

Coupling Minkowski's Theory with Maxwell's Equations for a Mechano-Driven Media System for Engineering Electromagnetism

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ABSTRACT: Minkowski's theory is regarded as the classical approach to describing the electromagnetism of uniformly moving objects by elegantly utilizing the form-invariance of Maxwell's equations in inertial reference frames under Lorentz transformations. However, in practice, we often encounter medium with boundaries that accelerate, such as a rotating dielectric disc or a magnetic object. To fully describe the electromagnetic behavior of such a system, we developed *Maxwell's equations for a mechano-driven medium system* (MEs-f-MDMS). This study expands Minkowski's approach to derive constitutive equations and boundary conditions for MEs-f-MDMS under the low-speed approximation ($v \ll c$), based on which equations can be properly solved to describe engineering electromagnetism, by considering the coupling among the electric field, magnetic field, and mechanical force field. In practice, mechanical action is not merely passive motion through space but an active source of electromagnetic generation and modulation of electromagnetic fields. Our theory makes it possible to treat electrodynamics by including medium deformation, strain gradients, rotational dynamics, and interfacial contact processes in the full calculation.

1. INTRODUCTION

The theoretical foundation of classical electrodynamics was first established by Maxwell in 1865 [1] and was later extensively developed and reformulated by Heaviside around 1900 [2]. These equations unify electric and magnetic phenomena into a self-consistent field theory, forming the foundation for wireless communication, signal transmission, radar, photonics, and many other applications. In material media, Maxwell's equations must be complemented by constitutive relations that describe how matter responds to electromagnetic fields. While conventional treatment assumes stationary or quasi-static media, many physical systems involve motion, deformation, and mechanical triggering that fundamentally alter electromagnetic behavior, especially if one is interested in near-field electromagnetic behavior.

Minkowski was the first to take a major step toward a relativistically consistent description of electrodynamics in moving matter in 1908 [3]. Minkowski reformulated macroscopic electrodynamics within four-dimensional spacetime, embedding electromagnetic fields and material responses in a covariant framework. In this theory, the motion of the medium directly influences electromagnetic excitation fields because the velocity of the medium is intrinsically included in constitutive equations; thus, Minkowski's formulation provides a conceptual starting point for understanding electrodynamics in moving continua [4, 5]. However, it primarily addresses uniform translational motion and does not fully incorporate internally driven mechanical processes such as strain, stress, rotation, dynamic

contact, or structural deformation. In modern mechano-driven media systems, mechanical action is not merely passive motion through space but an active source of electromagnetic generation and modulation. Mechanical degrees of freedom including deformation fields, strain gradients, rotational dynamics, and interfacial contact processes can directly induce polarization, magnetization, charge redistribution, and current generation.

Landau and Lifshitz systematically presented the macroscopic and continuum treatment of electromagnetic fields in matter [6], which remains the standard reference for moving dielectrics, stress tensors, and boundary conditions. Goto et al. in 2011 analyzed extensions to accelerated and rotating systems directly relevant to mechano-driven media in the covariant treatment of inhomogeneous rotating media [7]. In addition, de Groot and Suttrop systematically examined the coupling between electromagnetic fields and mechanical continua in the relativistic continuum framework [8], establishing the theoretical basis for incorporating deformation, strain, and stress into electromagnetic field equations. Ridgely used covariant field equations to derive general field equations in a rotating coordinate system [9]. Constitutive equations are then derived in the rotating and laboratory reference frames. Together, these works form the essential theoretical background, upon which the expanded Minkowski approach for mechano-driven media systems is constructed. These approaches mainly use Minkowski's approach without expanding Maxwell's equations even when the dielectric medium has accelerated motion.

Recently, Wang proposed a theory based on Maxwell's equations for a mechano-driven media system (MEs-f-MDMS) to

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systematically incorporate mechanical state variables into the electromagnetic framework [10–12]. In this generalized description, the material response depends not only on electromagnetic field variables and bulk velocity but also on non-equilibrium mechanical driving (accelerated motion). Therefore, mechanical excitation is elevated to the same theoretical status as electromagnetic excitation, leading to a unified mechano-electromagnetic field theory. As a result of the three-field coupling, the ME-f-MDMS is not format-invariant under Lorentz transformation.

Although the derivation of the ME-f-MDMS was reported in [10–12], solving this set of equations requires constitutive relations and associated boundary conditions to be precisely defined. This is important for applying the numerical solutions of the MEs-f-MDMS for engineering applications. In this study, by using the Minkowski approach, we derive constitutive equations and associated boundary conditions for the MEs-f-MDMS, providing a rigorous theoretical system for describing electromagnetic phenomena in dynamically driven media, so that the theory is complete and ready to be used for numerical calculations. This formulation not only advances the fundamental understanding of electrodynamics in non-equilibrium matter but also establishes a theoretical foundation for technologies based on mechanically driven electromagnetic processes, including deformable dielectrics, triboelectric systems, and other dynamic energy conversion platforms.

2. LORENTZ TRANSFORMATION FOR TREATING THE ELECTROMAGNETISM OF POINT CHARGES IN FREE-SPACE

In classical electrodynamics, because Maxwell's equations were first derived for a stationary medium system, the electromagnetism for a moving observer in free space is treated using the Lorentz transformation by assuming the form-invariance of Maxwell's equations, which means that the form of Maxwell's equations is preserved in two inertial reference frames that have a relatively uniform movement. This is the basic idea for dealing with the electromagnetic behavior in a moving reference frame by considering only the electromagnetic coupling [13].

The electromagnetic behavior of a moving observer is rather easy to determine in free space where there are no finite-size objects but distributed point charges; thus, no boundary conditions need to consider except at infinity. In Lorentz transformations, the invariance of the speed of light in free space is assumed. Therefore, electromagnetic fields observed in a moving frame (S') can be derived from those in the laboratory reference frame (S) by preserving the form-invariance of Maxwell's equations using the Lorentz transformation. Therefore, the electromagnetic phenomenon of a point charge in free space is observed by two independent observers in two inertial reference frames, which have a relative movement at a constant speed. Now we assume that an observer named Alice is in a moving inertial frame S' that moves at velocity v_0 relative to the non-moving frame S (lab frame), as shown in Fig. 1. If a point charge $+q$ is at rest in the Lab frame S , Bob in the Lab frame observes only

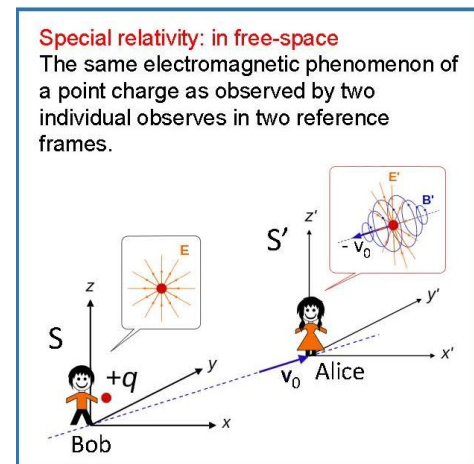


FIGURE 1. Classical approach for treating the electromagnetic behavior of point charges in free space as observed by two observers who have relative uniform motion using Lorentz transformation. This is the basis of special relativity.

a Coulomb field. As observed by Alice in the moving frame S' , the point charge is moving at a relative velocity of $-v_0$, so that she detects not only an electric field but also a magnetic field caused by the moving charge $+q$. The two relatively moving observers see different electromagnetic behaviors although the event is the same. To correlate the two sets of fields observed by Bob and Alice, a coordinate transformation is usually adopted. For simplicity, we assume that the moving direction is along the x -axis, and the Lorentz transformation is given as

$$x' = \gamma(x - v_0 t) \quad (1a)$$

$$y' = y \quad (1b)$$

$$z' = z \quad (1c)$$

$$t' = \gamma(t - x v_0 / c^2) \quad (1d)$$

where

$$\gamma = 1 / (1 - v_0^2 / c^2)^{1/2}, \quad (2a)$$

and

$$c = 1 / \sqrt{\epsilon_0 \mu_0} \quad (2b)$$

is the speed of light in free space, where ϵ_0 and μ_0 are electric and magnetic permittivities in free space, respectively.

3. ELECTRODYNAMICS OF A UNIFORMLY MOVING OBJECT — THE MINKOWSKI THEORY

3.1. General Approach

We now consider the case in which an observer is in the laboratory frame, and a medium/object with a finite size is moving at a uniform velocity (Fig. 2). In the moving frame affixed to the medium in the S' frame, in which the medium is stationary, the local fields satisfy the classical Maxwell's equations:

$$\nabla' \cdot \mathbf{D}' = \rho' \quad (3a)$$

$$\nabla' \cdot \mathbf{B}' = 0 \quad (3b)$$

$$\nabla' \times \mathbf{E}' = -\frac{\partial}{\partial t'} \mathbf{B}' \quad (3c)$$

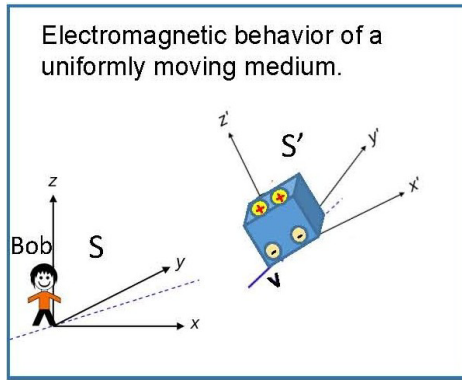


FIGURE 2. Schematic diagram showing the coordinate system for treating the electrodynamics of a uniformly moving dielectric medium in two inertial reference frames.

$$\nabla' \times \mathbf{H}' = \mathbf{J}' + \frac{\partial}{\partial t'} \mathbf{D}' \quad (3d)$$

For simplicity of derivation, we assume that the medium discussed in this paper is linear, isotropic, homogeneous, and non-dispersive. Therefore, the constitutive equations for a stationary medium case are given by:

$$\mathbf{B}' = \mu \mathbf{H}' \quad (4a)$$

$$\mathbf{D}' = \epsilon \mathbf{E}' \quad (4b)$$

where ϵ and μ are the electric permittivity and magnetic permeability, respectively, of the dielectric medium, which are assumed constant and independent of the medium's motion.

In the laboratory frame, the electromagnetic behavior is also assumed to be governed by Maxwell's equations if the medium's velocity is constant:

$$\nabla \cdot \mathbf{D} = \rho \quad (5a)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (5b)$$

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \mathbf{B} \quad (5c)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial}{\partial t} \mathbf{D} \quad (5d)$$

It means that the form-invariance of Maxwell's equations in inertial reference frames was also assumed in the Minkowski theory.

In general, the electromagnetic behavior observed in the Lab frame $(\mathbf{r}, t)$ ($\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H}$) and those fields in the moving frame affixed to the medium $(\mathbf{r}', t')$ ($\mathbf{E}', \mathbf{B}', \mathbf{D}', \mathbf{H}'$) are correlated by applying the Lorentz transformation to the Maxwell's equations in the moving frame (Eqs. (3a)–(3d)) and construct the final expression in the format of Maxwell's equations in the Lab frame (Eqs. (5a)–(5d)). For simplicity, we assume that the moving direction is along the x -axis, and the Lorentz transformation is

$$\frac{\partial}{\partial t'} = \gamma \left(\frac{\partial}{\partial t} + \mathbf{v}_0 \frac{\partial}{\partial x} \right) \quad (6a)$$

$$\frac{\partial}{\partial x'} = \gamma \left(\frac{\partial}{\partial x} + \mathbf{v}_0/c^2 \frac{\partial}{\partial t} \right) \quad (6b)$$

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y} \quad (6c)$$

$$\frac{\partial}{\partial z'} = \frac{\partial}{\partial z} \quad (6d)$$

After some mathematical work on the four equations, the fields in parallel and perpendicular to the moving direction of the reference frame are correlated by

$$\mathbf{E}'_{//} = \mathbf{E}_{//}, \quad \mathbf{B}'_{//} = \mathbf{B}_{//}, \quad \mathbf{D}'_{//} = \mathbf{D}_{//}, \quad \mathbf{H}'_{//} = \mathbf{H}_{//}, \quad (7a)$$

$$\mathbf{E}'_{\perp} = \gamma (\mathbf{E} + \mathbf{v}_0 \times \mathbf{B})_{\perp} \quad (7b)$$

$$\mathbf{B}'_{\perp} = \gamma (\mathbf{B} - \mathbf{v}_0 \times \mathbf{E}/c^2)_{\perp} \quad (7c)$$

$$\mathbf{D}'_{\perp} = \gamma (\mathbf{D} + \mathbf{v}_0 \times \mathbf{H}/c^2)_{\perp} \quad (7d)$$

$$\mathbf{H}'_{\perp} = \gamma (\mathbf{H} - \mathbf{v}_0 \times \mathbf{D})_{\perp} \quad (7e)$$

$$\mathbf{J}'_{//} = \gamma (\mathbf{J}_{//} - \rho \mathbf{v}_0) \quad (7f)$$

$$\mathbf{J}'_{\perp} = \mathbf{J}_{\perp} \quad (7g)$$

$$\rho' = \gamma (\rho - \mathbf{v}_0 \cdot \mathbf{J}/c^2) \quad (7h)$$

It should be noted that Maxwell's equations can truly be made form-invariant under the Lorentz transformation in free space, just as originally assumed by Minkowski, if the object is moving at a constant velocity $\mathbf{v}_0$.

The above theory was first proposed by Minkowski for a moving object that moves with a uniform velocity by assuming the form-invariance of Maxwell's equations. Because the fields satisfy the superposition principle, the fields generated by several moving objects can be solved individually and then added together, provided that the velocity of each object is uniform.

However, for an object that moves with acceleration, such as a rotating disc, the form-invariance of Maxwell's equations may not hold, as first pointed out by Feynman in his famous "anti-flux" examples [14]. These discussions are presented in Section 4.

3.2. The Constitutive Equations

In electromagnetic theory, there are four quantities: electric field $\mathbf{E}$, electric displacement vector $\mathbf{D}$, magnetic field $\mathbf{H}$, and magnetic induction $\mathbf{B}$. The equations that govern these relationships are called constitutive equations. We now derive constitutive equations for a moving medium. Eqs. (7a)–(7d) can be more generally written as [15]:

$$\mathbf{E}' = \gamma \left[\mathbf{E} - \frac{(\gamma - 1)}{\gamma} \frac{\mathbf{v}_0 (\mathbf{v}_0 \cdot \mathbf{E})}{v_0^2} + \mathbf{v}_0 \times \mathbf{B} \right] \quad (8a)$$

$$\mathbf{B}' = \gamma \left[\mathbf{B} - \frac{(\gamma - 1)}{\gamma} \frac{\mathbf{v}_0 (\mathbf{v}_0 \cdot \mathbf{B})}{v_0^2} - \frac{1}{c^2} \mathbf{v}_0 \times \mathbf{E} \right] \quad (8b)$$

$$\mathbf{D}' = \gamma \left[\mathbf{D} - \frac{(\gamma - 1)}{\gamma} \frac{\mathbf{v}_0 (\mathbf{v}_0 \cdot \mathbf{D})}{v_0^2} + \frac{1}{c^2} \mathbf{v}_0 \times \mathbf{H} \right] \quad (8c)$$

$$\mathbf{H}' = \gamma \left[\mathbf{H} - \frac{(\gamma - 1) \mathbf{v}_0 (\mathbf{v}_0 \cdot \mathbf{H})}{\gamma \mathbf{v}_0^2} - \mathbf{v}_0 \times \mathbf{D} \right] \quad (8d)$$

Substituting Eqs. (8a)–(8d) into constitutive Eqs. (4a)–(4b) in the moving frame:

$$\begin{aligned} & \mathbf{D} - \frac{(\gamma - 1) \mathbf{v}_0 (\mathbf{v}_0 \cdot \mathbf{D})}{\gamma \mathbf{v}_0^2} + \frac{1}{c^2} \mathbf{v}_0 \times \mathbf{H} \\ &= \varepsilon \left[\mathbf{E} - \frac{(\gamma - 1) \mathbf{v}_0 (\mathbf{v}_0 \cdot \mathbf{E})}{\gamma \mathbf{v}_0^2} + \mathbf{v}_0 \times \mathbf{B} \right] \end{aligned} \quad (9a)$$

$$\begin{aligned} & \mathbf{B} - \frac{(\gamma - 1) \mathbf{v}_0 (\mathbf{v}_0 \cdot \mathbf{B})}{\gamma \mathbf{v}_0^2} - \frac{1}{c^2} \mathbf{v}_0 \times \mathbf{E} \\ &= \mu \left[\mathbf{H} - \frac{(\gamma - 1) \mathbf{v}_0 (\mathbf{v}_0 \cdot \mathbf{H})}{\gamma \mathbf{v}_0^2} - \mathbf{v}_0 \times \mathbf{D} \right] \end{aligned} \quad (9b)$$

To solve Eqs. (9a)–(9b), we perform the following mathematical operations. The scalar product of $\mathbf{v}_0$ in Eqs. (9a)–(9b), we have

$$\mathbf{v}_0 \cdot \mathbf{D} = \varepsilon \mathbf{v}_0 \cdot \mathbf{E} \quad (10a)$$

$$\mathbf{v}_0 \cdot \mathbf{B} = \mu \mathbf{v}_0 \cdot \mathbf{H} \quad (10b)$$

Using Eqs. (10a)–(10b), Eqs. (9a)–(9b) are simplified as

$$\mathbf{D} + \frac{1}{c^2} \mathbf{v}_0 \times \mathbf{H} = \varepsilon (\mathbf{E} + \mathbf{v}_0 \times \mathbf{B}) \quad (11a)$$

$$\mathbf{B} - \frac{1}{c^2} \mathbf{v}_0 \times \mathbf{E} = \mu (\mathbf{H} - \mathbf{v}_0 \times \mathbf{D}) \quad (11b)$$

By making a cross of $\mathbf{v}_0 \times$ on Eqs. (11a)–(11b) and using the vector calculation formula:

$$\mathbf{v}_0 \times (\mathbf{v}_0 \times \mathbf{a}) = \mathbf{v}_0 (\mathbf{v}_0 \cdot \mathbf{a}) - \mathbf{v}_0^2 \mathbf{a},$$

for a general vector $\mathbf{a}$, constitutive equations can be received [15]:

$$\begin{aligned} \mathbf{D} &= \frac{1}{1 - \varepsilon \mu \mathbf{v}_0^2} \left\{ \left(1 - \frac{\mathbf{v}_0^2}{c^2} \right) \varepsilon \mathbf{E} + \left(\varepsilon \mu - \frac{1}{c^2} \right) \right. \\ &\quad \left. [\mathbf{v}_0 \times \mathbf{H} - \varepsilon \mathbf{v}_0 (\mathbf{v}_0 \cdot \mathbf{E})] \right\} \end{aligned} \quad (12a)$$

$$\begin{aligned} \mathbf{B} &= \frac{1}{1 - \varepsilon \mu \mathbf{v}_0^2} \left\{ \left(1 - \frac{\mathbf{v}_0^2}{c^2} \right) \mu \mathbf{H} - \left(\varepsilon \mu - \frac{1}{c^2} \right) \right. \\ &\quad \left. [\mathbf{v}_0 \times \mathbf{E} - \mu \mathbf{v}_0 (\mathbf{v}_0 \cdot \mathbf{H})] \right\} \end{aligned} \quad (12b)$$

It is interesting to note a factor of $(\varepsilon \mu - \frac{1}{c^2}) = (\frac{1}{c_m^2} - \frac{1}{c^2})$ appearing in the equation. This means that both the speeds of light in free space and in the medium appear in the same equation. This is a consequence of using the Lorentz transformation developed for free space to treat the electromagnetic behavior in a moving dielectric medium.

3.3. Low-Speed Approximated Results

To make discussions simple, we now consider the results under low-speed approximation, $\mathbf{v}_0 \ll c$, so that $\gamma \approx 1$. From Eqs. (8a)–(8d), the fields in the two reference frames are related approximately by [16]:

$$\mathbf{E}' \approx \mathbf{E} + \mathbf{v}_0 \times \mathbf{B} \quad (13a)$$

$$\mathbf{B}' \approx \mathbf{B} - \mathbf{v}_0 \times \mathbf{E}/c^2 \quad (13b)$$

$$\mathbf{D}' \approx \mathbf{D} + \mathbf{v}_0 \times \mathbf{H}/c^2 \quad (13c)$$

$$\mathbf{H}' \approx \mathbf{H} - \mathbf{v}_0 \times \mathbf{D} \quad (13d)$$

$$\mathbf{J}' \approx \mathbf{J} - \rho \mathbf{v}_0 \quad (13e)$$

$$\rho' \approx \rho - \mathbf{v}_0 \cdot \mathbf{J}/c^2 \quad (13f)$$

The corresponding constitutive equations are:

$$\mathbf{D} \approx \varepsilon \mathbf{E} + \alpha \varepsilon \mu [\mathbf{v}_0 \times \mathbf{H} - \varepsilon \mathbf{v}_0 (\mathbf{v}_0 \cdot \mathbf{E})] \quad (14a)$$

$$\mathbf{B} = \mu \mathbf{H} - \alpha \varepsilon \mu [\mathbf{v}_0 \times \mathbf{E} - \mu \mathbf{v}_0 (\mathbf{v}_0 \cdot \mathbf{H})] \quad (14b)$$

where we define a factor:

$$\alpha = 1 - \frac{\varepsilon_0 \mu_0}{\varepsilon \mu} = 1 - c_m^2/c^2 \quad (14c)$$

4. DERIVING MAXWELL'S EQUATIONS FOR A MECHANO-DRIVEN MEDIUM SYSTEM (MES-F-MDMS)

4.1. Galilean Space and Time

All of the original theories for expressing the physics laws were first given in Galilean space and time, in which space and time are independent:

$$x' = x - \mathbf{v}_0 t \quad (15a)$$

$$y' = y \quad (15b)$$

$$z' = z \quad (15c)$$

$$t' = t \quad (15d)$$

Later, after Einstein, the Lorentz transformation was used to derive electromagnetic theory in general. The Galilean and Lorentz transformations are inconsistent because they represent the classical and new space-time relationships, respectively. However, it has been proven that the Lorentz transformation is equivalent to the Galilean transformation under two approximations [17, 18]:

1. The relative speed between two inertial frames of reference is much smaller than the speed of light in vacuum: $\mathbf{v}_0 \ll c$, so $\gamma \approx 1$;
2. Electromagnetic phenomena occur in an arena whose spatial extension is much smaller than the distance traveled by light during the phenomenon: $x \ll ct$. From Eq. (1d), under the approximation of $\mathbf{v}_0 \ll c$, the Lorentz transformation for time becomes $t' \approx \left(t - \frac{x \mathbf{v}_0}{c^2} \right) = \frac{1}{c} \left(ct - x \frac{\mathbf{v}_0}{c} \right)$.

For time becomes $t' \approx \left(t - \frac{x \mathbf{v}_0}{c^2} \right) = \frac{1}{c} \left(ct - x \frac{\mathbf{v}_0}{c} \right)$.

If $x \ll ct$, the space-caused correction to time is so small that it can be ignored completely, resulting in $t' \approx t$, the result of the Galilean transformation.

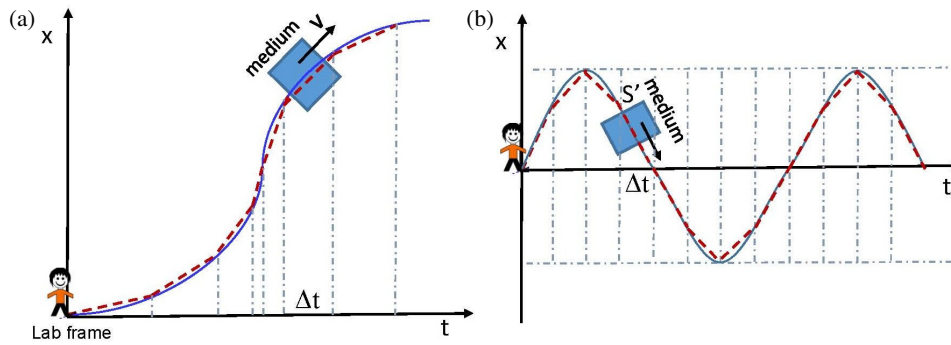


FIGURE 3. Approach for treating the electrodynamic behavior of an accelerated moving medium by using an integration of a segmented path, the movement velocity in each segment can be approximated as a constant so that we can use the constitutive relations derived using the Lorentz transformation for this specific moving velocity. (a) The medium has both translation motion and rotation motion; (b) the medium only has rotation motion.

These two conditions are usually met in engineering electrodynamics, so that the Galilean transformation can be used safely.

4.2. Expansion of the Minkowski Theory to a Low-Speed Moving Object with Acceleration

We now consider a case in which an observer is in the lab frame, and a medium/object with finite size is in accelerated motion, which occurs frequently in engineering electromagnetism, such as in power generators, wind blades, and extremely low-frequency electromagnetic wave generation. As presented in Fig. 3, we assume that the object can have accelerated motion, but its motion within a small time interval Δt can be viewed as moving at an instantaneous constant speed, so that we can treat the entire motion segment by segment, as illustrated in the trajectory of the moving medium. This is the general idea of using a straight line to represent an arc curve in calculus if the segment is short. In the moving frame affixed to the medium in the S' frame, the medium is momentarily stationary. Therefore, the field observed in a temporary reference frame S' with those in the lab frame can still be approximately correlated to the Minkowski theory, replacing $\mathbf{v}_0$ with the object's total moving speed $\mathbf{v}_t$ in Eqs. (13a)–(13f), and local fields are correlated as follows:

$$\mathbf{E}' \approx \mathbf{E} + \mathbf{v}_t \times \mathbf{B} \quad (16a)$$

$$\mathbf{B}' \approx \mathbf{B} - \mathbf{v}_t \times \mathbf{E}/c^2 \quad (16b)$$

$$\mathbf{D}' \approx \mathbf{D} + \mathbf{v}_t \times \mathbf{H}/c^2 \quad (16c)$$

$$\mathbf{H}' \approx \mathbf{H} - \mathbf{v}_t \times \mathbf{D} \quad (16d)$$

$$\mathbf{J}' \approx \mathbf{J} - \rho \mathbf{v}_t \quad (16e)$$

$$\rho' \approx \rho - \mathbf{v}_t \cdot \mathbf{J}/c^2 \quad (16f)$$

The total velocity $\mathbf{v}_t$ includes the translation velocity of the moving frame and the accelerated motion of the point charge in reference to the moving frame, which will be elaborated in more detail as given in Eq. (21). The Lorentz transformation theory has been extrapolated to treat the electrodynamics of a moving medium/object by assuming the following: 1. The applicability of the Lorentz transformation in the presence of a medium, which means the invariance of the speed of light; 2.

Form-invariance of Maxwell's equations in the presence of a medium.

We need to point out that the two assumptions are justifiable hypotheses and may not be exact as in the case of a medium. For example, the invariance of the speed of light holds in a vacuum but not in a medium, and in water, the speed of red light differs from that of green light. Such dispersion relations are ignored in relativity. Therefore, special relativity is exact in free space, but it may be subject to minor corrections in realistic medium cases.

4.3. Faraday's Law for an Accelerated Moving Medium

Faraday's law is the fundamental law for electromagnetic generators, which must have an external input energy to accelerate the magnets. The derivation of the ME-f-MDMS was systematically described previously. We now use coordinate transformations to re-derive the equations. A mathematical identity is used for calculations in Eq. (18): for a general function $\mathbf{G}(\mathbf{r}, t)$, whose boundary surface moves at a velocity field $\mathbf{v}(\mathbf{r}, t)$ [19], we have

$$\frac{d}{dt} \iint_C \mathbf{G} \cdot d\mathbf{s} = \iint_C \left[\frac{\partial}{\partial t} \mathbf{G} + (\mathbf{G} \cdot \nabla) \mathbf{v} + (\nabla \cdot \mathbf{G}) \mathbf{v} - (\nabla \cdot \mathbf{v}) \mathbf{G} \right] \cdot d\mathbf{s} - \oint_C (\mathbf{v} \times \mathbf{G}) \cdot d\mathbf{L} \quad (17)$$

We first start from Lenz's law to derive Faraday's law, which was derived from experiments. The physics law is expressed as follows (Eq. (18)) in integral form is usually given in Galilean space-time, so that it can be treated independently of the other terms. For a closed circuit that contains a large medium/object in its loop as shown in Fig. 4, the electromotive force around the circuit loop is the rate of change the magnetic flux [20]:

$$\begin{aligned} \xi_{\text{EMF}} &= -\frac{d}{dt} \iint_{s(t)} \mathbf{B} \cdot d\mathbf{s} = -\iint_{s(t)} \left[\frac{\partial}{\partial t} \mathbf{B} \cdot d\mathbf{s} + \mathbf{B} \cdot \frac{\partial}{\partial t} d\mathbf{s} \right] \\ &= -\iint_{s(t)} \left\{ \frac{\partial}{\partial t} \mathbf{B} - \nabla \times [\mathbf{v} \times \mathbf{B}] \right\} \cdot d\mathbf{s} \end{aligned} \quad (18)$$

where $\mathbf{v}$ is the translational velocity of the boundary surface $\mathbf{s}(t)$ of the medium for calculating the magnetic flux, and it depends

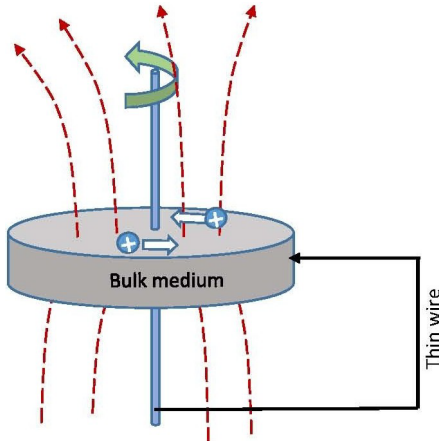


FIGURE 4. A schematic diagram showing a closed circuit loop that contains a rotating conductive disc with the contacting wire sliding on the surface of the disc. The disc may have surface charges and is rotating in an electromagnetic field. The electromagnetic behavior inside the disc is governed by the MEs-f-MDMS.

only on time, $\mathbf{v}(t)$, so that $(\mathbf{G} \cdot \nabla)\mathbf{v} = 0$, $(\nabla \cdot \mathbf{v})\mathbf{G} = 0$, and we use $\nabla \cdot \mathbf{B} = 0$.

The electromotive force can be calculated in the moving S' frame as the circulation of the electric field [21];

$$\xi_{\text{EMF}} = \oint_C \mathbf{E}' \cdot d\mathbf{L} = - \iint_{s(t)} \left\{ \frac{\partial}{\partial t} \mathbf{B} - \nabla \times [\mathbf{v} \times \mathbf{B}] \right\} \cdot d\mathbf{s} \quad (19)$$

where $\mathbf{E}'$ represents the electric field in the rest frame of each segment $d\mathbf{L}$ of the integration path on the circuit, which is stationary. The Lorentz force acting on a point charge q in the lab frame is $\mathbf{F} = q(\mathbf{E} + \mathbf{v}_t \times \mathbf{B})$, and the force acting on the charge in the moving frame is $\mathbf{F}' = q\mathbf{E}'$, where $\mathbf{v}_t$ is the total moving velocity of the point charge, which may be different from the moving velocity $\mathbf{v}$ of the circuit loop because of the motion of the point charge inside the large medium depending on its relative rotation/trajectory. If the inertia force is ignored, we may have $\mathbf{F}' \approx \mathbf{F}$. Therefore, the local electric field in the moving frame is related to that in the Lab frame for the instantaneous inertia frame as follows [22]:

$$\mathbf{E}' \approx \mathbf{E} + \mathbf{v}_t \times \mathbf{B}. \quad (20)$$

The total velocity $\mathbf{v}_t$ of a point charge is the sum of the circuit surface velocity $\mathbf{v}$ and the relative motion of the charge inside the large dielectric medium $\mathbf{v}_r$. More specifically, $\mathbf{v}$ is assumed to be time-dependent so that it can be viewed as a “rigid translation” of the circuit, and $\mathbf{v}_r$ is the relative moving velocity of the point charge in the S' frame, which is space- and time-dependent [10, 11]

$$\mathbf{v}_t = \mathbf{v}(t) + \mathbf{v}_r(\mathbf{r}, t) \quad (21)$$

Substituting Eqs. (20) and (21) into Eq. (19), we have

$$\oint_C (\mathbf{E} + \mathbf{v}_r \times \mathbf{B}) \cdot d\mathbf{L} = - \iint_s \frac{\partial}{\partial t} \mathbf{B} \cdot d\mathbf{s} \quad (22)$$

For the integral above, two cases are considered along the circuit loop, as shown in Fig. 4.

The first case is a thin wire, such that the moving velocity $\mathbf{v}_r$ of a point charge inside the wire is parallel to the wire; that is, $\mathbf{v}_r$ is parallel to the integral path $d\mathbf{L}$, and then the term $[\mathbf{v}_r \times \mathbf{B}] \cdot d\mathbf{L}$ vanishes automatically. In this case, using Stokes theorem, Faraday’s law is

$$\nabla \times \mathbf{E} = - \frac{\partial}{\partial t} \mathbf{B}. \quad (23)$$

This means that the standard form of Faraday’s law in the differential form is valid for all thin-wire segments. If the wire is thin enough, it can be considered outside the medium, in which case classical Maxwell’s equations are exact. Theoretically, a thin-wire circuit is a virtual circuit with an arbitrarily imaged shape. If the imaginary circuit can be expanded to any space, the thin-circuit assumption holds exactly if the circuit does not intercept any conductive medium boundary.

The second case is the presence of a large conductive medium, in which the relative velocity of the unit charge is not parallel to the integral path, such that $\oint_C [\mathbf{v}_r \times \mathbf{B}] \cdot d\mathbf{L} \neq 0$. Such cases occur in engineering because of the size and shape of the rotating media. When the distance between electromagnetic fields is considered, the medium size may not be negligible. This was first presented by Feynman as the “anti-flux” example arising from the rotation of a metal disk in a magnetic field. Now, we use Stokes theorem, and Eq. (22) becomes [11]

$$\nabla \times (\mathbf{E} + \mathbf{v}_r \times \mathbf{B}) = - \frac{\partial}{\partial t} \mathbf{B} \quad (24)$$

The term $\mathbf{v}_r \times \mathbf{B}$ represents the charge’s local Lorentz force that moves inside a moving medium. This is a key addition to this equation. It must be noted that this term was not present in Minkowski’s theory even for an accelerated moving object because form invariance was assumed in the first place.

4.4. The Ampere-Maxwell Law

Similarly, medium motion can be introduced in the Ampere-Maxwell law based on the same logic. In frame S' , the circulation of the magnetic field $\mathbf{H}'$ along a circuit is the total current, including conduction and displacement currents, which is given by

$$\oint_C \mathbf{H}' \cdot d\mathbf{L} \approx \iint_s \mathbf{J} \cdot d\mathbf{s} + \frac{d}{dt} \iint_s \mathbf{D} \cdot d\mathbf{s} \quad (25)$$

where $\mathbf{H}'$ is the magnetic field in the moving medium in the frame in which $d\mathbf{L}$ is at rest. Ampere-Maxwell’s law states that the circulation of the magnetic field around its looped edge is the total conductive current through an open surface plus the increasing rate of the electric flux through an open surface (e.g., the displacement current). Using the flux theorem in Eq. (17), assuming that the surface movement used for calculating the electric flux is a solid translation, $\mathbf{v}(t)$, and $\nabla \cdot \mathbf{D} \approx \rho$, we have

$$\oint_C \mathbf{H}' \cdot d\mathbf{L} = \iint_s (\mathbf{J} + \rho \mathbf{v}) \cdot d\mathbf{s} + \iint_s \frac{\partial}{\partial t} \mathbf{D} \cdot d\mathbf{s} - \oint_C (\mathbf{v} \times \mathbf{D}) \cdot d\mathbf{L} \quad (26)$$

Under a low-speed approximation, $D \gg |\mathbf{v}_t \times \mathbf{H}|/c^2$, we have $\mathbf{H}' \approx \mathbf{H} - \mathbf{v}_t \times \mathbf{D}$, and using Stokes theorem, under the first-order approximation, we have [10, 11]

$$\nabla \times (\mathbf{H} - \mathbf{v}_r \times \mathbf{D}) \approx \mathbf{J} + \rho \mathbf{v} + \frac{\partial}{\partial t} \mathbf{D} \quad (27)$$

Here, we only keep the velocity's first-order terms in the equation. The term $\mathbf{v}_r \times \mathbf{D}$ is the electric-field-induced magnetization due to media movement. This term is not present in Minkowski's theory even for an accelerated moving object because form-invariance was assumed in the first place.

4.5. The MES-f-MDMS

Now, putting all derivations together, the electrodynamic behavior inside an accelerated moving medium can be described by (see Fig. 5) [11, 12]

$$\nabla \cdot \mathbf{D} \approx \rho \quad (28a)$$

$$\nabla \cdot \mathbf{B} \approx 0 \quad (28b)$$

$$\nabla \times (\mathbf{E} + \mathbf{v}_r \times \mathbf{B}) \approx -\frac{\partial}{\partial t} \mathbf{B} \quad (28c)$$

$$\nabla \times (\mathbf{H} - \mathbf{v}_r \times \mathbf{D}) \approx \mathbf{J} + \rho \mathbf{v} + \frac{\partial}{\partial t} \mathbf{D} \quad (28d)$$

The charge conservation law is:

$$\nabla \cdot (\mathbf{J} + \rho \mathbf{v}) + \frac{\partial}{\partial t} \rho = 0 \quad (28e)$$

The Gauss law for electricity and magnetism (Eqs. (28a)–(28b)) holds if one ignores the high-order terms in velocity. MES-f-MDMS describes the coupling of force fields among electric, magnetic, and mechanical fields. The traditional Maxwell's

equations consider only the electromagnetic coupling in inertial reference frame.

The conditions under which Eqs. (28a)–(28d) were derived as follows: (1) low-speed approximation and keeping only first-order terms of the moving speed in the equations, and (2) $B \gg \mathbf{v}_t E/c^2$ (for Eq. (28b)) and $D \gg \mathbf{v}_t H/c^2$ (for Eq. (28d)), so that $\mathbf{D}' \approx \mathbf{D}$ and $\mathbf{B}' \approx \mathbf{B}$, thus, relativistic corrections made in $\mathbf{D}$ and $\mathbf{B}$ were ignored. This is the result of the Galilean transformation. The two conditions are equivalent to $\epsilon\mu/\epsilon_0\mu_0 \gg (\mathbf{v}_t/c)^2$, which means that these approximations are excellent in practice.

We now consider two special cases. Outside the medium in free space, $\mathbf{v}_r = 0$ and $\mathbf{v} = 0$, the MES-f-MDMS resumes the standard format of Maxwell's equations. Alternatively, if an object moves with uniform velocity without acceleration, such that $\mathbf{v}_r = 0$, $\mathbf{v} = \text{constant}$, the MES-f-MDMS decreases to the standard format of Maxwell's equations. This agrees with Minkowski's approach. Therefore, MES-f-MDMS is entirely consistent with Maxwell's equations under the considered conditions.

5. THE CONSTITUTIVE EQUATIONS FOR A MOVING MEDIUM THAT MOVES WITH ACCELERATION

We now derive constitutive equations for an accelerated moving medium. In the moving reference frame affixed to the medium, the conventional constitutive relationship in the S' frame is given by Eqs. (14a)–(14b). Replacing $\mathbf{v}_0$ by $\mathbf{v}_t$, we have the constitutive equations for MES-f-MDMS:

$$\begin{aligned} \mathbf{D} &\approx \epsilon \mathbf{E} + \alpha \epsilon \mu [\mathbf{v}_t \times \mathbf{H} - \epsilon \mathbf{v}_t (\mathbf{v}_t \cdot \mathbf{E})] \\ &\approx \epsilon \mathbf{E} + \alpha \epsilon \mu \mathbf{v}_t \times \mathbf{H} \end{aligned} \quad (29a)$$

$$\begin{aligned} \mathbf{B} &\approx \mu \mathbf{H} - \alpha \epsilon \mu [\mathbf{v}_t \times \mathbf{E} - \mu \mathbf{v}_t (\mathbf{v}_t \cdot \mathbf{H})] \\ &\approx \mu \mathbf{H} - \alpha \epsilon \mu \mathbf{v}_t \times \mathbf{E} \end{aligned} \quad (29b)$$

Please note that Eqs. (29a)–(29b) are given using the Lorentz transform for free space, in which the speed of light is assumed to be the speed in free space rather than that in the medium, that is, $c_m = 1/\sqrt{\epsilon\mu}$.

6. THE BOUNDARY CONDITIONS FOR MES-F-MDMS

As elaborated in Section 3, in the moving frame affixed to the medium, the instantaneous stationary boundary viewed in the moving frame can be treated using the classical approach, as elaborated in the literature. The boundary refers to the material boundary, which is usually the surface of the medium. Because the boundary is a medium boundary that is stationary as viewed in the moving frame, the boundary conditions for fields ($\mathbf{E}'$, $\mathbf{B}'$, $\mathbf{D}'$, $\mathbf{H}'$) can be derived using the integral form of Maxwell's equations (Eq. (3)) around a small loop or volume in the vicinity of the boundary. Following the method given in [21], we can obtain the boundary conditions in the moving frame as

$$\mathbf{n}' \cdot (\mathbf{D}'_2 - \mathbf{D}'_1) = \sigma' \quad (30a)$$

$$\mathbf{n}' \cdot (\mathbf{B}'_2 - \mathbf{B}'_1) = 0 \quad (30b)$$

$$\mathbf{n}' \times (\mathbf{E}'_2 - \mathbf{E}'_1) = 0 \quad (30c)$$

$$\mathbf{n}' \times (\mathbf{H}'_2 - \mathbf{H}'_1) = \mathbf{K}'_s \quad (30d)$$

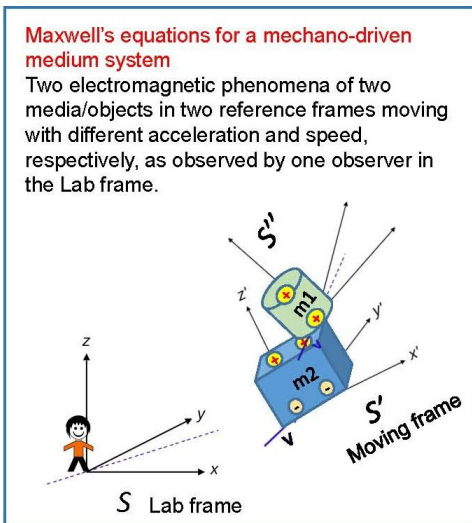


FIGURE 5. Schematic diagram shows the configuration for deriving the electrodynamics of moving media/objects that may have different movement velocities and accelerations using the MES-f-MDMS. Media M1 and M2 are affixed to the two moving reference frames S' and S'' , respectively.

where $\mathbf{n}'$ is the surface normal direction in the S' frame; σ' is the surface free charge density; $\mathbf{K}'_s$ is the surface current density; and subscripts 1 and 2 represent the two media adjacent to the boundary, as shown in Fig. 4.

However, boundary conditions for a moving medium may not be derived by following the classical approach of applying the integral form of Maxwell's equations around a small volume of a small loop at the boundary, because the boundary's configuration varies owing to medium motion. We now use the instantaneous coordinate transformation approach to derive the boundary conditions using the differential form of the equations. In the moving reference frame, boundary conditions are given by Eqs. (30a)–(30d), because the medium is instantaneously stationary. Therefore, we can substitute the field transformation as given in Eqs. (16a)–(16d) into (30a)–(30d). To illustrate the configuration clearly, Fig. 5 shows two moving frames, S' and S'' , which media m_1 and m_2 are affixed to, respectively. Both media can move with different velocities and may form a boundary. We use the constitutive equations derived under the Lorentz transformation to convert the boundary conditions in the lab frame. By retaining the first-order terms of v/c under the low-speed approximation, the boundary conditions between the two media are given by

$$\mathbf{n} \cdot [\mathbf{D}_2 - \mathbf{D}_1 + (\mathbf{v}_{t2} \times \mathbf{H}_2 - \mathbf{v}_{t1} \times \mathbf{H}_1)/c^2] \approx \sigma' \quad (31a)$$

$$\mathbf{n} \cdot [\mathbf{B}_2 - \mathbf{B}_1 - (\mathbf{v}_{t2} \times \mathbf{E}_2 - \mathbf{v}_{t1} \times \mathbf{E}_1)/c^2] \approx 0 \quad (31b)$$

$$\mathbf{n} \times [\mathbf{E}_2 - \mathbf{E}_1 + \mathbf{v}_{t2} \times \mathbf{B}_2 - \mathbf{v}_{t1} \times \mathbf{B}_1] \approx 0 \quad (31c)$$

$$\mathbf{n} \times [\mathbf{H}_2 - \mathbf{H}_1 - \mathbf{v}_{t2} \times \mathbf{D}_2 + \mathbf{v}_{t1} \times \mathbf{D}_1] \approx \mathbf{K}'_s \quad (31d)$$

and

$$\mathbf{K}'_s \approx \mathbf{K}_{s2} + \mathbf{K}_{s1} - (\sigma_2 \mathbf{v}_{t2} + \sigma_1 \mathbf{v}_{t1}) \quad (31e)$$

$$\sigma' \approx \sigma_2 + \sigma_1 - (\mathbf{v}_{t2} \cdot \mathbf{J}_2 + \mathbf{v}_{t1} \cdot \mathbf{J}_1)/c^2 \quad (31f)$$

where $\mathbf{n}$ is the surface normal direction that should be converted from $\mathbf{n}'$ using coordinate transformation; $\mathbf{K}'_s$ is the surface current density considering the contribution from the media adjacent to the boundary; subscripts 1 and 2 represent the corresponding quantities inside the dielectric media on the boundary's two sides; and $\mathbf{v}_{t1}$ and $\mathbf{v}_{t2}$ represent velocities of the two media, respectively, when they are in contact, sliding, or even deforming. The complete derivation of the constitutive relations, basic equations, and boundary conditions enables numerical calculations.

7. THE FIRST-ORDER APPROXIMATED RESULT OF MES-F-MDMS

The following discussions are provided by assuming that we only keep the first-order terms of $\mathbf{v}_t$ in the constitutive relations, MES-f-MDMS, and boundary conditions. From Eqs. (29a)–(29b), we obtain the first-order approximated result:

$$\mathbf{D} \approx \epsilon \mathbf{E} + \alpha \epsilon \mu \mathbf{v}_t \times \mathbf{H} \quad (32a)$$

$$\mathbf{B} \approx \mu \mathbf{H} - \alpha \epsilon \mu \mathbf{v}_t \times \mathbf{E} \quad (32b)$$

Both the speeds of light in free space and in the medium appear in the equation. This is a consequence of using the Lorentz transformation for free space to treat the electromagnetic behavior of a moving dielectric medium. The question is how

to expand the Lorentz transformation in space with finite-size media in the presence of boundaries.

Substituting Eqs. (32a)–(32b) into (28a)–(28d), the MES-f-MDMS inside the dielectric medium is

$$\epsilon \nabla \cdot (\mathbf{E} + \alpha \mu \mathbf{v}_t \times \mathbf{H}) \approx \rho \quad (33a)$$

$$\mu \nabla \cdot (\mathbf{H} - \alpha \epsilon \mathbf{v}_t \times \mathbf{E}) \approx 0 \quad (33b)$$

$$\nabla \times (\mathbf{E} + \mu \mathbf{v}_r \times \mathbf{H}) \approx -\mu \frac{\partial}{\partial t} (\mathbf{H} - \alpha \epsilon \mathbf{v}_t \times \mathbf{E}) \quad (33c)$$

$$\nabla \times (\mathbf{H} - \epsilon \mathbf{v}_r \times \mathbf{E}) \approx \mathbf{J} + \rho \mathbf{v} + \epsilon \frac{\partial}{\partial t} (\mathbf{E} + \alpha \mu \mathbf{v}_t \times \mathbf{H}) \quad (33d)$$

The solutions of Eqs. (33a)–(33d) can be obtained using the perturbation theory as presented in [12]. The corresponding boundary conditions under first-order approximation are given by Eqs. (26a)–(26d):

$$\mathbf{n} \cdot [\epsilon_2 \mathbf{E}_2 - \epsilon_1 \mathbf{E}_1 + \alpha_2 \epsilon_2 \mu_2 \mathbf{v}_{t2} \times \mathbf{H}_2 - \alpha_1 \epsilon_1 \mu_1 \mathbf{v}_{t1} \times \mathbf{H}_1] \approx \sigma' \quad (34a)$$

$$\mathbf{n} \cdot [\mu_2 \mathbf{H}_2 - \mu_1 \mathbf{H}_1 - \alpha_2 \epsilon_2 \mu_2 \mathbf{v}_{t2} \times \mathbf{E}_2 + \alpha_1 \epsilon_1 \mu_1 \mathbf{v}_{t1} \times \mathbf{E}_1] \approx 0 \quad (34b)$$

$$\mathbf{n} \times [\mathbf{E}_2 - \mathbf{E}_1 + \mu_2 \mathbf{v}_{t2} \times \mathbf{H}_2 - \mu_1 \mathbf{v}_{t1} \times \mathbf{H}_1] \approx 0 \quad (34c)$$

$$\mathbf{n} \times [\mathbf{H}_2 - \mathbf{H}_1 - \epsilon_2 \mathbf{v}_{t2} \times \mathbf{E}_2 + \epsilon_1 \mathbf{v}_{t1} \times \mathbf{E}_1] \approx \mathbf{K}'_s \quad (34d)$$

8. THE MES-F-MDMS FOR A ROTATING MEDIUM IN FREE SPACE

We now consider a medium rotating in free space, such as a disc-like rotating object. The electromagnetic fields inside the medium are governed by the MES-f-MDMS, as given by Eqs. (33a)–(33d) by taking $\mathbf{v} = 0$ and $\mathbf{v}_t = \mathbf{v}_r$:

$$\epsilon \nabla \cdot (\mathbf{E} + \alpha \mu \mathbf{v}_r \times \mathbf{H}) \approx \rho \quad (35a)$$

$$\mu \nabla \cdot (\mathbf{H} - \alpha \epsilon \mathbf{v}_r \times \mathbf{E}) \approx 0 \quad (35b)$$

$$\nabla \times (\mathbf{E} + \mu \mathbf{v}_r \times \mathbf{H}) \approx -\mu \frac{\partial}{\partial t} (\mathbf{H} - \alpha \epsilon \mathbf{v}_r \times \mathbf{E}) \quad (35c)$$

$$\nabla \times (\mathbf{H} - \epsilon \mathbf{v}_r \times \mathbf{E}) \approx \mathbf{J} + \epsilon \frac{\partial}{\partial t} (\mathbf{E} + \alpha \mu \mathbf{v}_r \times \mathbf{H}) \quad (35d)$$

The solution of Eqs. (35a)–(35d) will be presented in Section 10. Outside the dielectric medium, the electromagnetic fields are governed by

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho \quad (36a)$$

$$\mu_0 \nabla \cdot \mathbf{H} = 0 \quad (36b)$$

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial}{\partial t} \mathbf{H} \quad (36c)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \epsilon_0 \frac{\partial}{\partial t} \mathbf{E} \quad (36d)$$

The corresponding boundary conditions Eqs. (34a)–(34d) become:

$$\mathbf{n} \cdot [\epsilon_2 \mathbf{E}_2 - \epsilon_0 \mathbf{E}_1 + \alpha_2 \epsilon_2 \mu_2 \mathbf{v}_{r2} \times \mathbf{H}_2] \approx \sigma' \quad (37a)$$

$$\mathbf{n} \cdot [\mu_2 \mathbf{H}_2 - \mu_0 \mathbf{H}_1 - \alpha_2 \epsilon_2 \mu_2 \mathbf{v}_{r2} \times \mathbf{E}_2] \approx 0 \quad (37b)$$

$$\mathbf{n} \times [\mathbf{E}_2 - \mathbf{E}_1 + \mu_2 \mathbf{v}_{r2} \times \mathbf{H}_2] \approx 0 \quad (37c)$$

$$\mathbf{n} \times [\mathbf{H}_2 - \mathbf{H}_1 - \epsilon_2 \mathbf{v}_{r2} \times \mathbf{E}_2] \approx \mathbf{K}'_s \quad (37d)$$

9. THE MES-F-MDMS FOR HIGH REFRACTION INDEX MEDIUM WITH ROTATION MOTION

Now, we consider the case in which the medium's dielectric permittivity is rather large, such that $\varepsilon\mu \gg \varepsilon_0\mu_0$, such as in the cases of high-dielectric materials and permanent magnets. Then, $\alpha \approx 1$, and $\mathbf{v} = 0$, and the MEs-f-MDMS as shown in Eqs. (35a)–(35d) are simplified as [22]:

$$\varepsilon \nabla \cdot \mathbf{E}_{eff} \approx \rho \quad (38a)$$

$$\mu \nabla \cdot \mathbf{H}_{eff} \approx 0 \quad (38b)$$

$$\nabla \times \mathbf{E}_{eff} \approx -\mu \frac{\partial}{\partial t} \mathbf{H}_{eff} \quad (38c)$$

$$\nabla \times \mathbf{H}_{eff} \approx \mathbf{J} + \varepsilon \frac{\partial}{\partial t} \mathbf{E}_{eff} \quad (38d)$$

where effective fields are defined by:

$$\mathbf{E}_{eff} = \mathbf{E} + \mu \mathbf{v}_r \times \mathbf{H} \quad (39a)$$

$$\mathbf{H}_{eff} = \mathbf{H} - \varepsilon \mathbf{v}_r \times \mathbf{E} \quad (39b)$$

Equations (38a)–(38d) are identical to classical Maxwell's equations, except that the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ are replaced by effective fields $\mathbf{E}_{eff}$ and $\mathbf{H}_{eff}$, respectively. The electrodynamic behavior, as described by the effective fields, was described in detail previously. The condition for a valid effective medium is $\varepsilon\mu \gg \varepsilon_0\mu_0$, such as in permanent magnetic materials and ferroelectric or piezoelectric materials.

The corresponding boundary conditions under first-order approximation are given by:

$$\mathbf{n} \cdot [\varepsilon_2 \mathbf{E}_{eff,2} - \varepsilon_1 \mathbf{E}_{eff,1}] \approx \sigma' \quad (40a)$$

$$\mathbf{n} \cdot [\mu_2 \mathbf{H}_{eff,2} - \mu_1 \mathbf{H}_{eff,1}] \approx 0 \quad (40b)$$

$$\mathbf{n} \times [\mathbf{E}_{eff,2} - \mathbf{E}_{eff,1}] \approx 0 \quad (40c)$$

$$\mathbf{n} \times [\mathbf{H}_{eff,2} - \mathbf{H}_{eff,1}] \approx \mathbf{K}'_s \quad (40d)$$

The condition of $\varepsilon\mu \gg \varepsilon_0\mu_0$ applies to high refraction or permanent magnetic materials. For general materials, such conditions may not be strongly satisfied. The key test is to compare the results with experimental observations. One may include higher-order terms in Eqs. (40a)–(40d) if needed.

The condition of $\varepsilon\mu \gg \varepsilon_0\mu_0$ is equivalent to the total reflection of light inside a high-refraction medium, and it is most likely to be trapped in the medium. This means that the electromagnetic emission generated by motion of the medium inside the medium may not “escape the medium” due to total refraction, so that it is difficult to detect signals outside the medium. Therefore, we may build an electromagnetic fiber made of high-dielectric or magnetic materials so that low-frequency electromagnetic waves can be transported over long distances, similar to optical fibers. This can be an approach for transmitting electromagnetic waves for special applications, such as underwater.

10. SOLUTION OF THE MES-F-MDMS FOR A MEDIUM WITHOUT TRANSLATION MOTION

We now present an approach for solving Eqs. (35a)–(35d). We first define two quantities as follows:

$$\mathcal{E} = \mathbf{E} + \alpha \mu \mathbf{v}_r \times \mathbf{H} \quad (41a)$$

$$\mathcal{H} = \mathbf{H} - \alpha \varepsilon \mathbf{v}_r \times \mathbf{E} \quad (41b)$$

Using them, Eqs. (35a)–(35d) are written as

$$\varepsilon \nabla \cdot \mathcal{E} \approx \rho \quad (42a)$$

$$\mu \nabla \cdot \mathcal{H} \approx 0 \quad (42b)$$

$$\nabla \times \mathcal{E} \approx \mathbf{P}_M - \mu \frac{\partial}{\partial t} \mathcal{H} \quad (42c)$$

$$\nabla \times \mathcal{H} \approx \mathbf{P}_E + \varepsilon \frac{\partial}{\partial t} \mathcal{E} \quad (42d)$$

where we have defined two quantities, each of which is a function of the fields $\mathbf{E}$ and $\mathbf{H}$ and related to the medium's moving velocity.

$$\mathbf{P}_M = \mathbf{P}_M(\mathbf{E}, \mathbf{H}) = (\alpha - 1) \nabla \times (\mathbf{v}_r \times \mathbf{H}) \quad (43a)$$

$$\mathbf{P}_E = \mathbf{P}_E(\mathbf{E}, \mathbf{H}) = \mathbf{J} + \rho \mathbf{v} - (\alpha - 1) \nabla \times (\mathbf{v}_r \times \mathbf{E}) \quad (43b)$$

This notation is used to solve the equations using perturbation theory and the interaction method. By applying $\nabla \times$ on both Eqs. (42c)–(42d), we have:

$$-\nabla^2 \mathcal{E} + \mu \varepsilon \frac{\partial^2}{\partial t^2} \mathcal{E} \approx \nabla \times \mathbf{P}_M - \nabla \rho / \varepsilon - \mu \frac{\partial}{\partial t} \mathbf{P}_E \quad (44a)$$

$$-\nabla^2 \mathcal{H} + \mu \varepsilon \frac{\partial^2}{\partial t^2} \mathcal{H} \approx \nabla \times \mathbf{P}_E + \varepsilon \frac{\partial}{\partial t} \mathbf{P}_M \quad (44b)$$

These inhomogeneous wave equations can be solved using the iterative method reported previously [12].

By applying $\mathbf{v}_r \cdot$ and $\mathbf{v}_r \times$, respectively, to Eqs. (41a)–(41b), we can derive electric and magnetic fields as follows:

$$\begin{aligned} \mathbf{H} &= \frac{1}{1 - \alpha^2 \varepsilon \mu \mathbf{v}_r^2} [\mathcal{H} + \alpha \varepsilon \mathbf{v}_r \times \mathcal{E} - \alpha^2 \varepsilon \mu \mathbf{v}_r (\mathbf{v}_r \cdot \mathcal{H})] \\ &\approx \mathcal{H} + \alpha \varepsilon \mathbf{v}_r \times \mathcal{E} \end{aligned} \quad (45a)$$

$$\begin{aligned} \mathbf{E} &= \frac{1}{1 - \alpha^2 \varepsilon \mu \mathbf{v}_r^2} [\mathcal{E} - \alpha \varepsilon \mathbf{v}_r \times \mathcal{H} - \alpha^2 \varepsilon \mu \mathbf{v}_r (\mathbf{v}_r \cdot \mathcal{E})] \\ &\approx \mathcal{E} - \mu \varepsilon \mathbf{v}_r \times \mathcal{H} \end{aligned} \quad (45b)$$

The corresponding boundary conditions under the first-order approximation are:

$$\mathbf{n} \cdot [\varepsilon_2 \mathbf{E}_2 - \varepsilon_0 \mathbf{E}_1 + \alpha_2 \varepsilon_2 \mu_2 \mathbf{v}_{r2} \times \mathbf{H}_2] \approx \sigma' \quad (46a)$$

$$\mathbf{n} \cdot [\mu_2 \mathbf{H}_2 - \mu_0 \mathbf{H}_1 - \alpha_2 \varepsilon_2 \mu_2 \mathbf{v}_{r2} \times \mathbf{E}_2] \approx 0 \quad (46b)$$

$$\mathbf{n} \times [\mathbf{E}_2 - \mathbf{E}_1 + \mu_2 \mathbf{v}_{r2} \times \mathbf{H}_2] \approx 0 \quad (46c)$$

$$\mathbf{n} \times [\mathbf{H}_2 - \mathbf{H}_1 - \varepsilon_2 \mathbf{v}_{r2} \times \mathbf{E}_2] \approx \mathbf{K}'_s \quad (46d)$$

11. RECENT EXPERIMENTAL VERIFICATIONS

MEs-f-MDMS constitute an important theoretical framework for addressing complex electromagnetic problems involving accelerated moving media. The experimental validation of this framework is critical. A closely related issue is Feynman's anti-flux problem, which asks whether the voltage and current measured between two contact points on a rotating disk in a static magnetic field are intrinsic motion-induced signals or primarily caused by magnetic flux variation in the external measurement

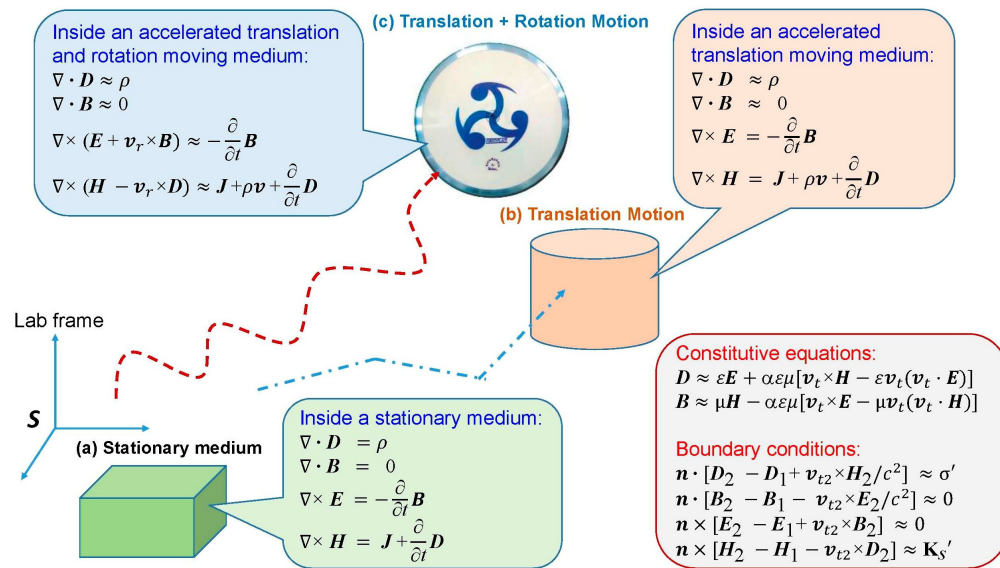


FIGURE 6. A summary on the theoretical schemes developed for quantitative description of the electrodynamics of various moving dielectric media in the lab reference frame.

circuit. To resolve this question, we developed a non-contact near-field electromagnetic measurement system and conducted two progressive experiments to test the theoretical predictions.

A non-contact near-field detection system was constructed to eliminate ambiguities caused by sliding electrodes and external contact loops [23]. Through control variable measurements, component-by-component noise identification, distance-dependent measurements, and material-dependent comparisons, we demonstrated that measurable low-frequency electromagnetic signals persisted around the rotating disk even after the complete removal of the conventional contact-based circuit. The observed response depends on rotation speed, probe distance, and sample material and thus cannot be fully attributed to magnetic flux changes in an external measuring loop. This experiment provides direct evidence that a rotating medium supports a genuine mechano-induced alternating charge-current process, which leaves observable electromagnetic radiation in the surrounding near field.

The electromagnetic behavior of a uniformly moving medium has been traditionally described by Minkowski's theory, which states that the electromagnetic (EM) emission from a rotating isotropic medium should be linear, which means that the frequency of the EM emission should be the same as that of the excitation source. However, we experimentally observed that the near-field EM emission from a rotating charged dielectric disk shows discrete multi-harmonics at frequencies of $n f_R$, with $n = 1$ to 6, where f_R is the disk's rotation frequency [24]. By reversing the disk's rotating direction, the phase shift for the observed magnetic field was π for odd harmonics, but zero for even harmonics. The experimental results may not be consistent with Minkowski theory, but the data can be well explained using the ME-f-MDMS. This study not only provides a solid proof of MEs-f-MDMS but also establishes a theory for describing near-field EM emissions from accelerated medium motion, which has many engineering applications.

12. CONCLUSIONS

Minkowski theory was developed using the Lorentz transformation for a uniformly moving object by constructing Maxwell's equations to be form-invariant in two inertial reference frames. In this study, under the instantaneous constant-speed approximation, we expanded Minkowski's theory to derive the constitutive equations and boundary conditions for a dielectric medium that has accelerated translation and rotation motion and even shape deformation, for example, $\mathbf{v}_t(\mathbf{r}, t)$. Using the Lorentz transformation under a low-speed approximation, we developed constitutive equations and boundary conditions for Maxwell's equations for a mechano-driven medium system (MEs-f-MDMS), so that the full theory for numerical calculations was completed.

Minkowski theory was derived using the Lorentz transformation, whereas MEs-f-MDMS was derived using Galilean transformation. However, under low speed and in the experimental environment of interest, the two transformations are equivalent. Therefore, nice coupling of Minkowski theory with the MEs-f-MDMS provides a systematic theory for treating the electrodynamics of accelerated moving objects.

Figure 6 shows a summary of the main theoretical results. For the cases of (a) a stationary medium or free space, standard Maxwell's equations should be used. In case (b), the medium is a solid object, and its motion is a solid translation without rotation, for example $\mathbf{v}_t = \mathbf{v}(t)$. The standard Maxwell's equations should be used, except that the additional current term $\rho \mathbf{v}$ is added. For case (c), where the object has both translation and rotation motion, $\mathbf{v}_t(\mathbf{r}, t)$, which depends on both space and time, the ME-f-MDMS should be used. The constitutive equations and boundary conditions remained the same for cases (b) and (c). In a mechano-driven media system, mechanical action is not merely passive motion through space but an active source of electromagnetic generation and modulation of electromagnetic waves. Our theory makes it possible to treat electrodynamics by including medium deformation, strain gradients, rotational

dynamics, and interfacial contact processes in the full calculation.

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