

Analytical Calculation of the Electrostatic Field of a Charged Conducting Spherical Shell with a Circular Aperture

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ABSTRACT: In electrostatic shielding, previous studies have focused on shielding external electric fields using electrically neutral shielding shells. Here, we consider a new situation in which the electric field generated by the charge on the shielding shell penetrates the shield's interior through the shell's apertures. As a preliminary exploration of this situation, we started with relatively simple yet analytically solvable configurations. Specifically, we focus on configuring a thin-walled conducting spherical shell that is charged and has a circular aperture. For this configuration, we derive a series-form analytical solution for the potential distribution using the method of variable separation and present simplified solutions for the potential and electric-field strength along the central axis. Based on this analytical solution, we analyzed the characteristics of the potential and electric field distributions inside the spherical shell for different aperture sizes and found that, compared with a single aperture, placing two apertures at diametrically opposite positions on the spherical shell can significantly reduce the electric field strength in the region near the center of the sphere.

1. INTRODUCTION

The rapid proliferation of electronic devices and wireless communication systems has made electromagnetic compatibility (EMC) an increasingly critical concern in modern engineering [1]. With the continuous increase in operating frequencies of integrated circuits and the density of electronic components, electromagnetic interference (EMI) has become an important factor affecting the reliability and performance of electronic equipment [2, 3].

Electromagnetic shielding aims to attenuate electromagnetic fields by introducing conductive or magnetic barriers between the interference source and protected region [4–6]. For static or low-frequency electric fields, the shielding mechanism relies mainly on redistributing free charges on the surface of the conducting enclosure, which generates an opposing field that cancels the external field within the enclosure's shielded volume. The electric field inside a closed conducting shell in an electrostatic equilibrium is identically zero. However, practical shielding enclosures are never perfectly closed. Apertures, slots, and seams are inevitably introduced for ventilation, optical windows, cable penetrations, and maintenance access, and these discontinuities can significantly degrade the shielding effectiveness by allowing electromagnetic fields to penetrate the shielded region [4, 7].

Although the problem of electrostatic field penetration through an aperture in a conducting shell can, in principle, be solved by numerical computation, developing analytical models remains essential. In addition to being convenient to use and physically transparent, analytical models serve as benchmarks for validating numerical models. Numerical models

require considerable user expertise and incur substantial costs in computational time and storage capacity. In particular, steps such as mesh generation and selecting boundary conditions affect the accuracy and convergence of the results.

The weakness of analytical models lies in their restriction to very limited geometric configurations. Nevertheless, the insights they provide can offer guidance on a broader range of problems. For example, the analytical solution for the penetration of an electrostatic field through a circular aperture in an infinitely thin conducting plate can also be approximately applied to the field penetration through a circular aperture in a rectangular cavity [8]. Likewise, the concept of an equivalent dipole moment of the aperture, distilled from the above analytical solution, has been incorporated into the theoretical framework for small-aperture coupling in complex structures such as waveguides and resonant cavities [9–11].

At present, analytically tractable problems of electrostatic-field penetration through small apertures are mainly limited to two geometric configurations: circular or elliptical apertures in an infinite plane and circular apertures in a conducting spherical shell [12, 13]. This study continues to focus on the case of a circular aperture in a spherical shell, but differs from existing studies. In previous studies [13], the electric field was externally applied, whereas in this study the field was produced by the charge on the conducting spherical shell itself. Specifically, the model is the electric potential and field distribution inside and outside a charged conducting spherical shell with a circular aperture. In practice, a conductor may become charged for various reasons. For instance, in a high-voltage environment, corona ion currents in the surrounding space can accumulate on the shell; electrostatic discharge may occur between the shell

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and nearby objects; or charge may be conducted to the shell when it is electrically connected to an external circuit.

This study analytically investigates the electrostatic field produced by a charged conducting spherical shell with a circular aperture. By employing a suitable coordinate system and exploiting the azimuthal symmetry of the configuration, we derived analytical expressions for the electric potential and field distribution inside and outside the shell. We performed a computational analysis to determine how the aperture size affects the field distribution. The results indicate that for small apertures, the electric potential inside the spherical shell is approximately uniform, whereas the electric field decays significantly with the depth. Furthermore, we verified that paired apertures can markedly reduce the electric-field strength at the center of the spherical shell.

The remainder of this paper is organized as follows. Section 2 describes the model's geometric configuration and derives the analytical theory. Section 3 presents the discussion and analysis, including the distributions of the electric potential and field, and the cancellation effect of paired apertures. Finally, Section 4 concludes the study.

2. MODEL AND THEORY

2.1. Model

As shown in Figure 1, the conducting spherical shell, which is considered infinitesimally thin, occupies the surface $r = R$, $0 \leq \theta < \alpha$ in spherical coordinates (r, θ, ϕ) with the origin at the center of the shell. The half-width central angle of the circular aperture θ_0 is related to α by $\theta_0 = \pi - \alpha$. The spherical shell carries a charge of Q . Both the interior and exterior of the spherical shell are considered free space with permittivity ϵ_0 . Furthermore, we assume that the shell's wall is thin and can be neglected, and there are no extraneous electric charges in the shell's inner and outer regions.

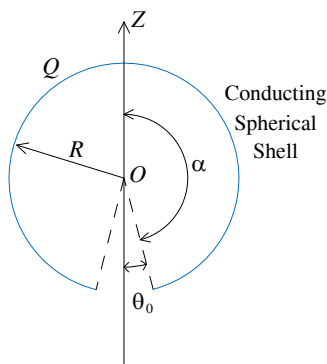


FIGURE 1. The sketch of a charged conducting spherical shell with a circular aperture. The radius of the spherical shell is R , the half-width central angular of the circular aperture is θ_0 , and the charge on the shell is Q .

2.2. The Analytical solutions

2.2.1. For an Arbitrary Field Point

Electric potential distributions inside and outside the spherical shell are denoted by scalar functions φ_i and φ_o , respectively.

Due to the axial symmetry of this model, both φ_i and φ_o depend only on r and θ . By solving Laplace's equation in spherical coordinates via the method of separation of variables and imposing the boundary conditions that the potential vanishes at infinity (taken as the potential reference) and remains finite at the center of the sphere, the potentials can be expressed as

$$\varphi_i(r, \theta) = Q \sum_{n=0}^{\infty} c_n \left(\frac{r}{R}\right)^n P_n(\cos \theta), \quad r \leq R \quad (1)$$

$$\varphi_o(r, \theta) = Q \sum_{n=0}^{\infty} c_n \left(\frac{r}{R}\right)^{-n-1} P_n(\cos \theta), \quad r \geq R \quad (2)$$

Coefficients c_n must be determined, and $P_n(\cdot)$ denotes the Legendre polynomial of degree n . Equations (1) and (2) automatically satisfy the continuity of potential at the spherical boundary $r = R$. To determine coefficients c_n , the following boundary conditions are required:

$$\varphi_i(R, \theta) = \varphi_o(R, \theta) = V_0, \quad 0 \leq \theta < \alpha \quad (3)$$

$$\frac{\partial \varphi_i}{\partial r} = \frac{\partial \varphi_o}{\partial r}, \quad \alpha < \theta \leq \pi \quad (4)$$

Note: In Equation (3), V_0 represents the potential of the spherical shell, whose value is to be determined; Equation (4) results from the continuity of the normal component of the electric field on the aperture surface.

Substituting Equations (1) and (2) into Equations (3) and (4) leads to

$$\sum_{n=0}^{\infty} c_n P_n(\cos \theta) = \frac{V_0}{Q}, \quad 0 \leq \theta < \alpha \quad (5)$$

$$\sum_{n=0}^{\infty} c_n (2n+1) P_n(\cos \theta) = 0, \quad \alpha < \theta \leq \pi \quad (6)$$

Equations (5) and (6) constitute what is known as the dual series equations [14, 15]. In [14], an integral-form solution is provided for the coefficients c_n , and performing the integration leads to

$$c_n = \frac{V_0}{\pi Q} \left[\frac{\sin n\alpha}{n} + \frac{\sin(n+1)\alpha}{(n+1)} \right] \quad (7)$$

When $n = 0$, the denominator in Equation (7) vanishes, leading to a computational singularity. In this case, the issue can be circumvented by evaluating the limit using L'Hôpital's rule. As a result,

$$c_0 = \frac{V_0}{\pi Q} (\alpha + \sin \alpha) \quad (8)$$

Because all points on the spherical surface are equidistant from the center, the potential at the center can be expressed as

$$\varphi_i(0, \theta) = \frac{1}{4\pi\epsilon_0} \frac{Q}{R} \quad (9)$$

The potential at the sphere's center yields:

$$c_0 = \frac{1}{4\pi\epsilon_0 R} \quad (10)$$

Combining Equations (8) and (10) results in

$$V_0 = \frac{Q}{4\epsilon_0 R (\alpha + \sin \alpha)} \quad (11)$$

Incidentally, the capacitance of the spherical shell with a circular aperture is $C = Q/V_0 = 4\epsilon_0 R(\alpha + \sin \alpha)$.

Substituting Equation (10) into Equations (1) and (2), we finally obtain potential expressions inside and outside the spherical shell as follows:

$$\varphi_i(r, \theta) = \frac{Q}{4\pi\epsilon_0 R (\alpha + \sin \alpha)} \sum_{n=0}^{\infty} \left[\frac{\sin n\alpha}{n} + \frac{\sin(n+1)\alpha}{(n+1)} \right] \left(\frac{r}{R} \right)^n P_n(\cos \theta), \quad r \leq R \quad (12)$$

$$\varphi_o(r, \theta) = \frac{Q}{4\pi\epsilon_0 R (\alpha + \sin \alpha)} \sum_{n=0}^{\infty} \left[\frac{\sin n\alpha}{n} + \frac{\sin(n+1)\alpha}{(n+1)} \right] \left(\frac{r}{R} \right)^{-n-1} P_n(\cos \theta), \quad r \geq R \quad (13)$$

It is apparent that the $n=0$ term in Equation (13) corresponds to the potential produced by a complete spherical shell carrying charge Q , while the $n=1$ term corresponds to the potential due to a dipole moment. Note that the dipole moment of this dipole term is not the same as the conventional dipole moment of a small aperture. This is because this dipole is at the center of the spherical shell, whereas the dipole corresponding to a small aperture is at the aperture's center.

The electric field can be obtained from the potential's negative gradient:

$$\mathbf{E} = -\frac{\partial \varphi}{\partial r} \mathbf{e}_r - \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \mathbf{e}_\theta \quad (14)$$

2.3. For Field Points on the Axis of Symmetry

The series expression for the potential can be further simplified for field points on the central axis. Specifically, inside the sphere, the potentials for field points with $z > 0$ and $z < 0$ are given by

$$\varphi_i(r, 0) = \frac{Q}{4\pi\epsilon_0 R (\alpha + \sin \alpha)} \left(\alpha + \tan^{-1} \frac{q \sin \alpha}{1 - q \cos \alpha} + \frac{R}{r} \tan^{-1} \frac{q \sin \alpha}{1 - q \cos \alpha} \right) \quad (15)$$

$$\varphi_i(r, \pi) = \frac{Q}{4\pi\epsilon_0 R (\alpha + \sin \alpha)} \left(\alpha + \tan^{-1} \frac{-q \sin \alpha}{1 + q \cos \alpha} + \frac{R}{r} \tan^{-1} \frac{q \sin \alpha}{1 + q \cos \alpha} \right) \quad (16)$$

For field points outside the sphere with $z < 0$,

$$\varphi_o(r, \pi) = \frac{Q}{4\pi\epsilon_0 R (\alpha + \sin \alpha)} \left[\frac{R}{r} \left(\alpha + \tan^{-1} \frac{-s \sin \alpha}{1 + s \cos \alpha} \right) + \tan^{-1} \frac{s \sin \alpha}{1 + s \cos \alpha} \right] \quad (17)$$

where $q = r/R$, and $s = 1/q$.

The key to deriving the above expressions lies in using the following identity:

$$\sum_{n=1}^{\infty} \frac{\sin n\alpha}{n} q^n = \tan^{-1} \frac{q \sin \alpha}{1 - q \cos \alpha} \quad (18)$$

The corresponding electric fields are:

$$\mathbf{E}_i(r, 0) = \mathbf{e}_r \frac{Q}{4\pi\epsilon_0 R^2 (\alpha + \sin \alpha)} \left[\frac{R^2}{r^2} \tan^{-1} \frac{q \sin \alpha}{1 - q \cos \alpha} - \left(1 + \frac{R}{r} \right) \frac{\sin \alpha}{1 - 2q \cos \alpha + q^2} \right] \quad (19)$$

$$\mathbf{E}_i(r, \pi) = \mathbf{e}_r \frac{Q}{4\pi\epsilon_0 R^2 (\alpha + \sin \alpha)} \left[\frac{(1 - R/r) \sin \alpha}{1 + 2q \cos \alpha + q^2} - q^{-2} \tan^{-1} \frac{-q \sin \alpha}{1 + q \cos \alpha} \right] \quad (20)$$

$$\mathbf{E}_o(r, \pi) = \mathbf{e}_r \frac{Q}{4\pi\epsilon_0 r^2 (\alpha + \sin \alpha)} \left[\alpha - \tan^{-1} \frac{s \sin \alpha}{1 + s \cos \alpha} + \frac{(1 - R/r) \sin \alpha}{1 + 2s \cos \alpha + s^2} \right] \quad (21)$$

Particularly, at the center of the sphere

$$\mathbf{E}_{i,0} = -\frac{Q \sin \alpha}{4\pi\epsilon_0 R^2 (\alpha + \sin \alpha)} (1 + \cos \alpha) \mathbf{e}_z \quad (22)$$

Note: It is difficult to derive Equation (22) directly from Equation (19) or (20), but it is relatively easy to derive it from the gradient of Equation (12). The validity of Equation (22) can be examined through two limiting cases: as α approaches π , the electric field should tend to zero; as α approaches 0, it should tend to $-\mathbf{e}_z Q / (4\pi\epsilon_0 R^2)$. Both conditions are satisfied.

3. RESULTS AND DISCUSSIONS

In subsequent calculations, consider that the coordinate r can be normalized with respect to the sphere radius R , and the spherical shell radius is taken as 1. In addition, the calculated potential results were normalized relative to the potential on the surface of the closed spherical shell, and the calculated electric field results were normalized relative to the electric field on the surface of the closed spherical shell. Specifically, using Equation (15) as an example, the corresponding normalized electric potential is

$$\varphi_{i,nor}(r, 0) = \varphi_i(r, 0) \frac{4\pi\epsilon_0 R}{Q} \quad (23)$$

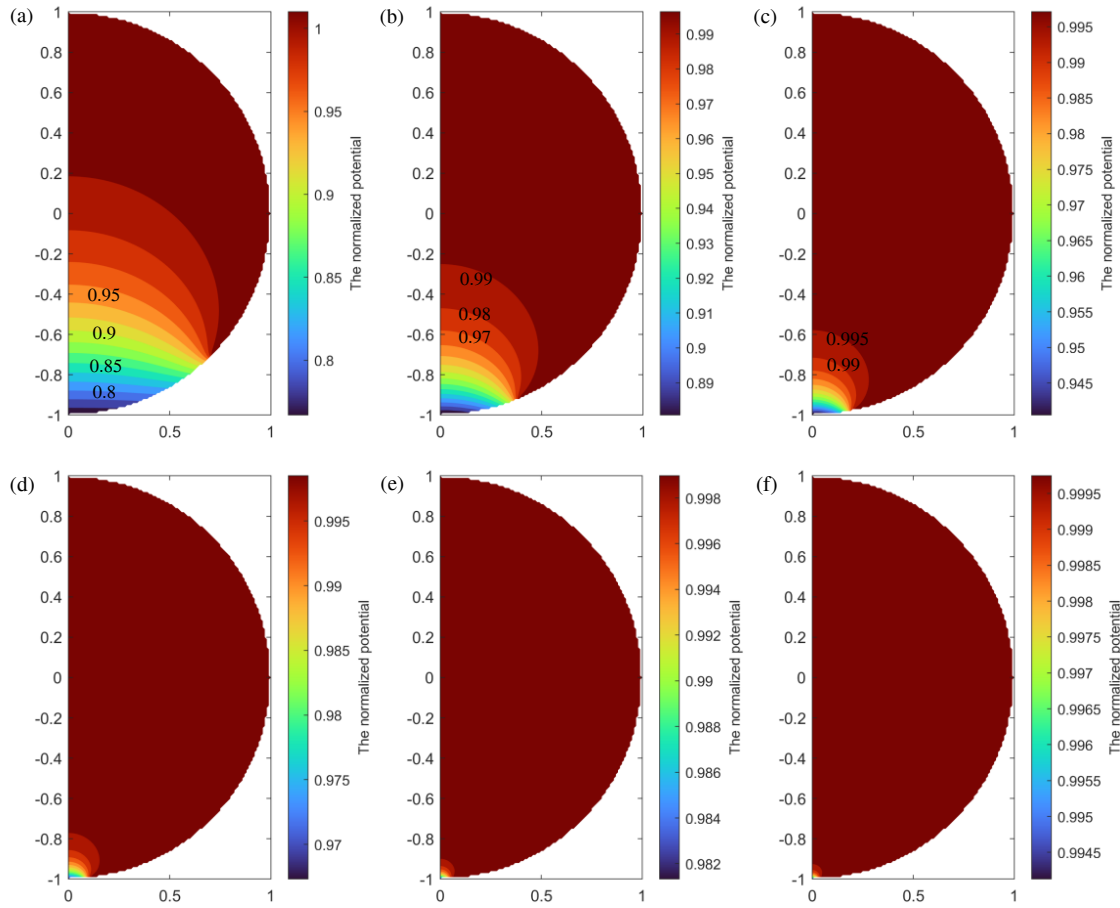


FIGURE 2. Distribution of the normalized electric potential inside the spherical shell under different opening angles of the aperture (a) $\theta_0 = \pi/4$, (b) $\theta_0 = \pi/8$, (c) $\theta_0 = \pi/16$, (d) $\theta_0 = \pi/32$, (e) $\theta_0 = \pi/64$, and (f) $\theta_0 = \pi/128$.

Also, taking Equation (19) as an example, the corresponding normalized electric field is

$$\mathbf{E}_{i,\text{nor}}(r, 0) = \mathbf{E}_i(r, 0) \frac{4\pi\epsilon_0 R^2}{Q} \quad (24)$$

Moreover, we used decibels to represent the field strength. The calculation formula for the decibel value is as follows:

$$E(\text{dB}) = 20 \log_{10}(E_{\text{nor}}) \quad (25)$$

3.1. The Potential Distribution

Figure 2 presents the normalized potential distribution inside the spherical shell for different aperture opening angles. Considering axial symmetry, only the potential distribution in a semicircular domain was provided. It can be observed that (1) the potential at the center of the aperture is the lowest; (2) the potential increases as the distance from the field point to the aperture increases; and (3) as the opening radius decreases, the potential inside the spherical shell becomes increasingly uniform and approaches the potential on the surface of the spherical shell.

3.2. The Electric-Field Distribution

Figure 3 shows the electric-field intensity distribution (in dB) inside the spherical shell for different aperture opening angles. It can be observed that (1) the electric field reaches its maximum at the edge of the opening and its minimum on the spherical shell surface directly opposite to the aperture; (2) the electric-field strength decreases as the distance from the field point to the aperture increases; and (3) the electric-field strength decreases significantly as the opening radius decreases.

3.3. The Dual-Aperture Effect

Occasionally, multiple apertures may be present on a shielding shell. Here, we show that when the spherical shell itself is charged, the electric field at the center of the sphere with two apertures opened at diametrically opposite positions on the shell surface can be much smaller than that with only a single aperture. In other words, the leakage electric fields from these two apertures can cancel each other out. Conversely, in the case of shielding an external electric field, the leakage electric fields from these two apertures will reinforce each other, meaning that the electric field at the center of the sphere will increase by a factor of two compared to that with a single aperture.

To demonstrate this effect, we first simulated the electric-field distributions under single- and dual-aperture configura-

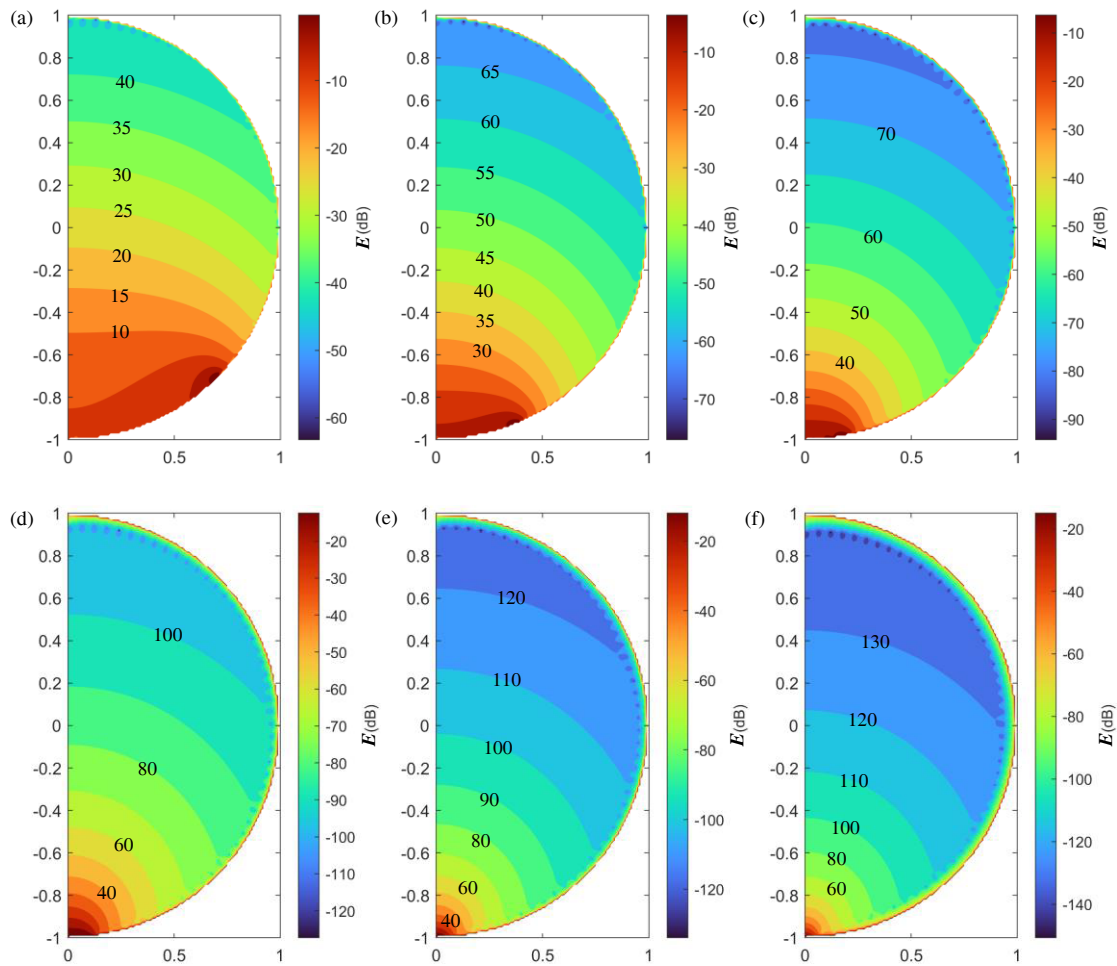


FIGURE 3. Distribution of the normalized electric field intensity (in dB) inside the spherical shell under different opening angles of the aperture: (a) $\theta_0 = \pi/4$, (b) $\theta_0 = \pi/8$, (c) $\theta_0 = \pi/16$, (d) $\theta_0 = \pi/32$, (e) $\theta_0 = \pi/64$, and (f) $\theta_0 = \pi/128$.

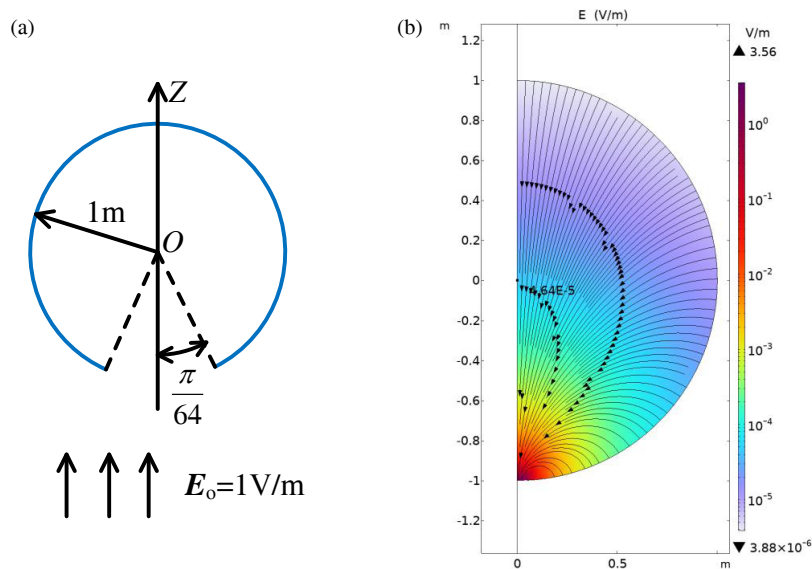


FIGURE 4. Penetration of external uniform electric field into a conducting spherical shell with a circular aperture: (a) The sketch of the single-aperture configuration and (b) the distribution of the electric field intensity obtained from finite element simulations.

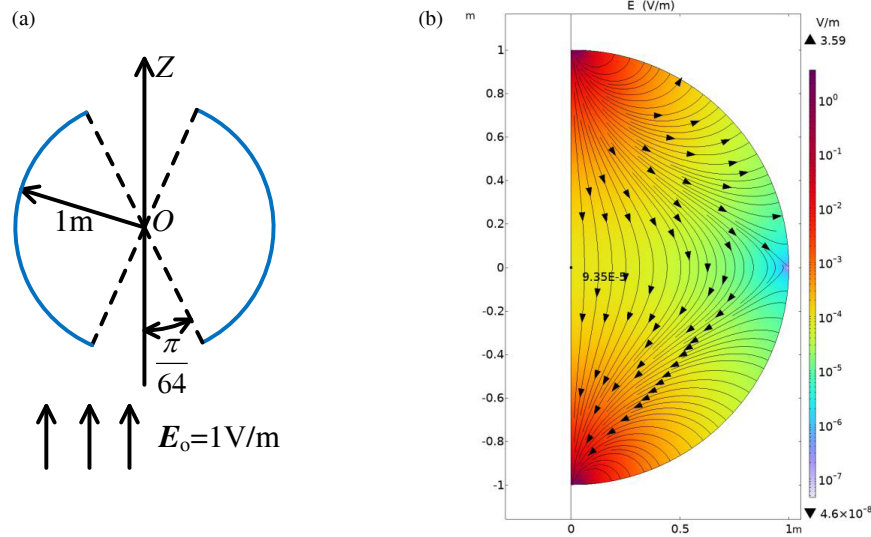


FIGURE 5. Penetration of an external uniform electric field into a conducting spherical shell with two circular apertures: (a) The sketch of the dual-aperture configuration and (b) the distribution of the electric-field intensity.

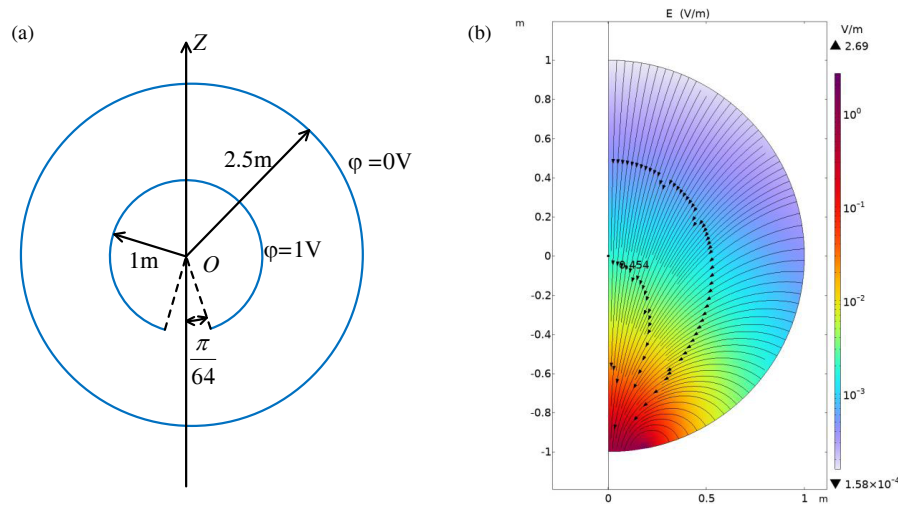


FIGURE 6. Electrostatic field of a charged conducting spherical shell with a circular aperture: (a) The sketch of the single-aperture configuration employed in finite element simulations and (b) the distribution of the electric-field intensity inside the spherical shell with a circular aperture obtained from finite element simulations.

tions for an externally applied uniform electric field using COMSOL finite element software, as shown in Figures 4 and 5, respectively. Comparing Figures 4(b) and 5(b) reveals that, compared with the single-aperture configuration, the dual-aperture configuration results in a doubled electric field at the center of the spherical shell, whereas the electric field near the apertures remains essentially unchanged. This is because the leakage electric fields produced by the two apertures are in the same direction and have equal magnitudes at the center of the sphere, thus doubling the field. Near each aperture, the leakage field from the other distant aperture is very small and can be neglected.

Second, based on COMSOL finite element software, we simulated the electric-field distributions in single- and dual-aperture configurations for the case in which the spherical shell itself is charged, as shown in Figures 6 and 7, respectively. In

Figures 6(a) and 7(a), the spherical boundary with a radius of 2.5 m and zero potential is set to approximate the natural boundary condition of zero potential at infinity. Theoretically, a larger radius is preferable, but this would also significantly increase computational costs. Our trials indicate that choosing 2.5 m as the radius achieves a good balance between computational accuracy and efficiency. Comparing Figures 6(b) and 7(b), it can be seen that compared to the single-aperture configuration, the dual-aperture configuration results in an electric field at the center of the sphere that is almost completely canceled out (less than 1/1000 of that with a single aperture), whereas the electric field near the apertures remains essentially unchanged. This is because the leakage electric fields produced by the two apertures are in opposite directions and have equal magnitudes at the center of the spherical shell, canceling each other out; near the apertures, the leakage field from the distant aperture is very

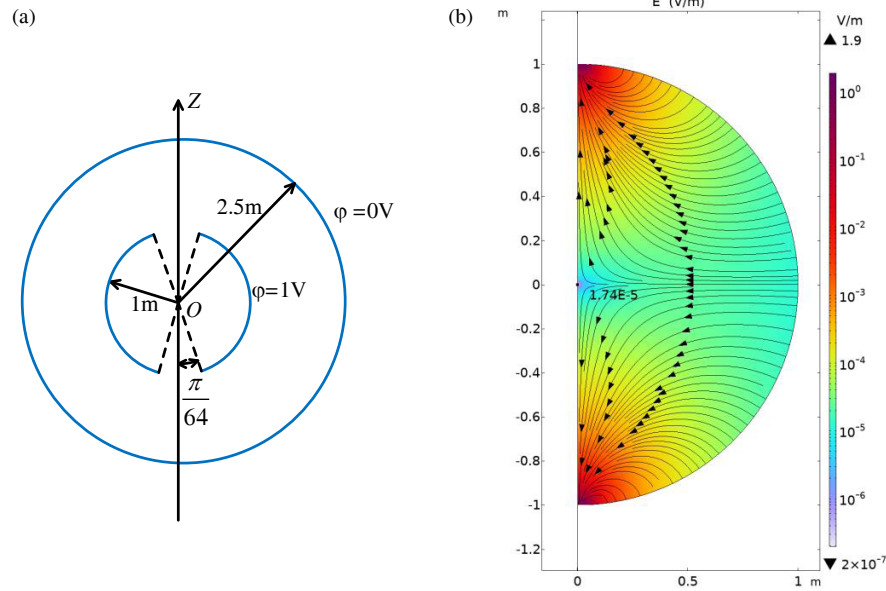


FIGURE 7. Electrostatic field of a charged conducting spherical shell with two circular apertures: (a) The sketch of the dual-aperture configuration employed in finite element simulations and (b) the distribution of the electric-field intensity inside the spherical shell with two circular apertures obtained from finite element simulations.

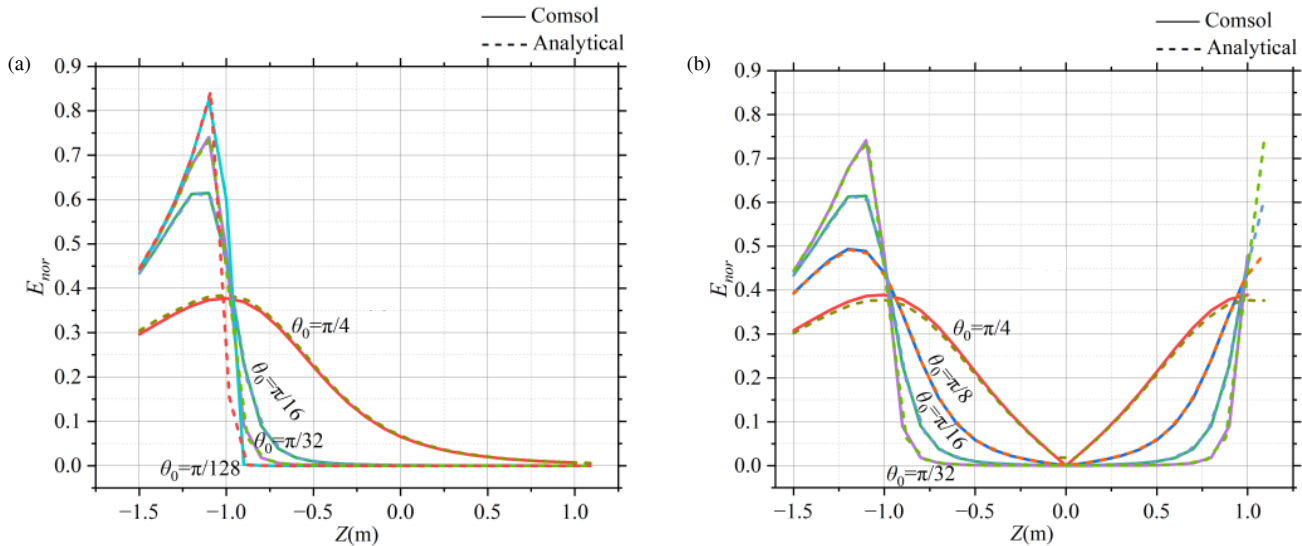


FIGURE 8. Comparison of the electric field strength (absolute value) along the central axis of a charged spherical shell with apertures obtained by the finite element simulations and the analytical solution: (a) single-aperture configuration and (b) dual-aperture configuration.

small and can be neglected. Theoretically, the electric field at the center of the sphere is zero for the dual-aperture configuration, but numerical calculations involve tiny errors.

The reason for choosing finite element simulations rather than directly using the analytical solution in the preceding discussion (Figures 4–7) is that the analytical solution is strictly applicable only to the case of a single aperture. However, as we shall see next, the agreement between analytical and finite element solutions is also very good for the dual-aperture configuration. Specifically, for the case where the spherical shell itself is charged, we present results for the normalized electric field along the central axis (the Z -axis in Figure 1), including

a comparison between finite element simulations and analytical solutions, as shown in Figure 8. Figure 8(a) and Figure 8(b) correspond to the single- and dual-aperture cases, respectively. The aperture's half-width central angle was $\theta_0 = \pi/64$ in both cases. For the dual-aperture case, the analytical solution presented is based on the direct superposition of the analytical results for individual apertures, neglecting the coupling between the two apertures. Given that the distance between the two apertures is much greater than the radius of apertures, such a coupling is negligible. The analytical results agree well with COMSOL finite element simulation results for both single- and dual-aperture cases.

4. CONCLUSION

This study derives analytical solutions for potential and electric-field distributions inside a thin-walled conducting spherical shell that is charged and has a circular aperture, presents their simplified forms along the central axis, and performs a computational analysis of the potential and electric-field distributions inside the spherical shell. The computational results indicate that:

(1) The minimum potential inside the spherical shell occurs at the center of the aperture; as the aperture decreases, the potential inside the spherical shell becomes increasingly uniform and approaches the potential of the spherical shell surface.

(2) The maximum electric field occurs at the edge of the aperture; as the aperture decreases or the distance from the field point to the aperture increases, the electric field inside the spherical shell decreases.

(3) Compared with a single aperture, placing two apertures at diametrically opposite positions on the spherical shell can significantly reduce the electric-field strength near the center of the sphere.

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