}'^\beta \left|\frac{\partial \mathbf{X}'}{\partial \xi'^1}\right|\left|\frac{\partial \mathbf{X}'}{\partial \xi'^2}\right| \end{aligned} \tag{B8}$$

having introduced unitary tangential four-vectors in K' in keeping with (A2) and as suggested in Figure 1. If, in addition, we conveniently choose orthogonal tangential vectors on S, then, by construction, we have

$$|\mathbf{N}'| = \left|\frac{\partial \mathbf{X}'}{\partial \xi'^1}\right|\left|\frac{\partial \mathbf{X}'}{\partial \xi'^2}\right| \tag{B9}$$

whereby

$$n'_\alpha := \frac{N'_\alpha}{|\mathbf{N}'|} = \frac{1}{c_0}\epsilon_{\alpha\nu\beta\eta}V'^\nu \underset{1}{s}'^\beta \underset{2}{s}'^\eta, \quad \mathbf{X}' \in \mathbb{R}\times S \tag{B10}$$

which passes over into

$$n_\alpha = \frac{1}{c_0}\epsilon_{\alpha\nu\beta\eta}V^\nu \underset{1}{s}^\beta \underset{2}{s}^\eta, \quad \mathbf{X} \in \mathbb{R}\times S \tag{B11}$$

in the frame K. This formula provides an alternative expression of the unit four-normal on the hyper-surface $\mathbb{R} \times S$ in terms of three orthogonal unit four-vectors tangential to $\mathbb{R} \times S$.

As a check, we verify that (B11) and (B7) return the same temporal component n_0. Since the unit tangential four-vectors in K have the structure outlined in (A8) and $V^i/c_0 = \gamma\varsigma^i$, we find

$$\begin{aligned} n_0 &= \epsilon_{0ijk}\gamma\varsigma^i \underset{1}{s}^j \underset{2}{s}^k = -\epsilon_{ijk}\gamma\varsigma^i \underset{1}{s}^j \underset{2}{s}^k \\ &= -\gamma^3\boldsymbol{\varsigma}\cdot\frac{\hat{\mathbf{s}}_1 + \boldsymbol{\varsigma}\times(\boldsymbol{\varsigma}\times\hat{\mathbf{s}}_1)}{\sqrt{1-|\hat{\mathbf{s}}_1\times\boldsymbol{\varsigma}|^2}}\times\frac{\hat{\mathbf{s}}_2 + \boldsymbol{\varsigma}\times(\boldsymbol{\varsigma}\times\hat{\mathbf{s}}_2)}{\sqrt{1-|\hat{\mathbf{s}}_2\times\boldsymbol{\varsigma}|^2}} \\ &= -\gamma^3\boldsymbol{\varsigma}\cdot\frac{\hat{\mathbf{s}}_1 + \boldsymbol{\varsigma}\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1 - \varsigma^2\hat{\mathbf{s}}_1}{\sqrt{1-|\hat{\mathbf{s}}_1\times\boldsymbol{\varsigma}|^2}}\times\frac{\hat{\mathbf{s}}_2 + \boldsymbol{\varsigma}\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_2 - \varsigma^2\hat{\mathbf{s}}_2}{\sqrt{1-|\hat{\mathbf{s}}_2\times\boldsymbol{\varsigma}|^2}} \\ &= -\frac{\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_1\times\hat{\mathbf{s}}_2/\gamma}{\sqrt{1-|\hat{\mathbf{s}}_1\times\boldsymbol{\varsigma}|^2}\sqrt{1-|\hat{\mathbf{s}}_2\times\boldsymbol{\varsigma}|^2}} \\ &= -\frac{\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}_{1\perp}\times\hat{\mathbf{s}}_{2\perp}/\gamma}{\sqrt{1-|\hat{\mathbf{s}}_1\times\boldsymbol{\varsigma}|^2}\sqrt{1-|\hat{\mathbf{s}}_2\times\boldsymbol{\varsigma}|^2}} = -\gamma\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}'_{1\perp}\times\hat{\mathbf{s}}'_{2\perp} \\ &= -\gamma\boldsymbol{\varsigma}\cdot\hat{\mathbf{s}}'_1\times\hat{\mathbf{s}}'_2 = -\gamma\boldsymbol{\varsigma}\cdot\hat{\mathbf{n}}' = -\gamma\frac{\boldsymbol{\varsigma}\cdot\mathbf{N}'}{|\mathbf{N}'|} = -\gamma\frac{\boldsymbol{\varsigma}\cdot\mathbf{N}'_\parallel}{|\mathbf{N}'|} \\ &= -\gamma\frac{\boldsymbol{\varsigma}\cdot\mathbf{N}_\parallel}{|\mathbf{N}'|} = -\gamma\frac{\boldsymbol{\varsigma}\cdot\mathbf{N}}{|\mathbf{N}'|} = -\frac{\boldsymbol{\varsigma}\cdot\hat{\mathbf{n}}}{\sqrt{1-(\hat{\mathbf{n}}\cdot\boldsymbol{\varsigma})^2}} \end{aligned} \tag{B12}$$

on account of (A7), (B5), and (B6).

APPENDIX C. DECOMPOSITION OF THE METRIC

The noteworthy tensor identity

$$g^{\eta\alpha} = \frac{1}{c_0^2}V^\eta V^\alpha - \underset{1}{s}^\eta \underset{1}{s}^\alpha - \underset{2}{s}^\eta \underset{2}{s}^\alpha - n^\eta n^\alpha \tag{C1}$$

provides the decomposition of $\mathbf{g}$ on the hyper-surface $\mathbb{R} \times S$ into the sum of tensors constructed out of mutually 'orthogonal' four-vectors, and can be regarded as the extension of (43) to the Lorentz space.

We can prove (C1) by inspection in K', and then invoke the principle of covariance. More generally, (C1) follows from the representation of an arbitrary four-vector on an orthonormal basis formed with n^η, V^η, $\underset{1}{s}^\eta$, and $\underset{2}{s}^\eta$.

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