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# Study of the Energy Storage Factor of a Cylindrical Dielectric Resonator by a Perturbation Method for Loss Tangent Measurement

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**ABSTRACT:** The relative value of the stored energy in the dielectric and that in the surroundings for a cylindrical dielectric resonator in a closed metal cavity was studied using a simple electromagnetic field theory model. The influence of this factor on the measurements of the loss tangents of dielectric samples with different dielectric properties and dimensions at their microwave resonant frequencies was discussed. In addition to the traditional calculation method, a perturbation method with a much simpler computation procedure was also adopted for the energy factor calculation, and its accuracy was compared with that of the traditional method.

## 1. INTRODUCTION

Disk-shaped resonators fabricated with low-loss dielectrics have been widely applied to various microwave-related applications, and one application is for the measurement of the dielectric constant and dielectric loss at microwave frequencies [1]. For the dielectric resonant configuration in Figure 1, metal shields are always introduced to prevent radiation loss. In the various resonant modes of resonators, $\mathrm{TE}_{01\delta}$ resonant peak is the most popular one to be used for the experimental measurement of the dielectric relative complex permittivity, $\varepsilon_r = \varepsilon' + j\varepsilon''$. Many theoretical and experimental studies have been reported for different designs of metal shield arrangements [2–11]. Precise but complicated analyses of the electromagnetic field distributions and frequencies of the resonant modes have been proposed. To simplify the analyses, some studies have suggested much simpler theoretical models [12–15]. However, these simple theory analyses often lack precise configurations of electromagnetic field distributions and resonant frequencies.

Based on one of the proposed simple models, Itoh and Rudokas theory [12], a modification to this simple theoretical structure has been suggested, and this modified theory can maintain the simplicity of the original model and dramatically improve the accuracy of measuring dielectric characteristics [16]. Itoh and Rudokas model was originally derived for the dielectric resonator without a sidewall, as shown in Figure 1; therefore, it is inadequate for the configuration of a closed cavity [12–15]. The suggested modification by [16] solves this problem and supplies a precise prediction of the resonant frequency of the $\mathrm{TE}_{01\delta}$ mode and reasonable reliability in the measurements of dielectric constants ($\varepsilon'$) on samples with various dielectric properties and dimensions. The loss tangents ($\tan\delta = \varepsilon''/\varepsilon'$) can also be measured using this new field model.

To calculate the loss tangent of a sample in experiments, the ratio of energies stored in the surrounding areas of the disk sample and inside the sample must be calculated. The primary objective of this article is to compare two different calculation methods for the energy ratio and provide suggestions for minimizing the measurement error of loss tangent resulting from this energy factor. Traditionally, this energy ratio is calculated based on the distributions of electromagnetic fields in the surrounding air regions and in the dielectric. The stored energies can be computed by the equations of volume integrals with complicated mathematical procedures. In addition to the traditional method, perturbation theory can also be applied to the computation of the energy factor, which is based on the fact that a small deviation in the resonant frequency will be caused by the small variation in the dielectric constant of the dielectric sample [17, 18]. This perturbation technique has a much simpler calculation procedure than the traditional method. However, the accuracy of this perturbation method based on the simple new field theory is still unclear. To verify the reliability of this perturbation technique, in this study, above two different approaches for calculating the energy factor are investigated for different dielectric constants of the samples and for various cavity and sample dimensions. The accuracy of this perturbation method was clarified. The influence of this energy factor on the measurement of the loss tangent at microwave frequencies was analyzed.

## 2. MODIFIED FIELD MODEL

In Figure 1, the diameter and thickness of the disk-shaped sample are indicated as $D$ and $L$. The sample has a relatively complex permittivity $\varepsilon_r = \varepsilon' + j\varepsilon''$. The height and diameter of

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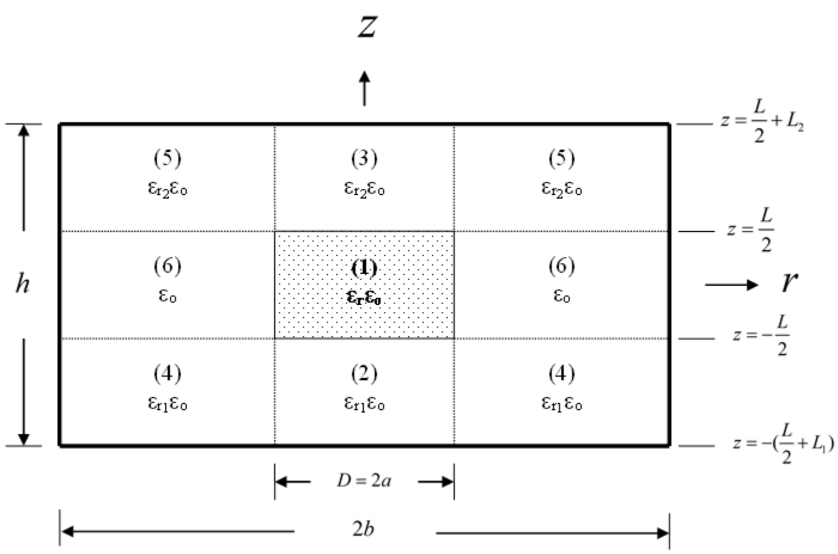


**FIGURE 1**. Side view of a cylindrical dielectric sample located inside a closed conducting shield.

the closed cylindrical cavity are indicated as $h$ and $2b$, shown in Figure 1. Noted by the subscripts 1 to 6, the equations of electric fields of the six areas are [16]:

$$E_{\phi 1} = E_o J_1(k_{ci}r)\cos(\beta_I z - \phi_1) \tag{1a}$$

$$E_{\phi 2} = E_o \frac{\cos(\beta_I \frac{L}{2} + \phi_1)}{\sinh(\alpha_1 L_1)} J_1(k_{ci}r)\sinh\alpha_1\left(z + L_1 + \frac{L}{2}\right) \tag{1b}$$

$$E_{\phi 3} = E_o \frac{\cos(\beta_I \frac{L}{2} + \phi_2)}{\sinh(\alpha_2 L_2)} J_1(k_{ci}r)\sinh\alpha_2\left(L_2 + \frac{L}{2} - z\right) \tag{1c}$$

$$E_{\phi 4} = E_o \frac{J_1(k_{ci}a)\cos(\beta_I \frac{L}{2} + \phi_1)}{K_1(k_{co}a)\sinh(\alpha_1 L_1)}$$

$$[K_1(k_{co}r) - K_1(k_{co}b)]\sinh\alpha_1\left(z + L_1 + \frac{L}{2}\right) \tag{1d}$$

$$E_{\phi 5} = E_o \frac{J_1(k_{ci}a)\cos(\beta_I \frac{L}{2} + \phi_2)}{K_1(k_{co}a)\sinh(\alpha_2 L_2)}$$

$$[K_1(k_{co}r) - K_1(k_{co}b)]\sinh\alpha_2\left(L_2 + \frac{L}{2} - z\right) \tag{1e}$$

$$E_{\phi 6} = E_o \frac{J_1(k_{ci}a)}{K_1(k_{co}a)}[K_1(k_{co}r) - K_1(k_{co}b)]$$

$$\cos(\beta_I z - \phi_1) \tag{1f}$$

with

$$\frac{J_0(k_{ci}a)}{J_1(k_{ci}a)} = -\frac{k_{co}a}{k_{ci}a}\frac{1}{K_1(k_{co}a)}\left[K_0(k_{co}a) + \frac{1}{k_{co}a}K_1(k_{co}b)\right] \tag{2a}$$

$$k_{co}^2 = \left(\frac{\pi}{h}\right)^2 - k_o^2 \tag{2b}$$

$$k_{ci}^2 = k_o^2\varepsilon' - \beta_I^2 \tag{2c}$$

$$k_o = \omega\sqrt{\mu_o\varepsilon_o} \tag{2d}$$

$$\phi_1 + \phi_2 = 0 \tag{2e}$$

$$\phi_{1,2} = \tan^{-1}\left(\frac{\alpha_{1,2}}{\beta_I}\coth\alpha_{1,2}L_{1,2}\right) - \beta_I\frac{L}{2} \tag{2f}$$

$$\alpha_{1,2}^2 = k_{ci}^2 - k_o^2\varepsilon_{r1,r2} \tag{2g}$$

where $E_0$ is the field amplitude; $k_{ci}$ and $k_{co}$ are the wavenumbers in the $r$ axis; $\beta_I$ represents the propagation constant of the $z$ axis; $\alpha_1$ and $\alpha_2$ indicate the attenuation constants in the $z$ axis of regions 2 and 3; and $J$ and $K$ are the Bessel and modified Bessel functions, respectively. $\phi_2$ and $\phi_1$ are the phase angles of the corner regions (areas 4 and 5 in Figure 1) and the phase angles of the rest areas, respectively. Both $\phi_1$ and $\phi_2$ are the parameters to be computed for experimental calculations of the dielectric constant. Equation (2a) represents the eigenvalue equation of the dielectric boundary [16]. $H_\phi$, $E_r$, and $E_z$ are all equal to zero for the simple $TE_{01\delta}$ resonance mode. The field expressions of the remaining magnetic fields can be obtained using Equation (2) from the relationships between the electric and magnetic fields [19].

## 3. CALCULATIONS OF DIELECTRIC PROPERTIES

If the inside height of the cavity is less than the cutoff condition, that is, smaller than one-half wavelength, the dielectric loss of the sample can be calculated using the following equations,

$$\tan\delta = A\left(\frac{1}{Q_u} - \frac{1}{Q_c}\right) \tag{3a}$$

$$Q_u = \frac{f_o}{\Delta f_{3\,\mathrm{dB}}}\frac{1}{|1 - S_{21}|} \tag{3b}$$

where $A$ in (3a) is the energy storage factor and the major topic studied in this article, which represents the proportion of the total storage energy to the storage energy inside the dielectric sample; $Q_u$ is the unloaded quality factor from the experimental measurement; $Q_c$ is the quality factor owing to conductor dissipation; $f_o$, $\Delta f_{3\,\text{dB}}$, and $S_{21}$ are the resonance frequency, half power (3 dB) bandwidth, and the transmission $S$ parameter measured from the resonant curve of the $\text{TE}_{01\delta}$ mode, respectively. The computations and discussions of $Q_c$ can be found elsewhere [19].

In addition to the measurement parameters of the resonant curve of the $\text{TE}_{01\delta}$ mode, the dimensions of the sample and cavity were input into a computer program to compute the real part of the complex permittivity, that is, the dielectric constant, using Equation (2). An example of the variation in the resonant frequency with the dielectric constant and cavity dimensions is presented in Figure 2, without considering the cutoff condition, to provide a simple overview of the resonant frequencies of the $\text{TE}_{01\delta}$ mode. The Newton-Raphson iterative method was executed in the computer program. The detailed procedure can also found elsewhere [20].

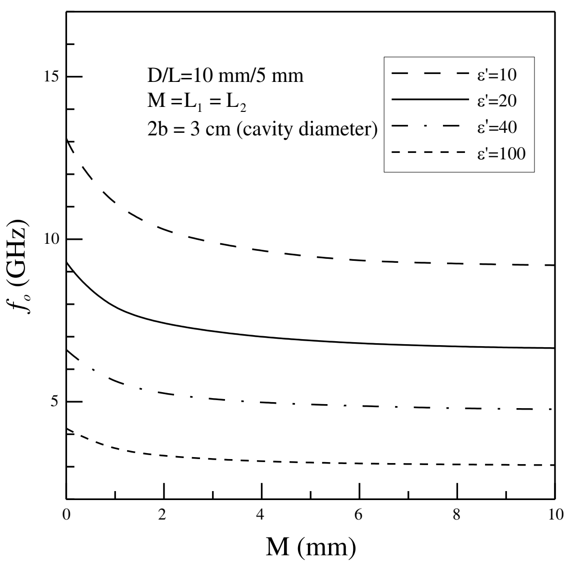


**FIGURE 2**. Relationship of resonant frequency with $\varepsilon'$ and cavity dimension.

## 4. CALCULATIONS OF FACTOR *A*

From Equation (3a), the accuracy of the dielectric loss measurement is strongly dependent on the computational accuracy of the quality factor of conducting loss $Q_c$ and energy factor $A$. Based on the definition of the energy factor, as mentioned, the ratio of the total storage energy to the storage energy inside the dielectric sample, the traditional calculation procedures for $A$ are as follows,

$$A = \frac{W}{W_d} = 1 + \frac{W_a}{W_d} \tag{4a}$$

$$W_d = \frac{1}{4}\varepsilon'\varepsilon_o \int_V |E_{\phi 1}|^2 dV \tag{4b}$$

$$W_a = \sum_{i=2}^{6} \frac{1}{4}\varepsilon_i\varepsilon_o \int_V |E_{\phi 1}|^2 dV, \quad \varepsilon_i = 1, \varepsilon_{r1}, \text{ or } \varepsilon_{r2} \tag{4c}$$

where $W_d$, $W_a$, and $W (W = W_a + W_d)$ are the stored energies inside the sample, in the surrounding areas of the dielectric sample, and the total storage energy, respectively. In Equations (4b) and (4c), the traditional calculation methods of $W_d$ and $W_a$ by volume integrals are presented.

Perturbation theory can also be applied to the computation of the factor $A$. Factor $A$ can be calculated using the relationship of a frequency shift ($\delta f_o$) caused by a small change in the $\varepsilon'(\delta\varepsilon')$ of a sample with the resonant frequency $f_o$ and real part $\varepsilon'$, by the following equation [17, 18],

$$A = -\frac{1}{2}\frac{f_o}{\delta f_o}\frac{\delta\varepsilon'}{\varepsilon'} \tag{5}$$

The advantage of the proposed perturbation technique is that it can avoid complex manipulations of the electromagnetic fields by adopting the basic concept of Equation (4). However, the main concern of Equation (5) is that a precise computation of the variation in the resonant frequency ($\delta f_o$) is important to the loss tangent calculation.

Figure 3 compares the $A$ factors calculated using Equations (4) and (5). Obviously, the calculation difference between Equations (4) and (5) increases as the dielectric constant $\varepsilon'$ decreases, as well as the diameter/thickness ratio $D/L$ increases. It is reasonable that the discrepancy is larger if the $\varepsilon'$ value is smaller. The stored energymin the surrounding air region $W_a$ will increase if the dielectric constant of the sample is lower because the electromagnetic fields outside the sample are stronger, which results in a larger $W_a/W_d$ ratio in Equation (4a) and causes a higher possibility of error for $A$ factor. About the rise of disagreement with increasing the $D/L$ ratio, the larger the diameter/thickness ratio is, the more the spaces are in areas 2 and 3 in Figure 1. $W_a$ will also increase, as will the possibility of error for the $A$ factor. Another possibility for the increase in differences between these two methods may be the decrease in the accuracy of the frequency calculation for this modified field model. The discrepancy also slightly increases for a lower $D/L$ ratio, especially for low $\varepsilon'$ conditions. This should also have something to do with the frequency accuracy.

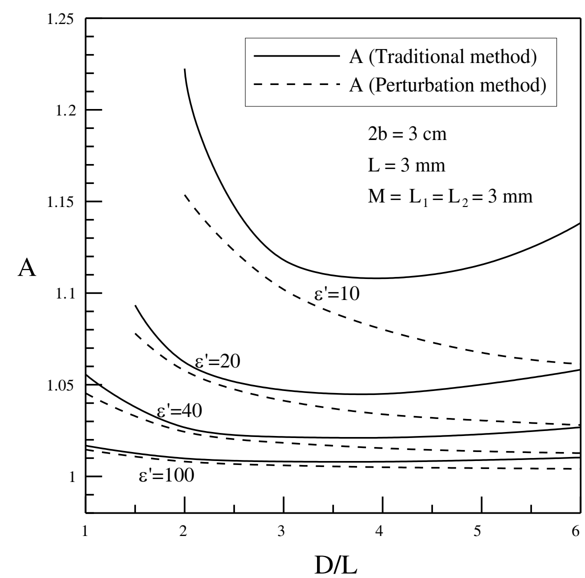


**FIGURE 3**. Comparisons of $A$ factors computed by Equations (4) and (5). Variations to the ratios of diameter/thickness of the dielectric sample are presented.

**TABLE 1**. Measurement errors of loss tangent due to the calculation error of energy factor $A$ by the perturbation method with cavity diameter $2b = 3$ cm, heights of the two air regions $L_1 = L_2 = 3$ mm, and dielectric sample thickness $L = 3$ mm.

(a) Sample diameter $D = 6$ mm ($D/L = 2$)

| Dielectric constant $\varepsilon'$ | Frequency (GHz) | $A$ (traditional method) | $A$ (perturbation method) | Measurement error of tan $\delta$ |
|---|---|---|---|---|
| $\varepsilon' = 10$ | 15.6 | 1.2225 | 1.1536 | 5.46% |
| $\varepsilon' = 20$ | 11.4 | 1.0623 | 1.0577 | 0.43% |
| $\varepsilon' = 40$ | 8.1 | 1.0267 | 1.0243 | 0.23% |
| $\varepsilon' = 100$ | 5.2 | 1.0098 | 1.0081 | 0.17% |

(b) Sample diameter $D = 9$ mm ($D/L$ =3)

| Dielectric constant $\varepsilon'$ | Frequency (GHz) | $A$ (traditional method) | $A$ (perturbation method) | Measurement error of tan $\delta$ |
|---|---|---|---|---|
| $\varepsilon' = 10$ | 12.2 | 1.1181 | 1.1020 | 1.44% |
| $\varepsilon' = 20$ | 8.8 | 1.0471 | 1.0413 | 0.55% |
| $\varepsilon' = 40$ | 6.3 | 1.0215 | 1.0183 | 0.31% |
| $\varepsilon' = 100$ | 4.0 | 1.0081 | 1.0060 | 0.21% |

(c) Sample diameter $D = 1.5$ cm ($D/L = 5$)

| Dielectric constant $\varepsilon'$ | Frequency (GHz) | $A$ (traditional method) | $A$ (perturbation method) | Measurement error of tan $\delta$ |
|---|---|---|---|---|
| $\varepsilon' = 10$ | 9.4 | 1.1155 | 1.0675 | 4.30% |
| $\varepsilon' = 20$ | 6.8 | 1.0501 | 1.0305 | 1.87% |
| $\varepsilon' = 40$ | 4.8 | 1.0230 | 1.0137 | 0.91% |
| $\varepsilon' = 100$ | 3.1 | 1.0089 | 1.0045 | 0.44% |

The greater the difference is, the higher the measurement error of the loss tangent is. From Equation (4b), the stored energy in the sample increases if $\varepsilon'$ is larger. The $A$ will approach 1 if the specimen is a high $\varepsilon'$ material. Therefore, for the calculation of a loss tangent measurement by Equation (3), for a sample with a high enough dielectric constant, the energy stored in the surrounding areas of the sample can be neglected directly, with no significant error [2, 3]. As shown in Figure 3, by simply letting $A = 1$, the error can be constrained to be approximately lower than 5% if the $\varepsilon'$ value is larger than 40. On one more observation about the diameter/thickness ratio, Figure 3 has also shown that the energy factor increases as the $D/L$ value decreases, which implies that for a smaller sample diameter, $W_a$ is mainly dominated by the side region (region 6). It should also be noted that the $A$ factor increases with increasing $L_1$ or $L_2$. Therefore, the $A$ factor is close to one if $L_1$ and $L_2$ are very close to zero.

The physical numerical values of the measurement errors of the loss tangents of three examples with sample thickness $L = 3$ mm are listed in Table 1 to understand the real impact of this calculation error caused by the perturbation method more precisely. The three tables present the errors under different sample dimensions with $D/L = 2, 3$, and 5, respectively, for dielectric constants varying from $\varepsilon' = 10$ to $\varepsilon' = 100$. From the tables, in most cases, these measurement errors caused by the perturbation method are all quite small; the errors are lower than 2% and can therefore be neglected. However, for $D/L = 2$ and 5, two high errors of 5.46% and 4.30% are observed for the lowest dielectric constant of $\varepsilon' = 10$. This phenomenon agrees with the observation about Figure 3, where the obvious differences between the traditional and perturbation curves happen on the two ends (the largest and smallest $D/L$) of the curves for $\varepsilon' = 10$. For lower dielectric constant samples ($\varepsilon' < 20$), the $D/L$ ratio should be handled carefully. The range of $2 < D/L < 5$ will be a better choice for the design of the sample dimension. For dielectric constants larger than 20, which is popular in the applications of dielectric resonators, this error can be controlled within 2%, which agrees with the previous discussion about Figure 3 that the calculation error decreases as the dielectric constant increases. Therefore, the perturbation method can be effectively adopted for high-dielectric-constant samples. It was also observed that the $A$ factor is very close to 1 for high $\varepsilon'$ samples, as mentioned before.

## 5. CONCLUSIONS

The energy storage factor calculated by the perturbation method for a cylindrical dielectric resonator in a closed metal cavity was studied using a simple modified field model. The calculated results of the perturbation technique were compared with those obtained using the traditional method based on the electromagnetic field distributions of this modified model. This perturbation method is simpler than the traditional method. In conclusion, for a dielectric constant $\varepsilon' < 20$, in order to have a low enough error caused by the perturbation method, the diameter/thickness ratio $D/L$ should be kept in the range of 2 to 5. For a dielectric constant larger than 20, the error of this perturbation method can be neglected. The energy factor was also very close to unit for high-dielectric-constant samples.

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