6

THE PROPAGATION OF LOW-FREQUENCY ELECTROMAGNETIC WAVES IN THE TERRESTRIAL ENVIRONMENT WITH SPECIAL REFERENCE TO WHISTLERS

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6.1 Introduction

The purpose of this chapter is to investigate the propagation of low-frequency electromagnetic pulses generated by thunderstorms which mainly occur in the troposphere at equatorial latitudes, and to clarify the misconcept often involved. It is well-known that low-frequency electromagnetic waves can propagate in the terrestrial waveguide between the Earth's surface and the bottom boundary of the ionosphere. It is also well-known that part of the low-frequency energy can penetrate into the ionosphere under appropriate conditions, travel approximately along the Earth's magnetic field lines until the bottom boundary is reached in its return path and then re-enter into the terrestrial waveguide, constituting what are called whistlers. People interested in low-frequency electromagnetic wave propagation in the terrestrial environment belong to two categories. People of one category are interested in it because of its applications in communications, navigation, standard time signal broadcasting, etc. People of this category generally consider only the terrestrial waveguide propagation, neglecting the whistler mode propagation altogether. People of the other category are interested in whistlers and their propagation because of their relevance to the study of the ionosphere and magnetosphere, which will henceforth often be referred to collectively as the ionized atmosphere. They generally consider only the whistler mode propagation, neglecting the terrestrial waveguide propagation altogether. This overlook leads to the erroneous conclusion that whistlers cannot be received below certain latitudes. If the terrestrial waveguide propagation is taken into proper consideration, the usual conviction that whistlers cannot be received below certain latitudes is clearly unwarranted.*Also if this conviction were true, then by reciprocity, we would get the conclusion that the sources of whistlers must occur at sufficiently high latitudes, but factually, the thunderstorms which generate whistlers mostly occur in equatorial regions. Moreover, if the terrestrial waveguide propagation is considered, it is clear that the source and the receiving point need not to be mutually magnetic-conjugate. In whistler investigation, it is therefore necessary to study the combined effect of the terrestrial waveguide propagation and the whistler mode propagation, and in this study, it is crucial to examine the conditions and regions of the boundary surface suitable for the penetration of the whistler component into the ionosphere and for its re-entry from the ionosphere back into the terrestrial waveguide. Penetration from the terrestrial waveguide into the ionosphere is possible only at regions on the boundary surface where the matching conditions ensure that the square of the radial component of the propagation vector is greater than zero. Otherwise, the wave will be evanescent and decrease exponentially. Furthermore, only those component whose refractive indices squared are greater than zero along its whole path of propagation in the ionized atmosphere can constitute whistlers receivable at

* B. X. Liang and C. T. Bao of Wuhan University, China, have done extensive work in receiving and analyzing whistler signals at many places in southern China. Their results fully testify the feasibility of whistler detection at low latitudes.

the magnetic-conjugate point. It is often assumed that there are filamentary structures of electron density, the so-called ducts, along the Earth's magnetic field lines to account for the quasi-longitudinal propagation of whistlers. However there has been no direct experimental evidence of such structures obtained as yet. According to our view, it is the requirement on the matching conditions that causes the apparent existence of ducts. The difficulty of the matching problem lies in the fact that in the terrestrial waveguide, the field is convenient to be expressed as the summation of wave modes but not convenient to be expressed in rays, while in the ionized atmosphere it is convenient to use the magneto-ionic theory which in essence a ray theory, but it is not convenient to express the field in modes. A method of matching the fields in the regions on the two sides of the bottom boundary of the ionosphere needs careful consideration. The whole treatment involves several intricate points.

6.2 Electromagnetic Field in the Terrestrial Waveguide

Let us consider a coordinate system $S(x, y, z)$ or equivalently (r, θ, ϕ) with $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ such that the z-axis is coincident with the Earth's magnetic polar axis pointing northward, and another coordinate system $S'(x', y', z')$ or equivalently (r, θ', ϕ') with $x' = r\sin\theta'\cos\phi'$, $y' = r\sin\theta'\sin\phi'$, $z' = r\cos\theta'$, which is obtained by first rotating the coordinate system (x, y, z) about the z-axis through an angle ϕ_0 and then rotating the coordinate system thus obtained about the new y'-axis through angle θ_0, as shown in Fig. 6.1. The transformation relation is as follows.

$$\begin{bmatrix} \sin\theta'\cos\phi' \\ \sin\theta'\sin\phi' \\ \cos\theta' \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta_0 & 0 & -\sin\theta_0 \\ 0 & 1 & 0 \\ \sin\theta_0 & 0 & \cos\theta_0 \end{bmatrix} \begin{bmatrix} \cos\phi_0 & \sin\phi_0 & 0 \\ -\sin\phi_0 & \cos\phi_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta_0\cos\phi_0 & \cos\theta_0\sin\phi_0 & -\sin\theta_0 \\ -\sin\phi_0 & \cos\phi_0 & 0 \\ \sin\theta_0\cos\phi_0 & \sin\theta_0\sin\phi_0 & \cos\theta_0 \end{bmatrix} \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix} \tag{1}$$

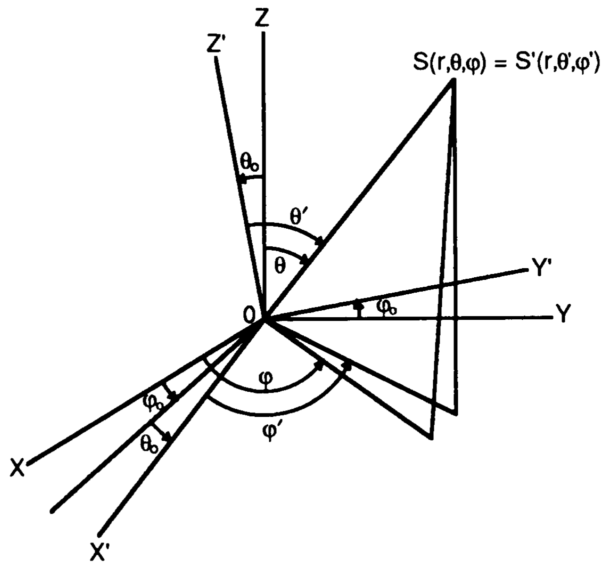

Figure 6.1 Transformation of coordinate systems.

Hence,

$$\left.\begin{aligned} \sin\theta'\cos\phi' &= \cos\theta_0\sin\theta\cos(\phi-\phi_0)-\sin\theta_0\cos\theta \\ \sin\theta'\sin\phi' &= \sin\theta\sin(\phi-\phi_0) \\ \cos\theta' &= \sin\theta_0\sin\theta\cos(\phi-\phi_0)+\cos\theta_0\cos\theta \end{aligned}\right\} \qquad (2)$$

We assume that the vertical electric dipole of moment p generated by the thunderstorm is situated on the z'-axis at a distance r_0 from the origin, i.e., at $S'(r_0,0,0)$. In the terrestrial waveguide taking into consideration of only one Fourier component with $e^{-i\omega t}$ time dependence, in MKS units, we have

$$\left.\begin{aligned} \nabla\times\overline{E}-i\omega\mu_0\overline{H} &= 0 \\ \nabla\times\overline{H}+i\omega\epsilon_0\overline{E} &= \overline{J} \end{aligned}\right\} \qquad (3)$$

where $\overline{J}$ is the current density of excitation due to the vertical electric dipole of moment p at $\overline{r}_0$, and the dielectric constant of the neutral atmosphere is approximately assumed to be equal to that of vacuum ϵ_0. We can let

$$\overline{H} = -i\omega\epsilon_0 \nabla \times \overline{\Pi} \tag{4}$$

where $\overline{\Pi}$ is the Hertz vector potential. Then,

$$\nabla \times (\overline{E} - k_0^2 \overline{\Pi}) = 0 \tag{5}$$

where $k_0^2 = \omega^2 \epsilon_0 \mu_0$. We can let

$$\overline{E} = k_0^2 \overline{\Pi} + \nabla\phi \tag{6}$$

where ϕ is an arbitrary scalar function of $\overline{r}$ still free to be specified. Except at the source point,

$$\overline{E} = \nabla \times \nabla \times \overline{\Pi} \tag{7}$$

Let us stipulate that $\overline{\Pi}$ has only r component and write

$$\overline{\Pi} = \overline{r}\Gamma \tag{8}$$

Then,

$$\overline{H} = -i\omega\epsilon_0 \nabla \times (\overline{r}\Gamma) \tag{9}$$

$$\begin{aligned} \overline{E} &= k_0^2 \overline{r}\Gamma + \nabla\phi \\ &= \nabla \times \nabla \times (\overline{r}\Gamma) \qquad \text{(except at the source point)} \end{aligned} \tag{10}$$

Now,

$$\begin{aligned} \nabla \times (\overline{r}\Gamma) &= \hat{\theta}\frac{1}{\sin\theta}\frac{\partial \Gamma}{\partial \phi} - \hat{\phi}\frac{\partial \Gamma}{\partial \theta} \\ \nabla \times \nabla \times (\overline{r}\Gamma) &= -\hat{r}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Gamma}{\partial \theta}\right) + \frac{1}{r\sin^2\theta}\frac{\partial^2 \Gamma}{\partial \phi^2}\right] \\ &\quad + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\frac{\partial}{\partial r}(r\Gamma) + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\frac{\partial}{\partial r}(r\Gamma) \end{aligned} \tag{11}$$

where $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$ are unit vectors in the directions of increasing r, θ, ϕ respectively. Letting $\phi = \partial(r\Gamma)/\partial r$, we have

$$\begin{aligned} \overline{r}(\nabla^2\Gamma + k_0^2\Gamma) &= \overline{r}\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Gamma) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \Gamma}{\partial \theta}\right) \right. \\ &\quad \left. + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \Gamma}{\partial \phi^2} + k_0^2\Gamma\right] \\ &= \frac{\overline{J}}{i\omega\mu_0} \end{aligned} \tag{12}$$

We see therefore that our stipulation that $\overline{\Pi} = \bar{r}\Gamma$ is justified, and

$$\nabla^2\Gamma + k_0^2\Gamma = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Gamma) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Gamma}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Gamma}{\partial\phi^2} + k_0^2\Gamma = \frac{J}{i\omega\epsilon_0 r_0} \tag{13}$$

$$\begin{aligned}\overline{H} &= -\hat{\theta}\frac{i\omega\epsilon_0}{\sin\theta}\frac{\partial\Gamma}{\partial\phi} + \hat{\phi}\, i\omega\epsilon_0\frac{\partial\Gamma}{\partial\theta}\\ \overline{E} &= -\hat{r}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\Gamma}{\partial\theta}) + \frac{1}{r\sin^2\theta}\frac{\partial^2\Gamma}{\partial\phi^2}\right] + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}(r\Gamma)\\ &\quad + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}(r\Gamma)\\ &= \hat{r}\left[\frac{\partial^2}{\partial r^2}(r\Gamma) + k_0^2 r\Gamma\right] + \hat{\theta}\,\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}(r\Gamma) + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}(r\Gamma)\end{aligned} \tag{14}$$

It is expedient to write Γ as the sum of Γ_p pertaining to the primary field, that is the field which would be produced by the source if the medium were infinite in extent and no other objects present, and Γ_s pertaining to the secondary field due to the presence of the Earth and the ionized atmosphere. It is well-known that the primary field can be expressed as

$$\begin{aligned}\Gamma_p &= \frac{p}{4\pi\epsilon_0 r_0}\frac{e^{ik_0|\bar{r}-\bar{r}_0|}}{|\bar{r}-\bar{r}_0|}\\ &= \frac{p}{4\pi\epsilon_0}\frac{ik_0}{r_0}\sum_{\ell=0}^{\infty}(2\ell+1)\left\{\begin{array}{l} j_\ell(k_0r_0)h_\ell^{(1)}(k_0r)\\ h_\ell^{(1)}(k_0r_0)j_\ell(k_0r)\end{array}\right\}P_\ell(\cos\theta'), \quad \begin{array}{l} r>r_0\\ r<r_0\end{array}\end{aligned} \tag{15}$$

We have already shown that $\cos\theta' = \sin\theta_0\sin\theta\cos(\phi-\phi_0)+\cos\theta_0\cos\theta$ and we can prove that

$$\begin{aligned}P_\ell(\cos\theta') &= \sum_{m=-\ell}^{\ell}\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(\cos\theta_0)P_\ell^m(\cos\theta_0)\, e^{im(\phi-\phi_0)}\\ &= \sum_{m=-\ell}^{\ell}(-1)^m P_\ell^{-m}(\cos\theta)P_\ell^m(\cos\theta)\, e^{im\phi-\phi_0}\end{aligned} \tag{16}$$

Where

$$P_\ell^{-m}(\cos\theta_0) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(\cos\theta_0) \tag{17}$$

Hence,

$$\Gamma_p = \begin{cases} \dfrac{p}{4\pi\epsilon_0}\dfrac{ik_0}{r_0}\displaystyle\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(-1)^m(2\ell+1)\,j_\ell(k_0r_0)\,h_\ell^{(1)}(k_0r) \\ \qquad P_\ell^{-m}(\cos\theta_0)P_\ell^m(\cos\theta)e^{im(\phi-\phi_0)}, & r > r_0 \\ \dfrac{p}{4\pi\epsilon_0}\dfrac{ik_0}{r_0}\displaystyle\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(-1)^m(2\ell+1)\,h_\ell^{(1)}(k_0r_0)\,j_\ell(k_0r) \\ \qquad P_\ell^{-m}(\cos\theta_0)P_\ell^m(\cos\theta)\,e^{im(\phi-\phi_0)}, & r < r_0 \end{cases} \tag{18}$$

Γ_s satisfies the homogeneous equation, that is, (13) with $J = 0$. By the method of separation of variables, the solution can be written

$$\Gamma_s = \frac{p}{4\pi\epsilon_0}\frac{ik_0}{r_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(-1)^m(2\ell+1)\left[A_{\ell m}h_\ell^{(1)}(k_0r) + B_{\ell m}h_\ell^{(2)}(k_0r)\right]$$
$$P_\ell^{-m}(\cos\theta_0)P_\ell^m(\cos\theta)\,e^{im(\phi-\phi_0)} \tag{19}$$

where $A_{\ell m}$ and $B_{\ell m}$ are constants to be determined. Hence,

$$\Gamma = \frac{p}{4\pi\epsilon_0}\frac{ik_0}{r_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(-1)^m(2\ell+1)$$
$$\cdot\left\{\begin{matrix} j_\ell(k_0r_0)\,h_\ell^{(1)}(k_0r) \\ h_\ell^{(1)}(k_0r_0)\,j_\ell(k_0r) \end{matrix} + A_{\ell m}h_\ell^{(1)}(k_0r) + B_{\ell m}h_\ell^{(2)}(k_0r)\right\}$$
$$\cdot P_\ell^{-m}(\cos\theta_0)P_\ell^m(\cos\theta)e^{im(\phi-\phi_0)}, \qquad \begin{matrix} r > r_0 \\ r < r_0 \end{matrix} \tag{20}$$

At low frequencies, the Earth's surface can be approximately considered as perfectly conducting, and therefore, at $r = a$, where a is the Earth's radius, $E_\theta = E_\phi = 0$, i.e.,

$$\left.\frac{\partial}{\partial r}(r\Gamma)\right|_{r=a} = 0 \tag{21}$$

From (20), we have

$$A_{\ell m}\left[h_\ell^{(1)'}(k_0a)+\frac{1}{k_0a}h_\ell^{(1)}(k_0a)\right]+B_{\ell m}\left[h_\ell^{(2)'}(k_0a)+\frac{1}{k_0a}h_\ell^{(2)}(k_0a)\right]$$
$$=-h_\ell^{(1)}(k_0r_0)\left[j_\ell'(k_0a)+\frac{1}{k_0a}j_\ell(k_0a)\right] \quad (22)$$

Substituting (22) into (20) for the case $r>r_0$, we have

$$\Gamma=\frac{p}{4\pi\epsilon_0}\frac{ik_0}{r_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(-1)^m(2\ell+1)$$
$$\cdot\left\{j_\ell(k_0r_0)h_\ell^{(1)}(k_0r)\right.$$
$$-\frac{j_\ell'(k_0a)+\frac{1}{k_0a}j_\ell(k_0a)}{h_\ell^{(2)'}(k_0a)+\frac{1}{k_0a}h_\ell^{(2)}(k_0a)}h_\ell^{(1)}(k_0r_0)\,h_\ell^{(2)}(k_0r)$$
$$\left.+A_{\ell m}\left[h_\ell^{(1)}(k_0r)-\frac{h_\ell^{(1)'}(k_0a)+\frac{1}{k_0a}h_\ell^{(1)}(k_0a)}{h_\ell^{(2)'}(k_0a)+\frac{1}{k_0a}h_\ell^{(2)}(k_0a)}h_\ell^{(2)}(k_0r)\right]\right\}$$
$$\cdot P_\ell^{-m}(\cos\theta_0)P_\ell^m(\cos\theta)\,e^{im(\phi-\phi_0)} \quad (23)$$

In the investigation of whistler propagation, we are only interested in the part of Γ, $\Gamma^\dagger$, which corresponds to the field components capable of penetrating into the ionosphere at some regions of its bottom boundary and capable of propagating through the ionized atmosphere to the magnetic conjugate point without becoming evanescent.* $\Gamma^\dagger$ can be written

* As far as the field in the ionized atmosphere is concerned, it would not be changed if the bottom boundary of the ionosphere is replaced by an artificial boundary layer which is totally obsorbing at regions where the wave cannot penetrate but selectively transparent according to the sense of polarization at the remaining regions. The field in the terrestrial waveguide, however, is changed by such artificial boundary replacement.

$$\Gamma^\dagger = \frac{p}{4\pi\mu_0}\frac{ik_0}{r_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(-1)^m(2\ell+1)$$

$$\cdot\left\{\begin{matrix} j_\ell(k_0r_0)h_\ell^{(1)}(k_0r) \\ h_\ell^{(1)}(k_0r_0)j_\ell(k_0r)\end{matrix} + A_{\ell m}^\dagger h_\ell^{(1)}(k_0r) + B_{\ell m}^\dagger h_\ell^{(2)}(k_0r)\right\}$$

$$\cdot P_\ell^{-m}(\cos\theta_0)P_\ell^m(\cos\theta)e^{im(\phi-\phi_0)}, \qquad \begin{matrix} r>r_0 \\ r<r_0\end{matrix} \tag{24}$$

Equation (22) still holds if Γ, $A_{\ell m}$ and $B_{\ell m}$ are replaced by $\Gamma^\dagger$, $A_{\ell m}^\dagger$, and $B_{\ell m}^\dagger$ respectively, so that for the case $r > r_0$,

$$\Gamma^\dagger = \frac{p}{4\pi\epsilon_0}\frac{ik_0}{r_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(-1)^m(2\ell+1)$$

$$\cdot\Bigg\{j_\ell(k_0r_0)h_\ell^{(1)}(k_0r)$$

$$-\frac{j_\ell'(k_0a)+\frac{1}{k_0a}j_\ell(k_0a)}{h_\ell^{(2)'}(k_0a)+\frac{1}{k_0a}h_\ell^{(2)}(k_0a)}h_\ell^{(1)}(k_0r_0)h_\ell^{(2)}(k_0r)$$

$$+A_{\ell m}^\dagger\left[h_\ell^{(1)}(k_0r)-\frac{h_\ell^{(1)'}(k_0a)+\frac{1}{k_0a}h_\ell^{(1)}(k_0a)}{h_\ell^{(2)'}(k_0a)+\frac{1}{k_0a}h_\ell^{(2)}(k_0a)}h_\ell^{(2)}(k_0r)\right]\Bigg\}$$

$$\cdot P_\ell^{-m}(\cos\theta_0)P_\ell^m(\cos\theta)\,e^{im(\phi-\phi_0)} \tag{25}$$

and

$$H_r^\dagger = 0 \tag{26}$$

$$H_\theta^\dagger = -\frac{i\omega\epsilon_0}{\sin\theta}\frac{\partial\Gamma^\dagger}{\partial\phi}$$

$$= i\frac{p}{4\pi}\frac{\omega k_0}{r_0}\frac{1}{\sin\theta}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(-1)^m(2\ell+1)m$$

$$\cdot\Bigg\{j_\ell(k_0r_0)h_\ell^{(1)}(k_0r)$$

$$-\frac{j_\ell'(k_0a)+\frac{1}{k_0a}j_\ell(k_0a)}{h_\ell^{(2)'}(k_0a)+\frac{1}{k_0a}h_\ell^{(2)}(k_0a)}h_\ell^{(1)}(k_0r_0)h_\ell^{(2)}(k_0r)$$

$$+ A^{\dagger}_{\ell m}\left[h^{(1)}_{\ell}(k_0 r) - \frac{h^{(1)'}_{\ell}(k_0 a) + \frac{1}{k_0 a}h^{(1)}_{\ell}(k_0 a)}{h^{(2)'}_{\ell}(k_0 a) + \frac{1}{k_0 a}h^{(2)}_{\ell}(k_0 a)} h^{(2)}_{\ell}(k_0 r)\right]\Bigg\}$$

$$\cdot P^{-m}_{\ell}(\cos\theta_0) P^{m}_{\ell}(\cos\theta)\, e^{im(\phi-\phi_0)} \tag{27}$$

$$H^{\dagger}_{\phi} = i\omega\epsilon_0 \frac{\partial \Gamma^{\dagger}}{\partial \theta}$$

$$= \frac{p}{4\pi}\frac{\omega k_0}{r_0}\sin\theta \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(-1)^m(2\ell+1)$$

$$\cdot\Bigg\{ j_{\ell}(k_0 r_0) h^{(1)}_{\ell}(k_0 r)$$

$$- \frac{j'_{\ell}(k_0 a) + \frac{1}{k_0 a} j_{\ell}(k_0 a)}{h^{(2)'}_{\ell}(k_0 a) + \frac{1}{k_0 a}h^{(2)}_{\ell}(k_0 a)} h^{(1)}_{\ell}(k_0 r_0) h^{(2)}_{\ell}(k_0 r)$$

$$+ A^{\dagger}_{\ell m}\left[h^{(1)}_{\ell}(k_0 r) - \frac{h^{(1)'}_{\ell}(k_0 a) + \frac{1}{k_0 a}h^{(1)}_{\ell}(k_0 a)}{h^{(2)'}_{\ell}(k_0 a) + \frac{1}{k_0 a}h^{(2)}_{\ell}(k_0 a)} h^{(2)}_{\ell}(k_0 r)\right]\Bigg\}$$

$$\cdot P^{-m}_{\ell}(\cos\theta_0) P^{m'}_{\ell}(\cos\theta)\, e^{im(\phi-\phi_0)} \tag{28}$$

$$E^{\dagger}_{r} = -\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Gamma^{\dagger}}{\partial\theta}\right) + \frac{1}{r\sin^2\theta}\frac{\partial^2\Gamma^{\dagger}}{\partial\phi^2}\right]$$

$$= \frac{\partial^2}{\partial r^2}(r\Gamma^{\dagger}) + k_0^2 r\Gamma^{\dagger}$$

$$= \frac{p}{4\pi\epsilon_0}\frac{ik_0}{r_0 r}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(-1)^m(2\ell+1)\ell(\ell+1)$$

$$\Bigg\{ j_{\ell}(k_0 r_0) h^{(1)}_{\ell}(k_0 r)$$

$$- \frac{j'_{\ell}(k_0 a) + \frac{1}{k_0 a} j_{\ell}(k_0 a)}{h^{(2)'}_{\ell}(k_0 a) + \frac{1}{k_0 a}h^{(2)}_{\ell}(k_0 a)} h^{(1)}_{\ell}(k_0 r_0) h^{(2)}_{\ell}(k_0 r)$$

$$+ A^{\dagger}_{\ell m}\left[h^{(1)}_{\ell}(k_0 r) - \frac{h^{(1)'}_{\ell}(k_0 a) + \frac{1}{k_0 a}h^{(1)}_{\ell}(k_0 a)}{h^{(2)}_{\ell}(k_0 a) + \frac{1}{k_0 a}h^{(2)}_{\ell}(k_0 a)} h^{(2)}_{\ell}(k_0 r)\right]\Bigg\}$$

$$\cdot P^{-m}_{\ell}(\cos\theta_0) P^{m}_{\ell}(\cos\theta)\, e^{im(\phi-\phi_0)} \tag{29}$$

$$E_\theta^\dagger = \frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}(r\Gamma^\dagger)$$

$$= -i\,\frac{p}{4\pi\epsilon_0}\frac{k_0^2}{r_0}\sin\theta\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(-1)^m(2\ell+1)$$

$$\Bigg\{ j_\ell(k_0r_0)\left[h_\ell^{(1)\prime}(k_0r)+\frac{1}{k_0r}h_\ell^{(1)}(k_0r)\right]$$

$$-\frac{j_\ell'(k_0a)+\frac{1}{k_0a}j_\ell(k_0a)}{h_\ell^{(2)\prime}(k_0a)+\frac{1}{k_0a}h_\ell^{(2)}(k_0a)}h_\ell^{(1)}(k_0r_0)$$

$$\left[h_\ell^{(2)\prime}(k_0r)+\frac{1}{k_0r}h_\ell^{(2)}(k_0r)\right]$$

$$+A_{\ell m}^\dagger\Bigg[h_\ell^{(1)\prime}(k_0r)+\frac{1}{k_0r}h_\ell^{(1)}(k_0r)$$

$$-\frac{h_\ell^{(1)\prime}(k_0a)+\frac{1}{k_0a}h_\ell^{(1)}(k_0a)}{h_\ell^{(2)\prime}(k_0a)+\frac{1}{k_0a}h_\ell^{(2)}(k_0a)}\left[h_\ell^{(2)\prime}(k_0r)+\frac{1}{k_0r}h_\ell^{(2)}(k_0r)\right]\Bigg]\Bigg\}$$

$$\cdot P_\ell^{-m}(\cos\theta_0)P_\ell^{m\prime}(\cos\theta)\,e^{im(\phi-\phi_0)} \tag{30}$$

$$E_\phi^\dagger = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}(r\Gamma^\dagger)$$

$$= -\frac{p}{4\pi\mu_0}\frac{k_0^2}{r_0\sin\theta}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(-1)^m(2\ell+1)m$$

$$\Bigg\{ j_\ell(k_0r_0)\qquad\left[h_\ell^{(1)\prime}(k_0r)+\frac{1}{k_0r}h_\ell^{(1)}(k_0r)\right]$$

$$-\frac{j_\ell'(k_0a)+\frac{1}{k_0a}j_\ell(k_0a)}{h_\ell^{(2)\prime}(k_0a)+\frac{1}{k_0a}h_\ell^{(2)}(k_0a)}h_\ell^{(1)}(k_0r_0)$$

$$\left[h_\ell^{(2)\prime}(k_0r)+\frac{1}{k_0r}h_\ell^{(2)}(k_0r)\right]$$

$$+ A_{\ell m}^{\dagger} \left[h_{\ell}^{(1)'}(k_0 r) + \frac{1}{k_0 r} h_{\ell}^{(1)}(k_0 r) \right.$$

$$\left. \left. - \frac{h_{\ell}^{(1)'}(k_0 a) + \frac{1}{k_0 a} h_{\ell}^{(1)}(k_0 a)}{h_{\ell}^{(2)'}(k_0 a) + \frac{1}{k_0 a} h_{\ell}^{(2)}(k_0 a)} \left[h_{\ell}^{(2)'}(k_0 r) + \frac{1}{k_0 r} h_{\ell}^{(2)}(k_0 r) \right] \right] \right\}$$

$$\cdot P_{\ell}^{-m}(\cos\theta_0) P_{\ell}^{m}(\cos\theta)\, e^{im(\phi-\phi_0)} \tag{31}$$

It is well-known that functions expressed in series expansions in terms of spherical harmonics converge very slowly and that under certain conditions they can be converted by the so-called Watson transformation into new series expansions which converge much more rapidly in the shadow region than the original expansions. In our case, since associated Legendre polynomials are involved and if we write $\ell = \nu - 1/2$, $P_{\ell}^{m}(\cos\theta) = P_{\nu-1/2}^{m}(\cos\theta)$ are not even functions of ν, the Watson transformation cannot be applied and (24)–(31) have to be used as the basis for numerical computations.

6.3 The Propagation of Low-Frequency Electromagnetic Waves in the Ionized Atmosphere

The ionized atmosphere can be considered as a cold magnetoplasma. For simplicity, we assume that only electrons need to be considered and that the effect of collisions can be neglected. Taking into consideration of only one Fourier component with $e^{-i\omega t}$ time dependence, we have the following fundamental equations

$$\left. \begin{aligned} &\nabla \times \overline{E} - i\omega\mu_0 \overline{H} = 0 \\ &\nabla \times \overline{H} + i\omega\epsilon_0 \overline{E} = -Ne\overline{v} \\ &-im\omega\overline{v} = -e(\overline{E} + \overline{v} \times \overline{B}_0) \end{aligned} \right\} \tag{32}$$

where $-e$ is the charge of the electron, m its mass, $\overline{v}$ the velocity of electrons, N the electron density and $\overline{B}_0$ the magnetic induction of the Earth's magnetic field. We further assume that the Earth's magnetic field and, except near the bottom boundary of the ionosphere, the electron density of the ionized atmosphere can be considered as locally homogeneous. From the third equation of (32)

$$\left.\begin{aligned} im\omega v_r - eB_{0\phi}v_\theta + eB_{0\theta}v_\phi &= eE_r \\ eB_{0\phi}v_r + im\omega v_\theta - eB_{0r}v_\phi &= eE_\theta \\ -eB_{0\theta}v_r + eB_{0r}v_\theta + im\omega v_\phi &= eE_\phi \end{aligned}\right\} \tag{33}$$

$$\begin{aligned} v_r &= -i\frac{e}{m\omega}\frac{1}{Y^2-1}\left[(Y_r^2-1)E_r + (Y_rY_\theta + iY_\phi)E_\theta + (Y_\phi Y_r - iY_\theta)E_\phi\right] \\ v_\theta &= -i\frac{e}{m\omega}\frac{1}{Y^2-1}\left[(Y_rY_\theta - iY_\phi)E_r + (Y_\theta^2-1)E_\theta + (Y_\theta Y_\phi + iY_\phi)E_\phi\right] \\ v_\phi &= -i\frac{e}{m\omega}\frac{1}{Y^2-1}\left[(Y_\phi Y_r + iY_\theta)E_r + (Y_\theta Y_\phi - iY_r)E_\theta + (Y_\phi^2-1)E_\phi\right] \end{aligned} \tag{34}$$

$$\begin{aligned} \overline{J} &= Ne\overline{v} \\ &= i\omega\epsilon_0\frac{X}{Y^2-1} \\ &\quad \cdot\Big\{\hat{r}\left[(Y_r^2-1)E_r + (Y_rY_\theta + iY_\phi)E_\theta + (Y_\phi Y_r - iY_\theta)E_\phi\right] \\ &\quad + \hat{\theta}\left[(Y_rY_\theta - iY_\phi)E_r + (Y_\theta^2-1)E_\theta + (Y_\theta Y_\phi + iY_r)E_\phi\right] \\ &\quad + \hat{\phi}\left[(Y_\phi Y_r + iY_\theta)E_r + (Y_\theta Y_\phi - iY_r)E_\theta + (Y_\phi^2-1)E_\phi\right]\Big\} \end{aligned} \tag{35}$$

where

$$\left.\begin{aligned} X &= \omega_p^2/\omega^2 = Ne^2/m\epsilon_0\omega^2 \\ Y &= \Omega/\omega = eB_0/m\omega \\ Y_r &= eB_{0r}/m\omega \\ Y_\theta &= eB_{0\theta}/m\omega \\ Y_\phi &= eB_{0\phi}/m\omega \end{aligned}\right\} \tag{36}$$

Hence

$$\left.\begin{aligned} \nabla\times\overline{E} - i\omega\mu_0\overline{H} &= 0 \\ \nabla\times\overline{H} + i\omega\epsilon_0\overline{\overline{\epsilon}}\cdot\overline{E} &= 0 \end{aligned}\right\} \tag{37}$$

and

$$\nabla\times\nabla\times\overline{E} - k_0^2\,\overline{\overline{\epsilon}}\cdot\overline{E} = 0 \tag{38}$$

where

$$\bar{\bar{\epsilon}} = \begin{bmatrix} 1 - X\frac{Y_r^2-1}{Y^2-1} & -X\frac{Y_rY_\theta+iY_\phi}{Y^2-1} & -X\frac{Y_\phi Y_r-iY_\theta}{Y^2-1} \\ -X\frac{Y_rY_\theta-iY_\phi}{Y^2-1} & 1 - X\frac{Y_\theta^2-1}{Y^2-1} & -X\frac{Y_\theta Y_\phi+iY_r}{Y^2-1} \\ -X\frac{Y_\phi Y_r+iY_\theta}{Y^2-1} & -X\frac{Y_\theta Y_\phi-iY_r}{Y^2-1} & 1 - X\frac{Y_\phi^2-1}{Y^2-1} \end{bmatrix} \tag{39}$$

In the ionosphere, $Y_\phi \cong 0$, as we shall see later, we have approximately

$$\bar{\bar{\epsilon}} = \begin{bmatrix} 1 - X\frac{Y_r^2-1}{Y^2-1} & -X\frac{Y_rY_\theta}{Y^2-1} & i\frac{XY_\theta}{Y^2-1} \\ -X\frac{Y_rY_\theta}{Y^2-1} & 1 - X\frac{Y_\theta^2-1}{Y^2-1} & -i\frac{XY_r}{Y^2-1} \\ -i\frac{XY_\theta}{Y^2-1} & i\frac{XY_r}{Y^2-1} & 1 + \frac{X}{Y^2-1} \end{bmatrix} \tag{40}$$

$\overline{E}(\bar{r})$ can be expressed approximately as

$$\overline{E}(\bar{r}) = \overline{E}_o(\bar{r})\, e^{i\int \overline{K}(r)\cdot d\bar{r}} \tag{41}$$

Then,

$$\frac{\partial}{\partial x_k}E_j(\bar{r}) = \left[iK_k(\bar{r})\frac{\partial}{\partial x_k}\ln E_{oj}(\bar{r})\right]E_j(\bar{r}) \cong iK_k(\bar{r})E_j(\bar{r}) \tag{42}$$

where it is assumed that $\frac{\partial}{\partial x_k}\ln E_{oj}(\bar{r})$ can be neglected in comparison to $iK_k(\bar{r})$. Therefore

$$\nabla \cong i\overline{K}(\bar{r}) = ik_0 n(\bar{r})\hat{K}(\bar{r}) \tag{43}$$

where $n = K/\epsilon_0$ and $\hat{K}(r) = \overline{K}/K$. Hence, we have

$$\begin{aligned}
&\left[1 - X\frac{Y_r^2-1}{Y^2-1} - (1-\hat{K}_r^2)n^2\right]E_r + \left(-X\frac{Y_rY_\theta+iY_\phi}{Y^2-1} + \hat{K}_r\hat{K}_\theta n^2\right)E_\theta \\
&\quad + \left(-X\frac{Y_\phi Y_r - iY_\phi}{Y^2-1} + \hat{K}_\phi\hat{K}_r n^2\right)E_\phi = 0 \\
&\left(-X\frac{Y_rY_\theta - iY_\phi}{Y^2-1} + \hat{K}_r\hat{K}_\theta n^2\right)E_r + \left[1 - X\frac{Y_\theta^2-1}{Y^2-1} - (1-\hat{K}_\theta^2)n^2\right]E_\theta \\
&\quad + \left(-X\frac{Y_\theta Y_\phi + iY_r}{Y^2-1} + \hat{K}_\theta\hat{K}_\phi n^2\right)E_\phi = 0 \\
&\left(-X\frac{Y_\phi Y_r + iY_\theta}{Y^2-1} + \hat{K}_\phi\hat{K}_r n^2\right)E_r + \left(-X\frac{Y_\theta Y_\phi - iY_r}{Y^2-1} + \hat{K}_\theta\hat{K}_\phi n^2\right)E_\theta \\
&\quad + \left[1 - X\frac{Y_\phi^2-1}{Y^2-1} - (1-\hat{K}_\phi^2)n^2\right]E_\phi = 0
\end{aligned} \tag{44}$$

Therefore

$$\begin{vmatrix} 1 - X\frac{Y_r^2-1}{Y^2-1} & -X\frac{Y_rY_\theta+iY_\phi}{Y^2-1} & -X\frac{Y_\theta Y_r-iY_\phi}{Y^2-1} \\ -(1-\hat{K}_r^2)n^2 & +\hat{K}_r\hat{K}_\theta n^2 & +\hat{K}_\phi\hat{K}_r n^2 \\ -X\frac{Y_rY_\theta-iY_\phi}{Y^2-1} & 1 - X\frac{Y_\theta^2-1}{Y^2-1} & -X\frac{Y_\theta Y_\phi+iY_r}{Y^2-1} \\ +\hat{K}_r\hat{K}_\theta n^2 & -(1-\hat{K}_\theta^2)n^2 & +\hat{K}_\theta\hat{K}_\phi n^2 \\ -X\frac{Y_\phi Y_r+iY_\theta}{Y^2-1} & -X\frac{Y_\theta Y_\phi-iY_r}{Y^2-1} & 1 - X\frac{Y_\phi^2-1}{Y^2-1} \\ +\hat{K}_\phi\hat{K}_r n^2 & +\hat{K}_\theta\hat{K}_\phi n^2 & -(1-\hat{K}_\phi^2)n^2 \end{vmatrix} = 0 \qquad (45)$$

$$n^4 + 2\frac{(X-1)^2 - Y^2 + XY_L^2 + \frac{1}{2}XY_T^2}{X-1+Y^2-XY_L^2}n^2 + \frac{(X-1)\left[(X-1)^2-Y^2\right]}{X-1+Y^2-XY_L^2} = 0 \qquad (46)$$

$$n^2 = 1 - \frac{X(X-1)}{X-1+\frac{1}{2}Y_T^2 \mp \left[\frac{1}{4}Y_T^4 + (X-1)^2Y_L^2\right]^{\frac{1}{2}}}$$

$$= \frac{\pm\left[\frac{1}{4}Y_T^4 + (X-1)^2Y_L^2\right]^{\frac{1}{2}} + (X-1)^2 - \frac{1}{2}Y_T^2}{\pm\left[\frac{1}{4}Y_T^4 + (X-1)^2Y_L^2\right]^{\frac{1}{2}} - (X-1) - \frac{1}{2}Y_T^2} \qquad (47)$$

where $Y_L^2 = (\overline{Y}\cdot\hat{K})^2$, $Y_T^2 = Y^2 - Y_L^2$. The upper and lower signs correspond to the ordinary wave and the extraordinary wave respectively. Equation (44) is the well-known Appleton dispersion formula.

n^2 must be positive in order to make propagation possible. If $n^2 < 0$, we have evanescent wave which diminishes exponentially. For the ordinary wave, the numerator of (47) is always positive and propagation is possible if and only if $[1/4Y_T^4+(X-1)^2Y_L^2]^{1/2} > X-1+1/2Y_T^2$, i.e., if $0 < X < 1$ or

$$X > \frac{Y^2-1}{Y_L^2-1} \qquad (48)$$

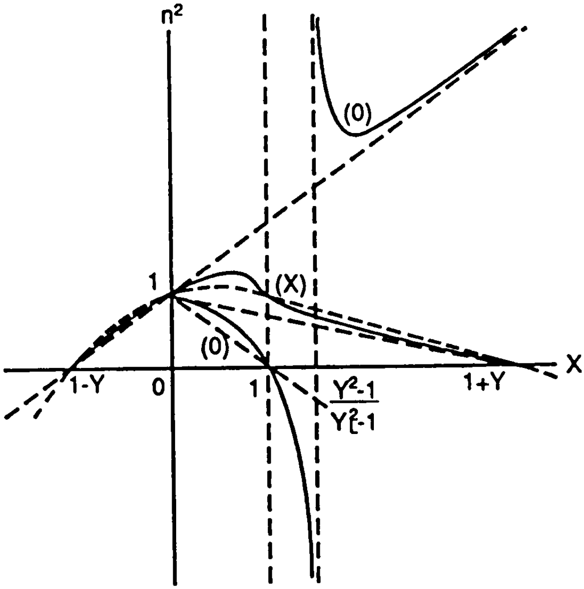

Figure 6.2a

provided $X > 1$. Let τ be the acute angle between $\overline{K}$ and $\overline{B}_o$ or $-\overline{B}_o$, then

$$\tan^2\tau = \frac{Y_\tau^2}{Y_L^2} < (X-1)\left(1 - \frac{1}{Y_L^2}\right) \tag{49}$$

For the extraordinary wave, if $X > 1$, the denominator of (47) is always negative and propagation is possible if and only if $[1/4Y_T^4 + (X-1)^2Y_L^2]^{\frac{1}{2}} > (X-1)^2 - 1/2Y_T^2$, i.e.,

$$X - 1 < Y \tag{50}$$

Using (47), we can draw the curve of n^2 vs X for both the ordinary wave and the extraordinary wave. For $Y > 1$, if $Y_T \neq 0$, $Y_L \neq 0$, the curve is as shown in Fig. 6.2a and if $Y_T = 0$, $Y_L = Y$, the curve is as shown in Fig. 6.2b. The whistler wave corresponds to the part of the curve marked by (O) with $X > (Y^2-1)/(Y_L^2-1)$ in the upper right portion of Fig. 6.2a. It has an asymptote whose equation is $n^2 = 1 + X/(Y-1)$. This part of the curve is separated from the part of the curve marked by (X) with $0 \leq X \leq 1$. The two parts of the curve become closer as propagation becomes more longitudinal. As propagation becomes truly longitudinal, i.e., if $Y_T = 0$, $Y_L = Y$, as

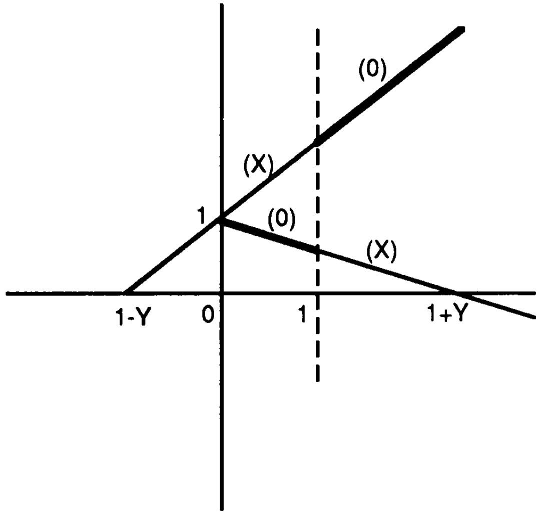

Figure 6.2b

shown in Fig. 6.2b, these two parts of the curve form a straight line with $0 \leq X \leq \infty$. Under the condition of quasi-longitudinal propagation, if the longitudinal approximation is sufficiently good, there may be appreciable coupling between the waves corresponding to the two parts of the curve due to the so-called tunnel effect as the wave nature and the gradient of electron density are considered. When X is greater than $(Y^2 - 1)/(Y_L^2 - 1)$, the curve of n^2 vs. X can be approximated by its asymptote and propagation is essentially longitudinal with

$$n^2 \simeq 1 + \frac{X}{|Y_L| - 1} \simeq 1 + \frac{X}{Y - 1} \tag{51}$$

Considering the variation of electron density with altitude above the bottom boundary of the ionosphere, we see the following picture during the process of penetration of the wave from the terrestrial waveguide into the ionosphere. We distinguish between two cases. In Case 1, at the boundary the electron density jumps abruptly (in comparison with the wavelength) to a value such that $X > (Y^2 - 1)/(Y_L^2 - 1)$, then, if the boundary condition suitable for penetration is satisfied, a portion of the wave will penetrate from the terrestrial waveguide into the ionosphere as an ordinary wave propagating approximately longitudinally. In Case 2, the electron density at the bottom boundary of the ionosphere is such that $X < 1$, then, if the boundary condition suitable for penetration is satisfied, a portion of the wave will penetrate from the terrestrial waveguide into the ionosphere as an extraordinary wave, but in order to ensure further propagation to form a whistler wave, the propagation must be very close to longitudinal (if not exactly so) so that the tunnel barriers is sufficiently thin to give appreciable coupling.

In the lower ionized atmosphere, i.e., in the ionosphere, we assume that the Earth's magnetic field can be approximated by the field produced by a magnetic dipole at the Earth's center parallel to the magnetic polar axis pointing southward. Then, at point (r, θ, ϕ)

$$\overline{B}_o = -\hat{r}\frac{2\mu_0 M\cos\theta}{4\pi r^3} - \hat{\theta}\,\frac{\mu_0 M\sin\theta}{4\pi r^3} \tag{52}$$

where M is the moment of the magnetic dipole. Let δ be the acute angle between $\overline{B}_o$ and $-\overline{r}$ or $\overline{r}$. Then

$$\left.\begin{aligned}
Y_r &= -\frac{2\mu_0 eM\cos\theta}{4\pi r^3 m\omega} \\
Y_\theta &= -\frac{\mu_0 eM\sin\theta}{4\pi r^3 m\omega} \\
Y_\phi &= 0 \\
Y &= \frac{\mu_0 eM}{4\pi r^3 m\omega}\sqrt{1+3\cos^2\theta} \\
\tan\delta &= \pm\frac{B_{0\theta}}{B_{0r}} = \pm\frac{1}{2}\tan\theta \\
\sin\delta &= \frac{\sin\theta}{\sqrt{1+3\cos^2\theta}} \\
\cos\delta &= \pm\frac{2\cos\theta}{\sqrt{1+3\cos^2\theta}}
\end{aligned}\right\} \tag{53}$$

where the $(+)$ and $(-)$ signs correspond to the northern and southern hemispheres respectively. We also have

$$\left.\begin{aligned}|Y_L| = Y_L &= Y_r\hat{K}_r + Y_\theta \hat{K}_\theta, && \text{for propagation from the southern to the northern hemisphere}\\ &= -Y_L = -(Y_r\hat{K}_r + Y_\theta\hat{K}_\theta), && \text{for propagation from the northern to the southern hemisphere}\\ Y_T^2 &= Y^2 - (Y_r\hat{K}_r + Y_\theta\hat{K}_\theta)^2\\ &= (Y_r\hat{K}_\theta - Y_\theta\hat{K}_r)^2 + Y^2\hat{k}_\phi^2\\ \hat{K}_r^2 &+ \hat{K}_\theta^2 + \hat{K}_\phi^2 = 1\end{aligned}\right\} \quad (54)$$

In the following, as the criteria of quasi-longitudinal approximation, we assume that

$$\left.\begin{aligned}&Y_T^2/Y^2 \ll 1\\ &(Y_r\hat{K}_\theta - Y_\theta\hat{K}_r)^2/Y^2 \ll 1\\ &\hat{K}_\phi^2 \ll 1\\ &|(Y_r\hat{K}_\theta - Y_\theta\hat{K}_r)\hat{K}_\phi|/Y \ll 1\end{aligned}\right\} \quad (55)$$

so that

$$\left.\begin{aligned}&|Y_L| = (Y^2 - Y_T^2)^{\frac{1}{2}} = Y\left(1 - \frac{Y_T^2}{2Y^2} + \cdots\right) \cong Y\\ &\hat{K}_r^2 + \hat{K}_\theta^2 = 1 - \hat{K}_\phi^2 \cong 1\end{aligned}\right\} \quad (56)$$

In the ionosphere, (44) can be rewritten approximately as

$$\begin{aligned}&\left[1 - X\frac{Y_r^2 - 1}{Y - 1} - \left(1 + \frac{X}{Y - 1}\right)(1 - \hat{K}_r^2)\right]E_r\\ &\qquad + \left[-X\frac{Y_rY_\theta}{Y^2 - 1} + \left(1 + \frac{X}{Y - 1}\right)\hat{K}_r\hat{K}_\theta\right]E_\theta\\ &\qquad + \left[i\frac{XY_\theta}{Y^2 - 1} + \left(1 + \frac{X}{Y - 1}\right)\hat{K}_\phi\hat{K}_r\right]E_\phi = 0\\ &\left[-X\frac{Y_rY_\theta}{Y^2 - 1} + \left(1 + \frac{X}{Y - 1}\right)\hat{K}_r\hat{K}_\theta\right]E_r\\ &\qquad + \left[1 - X\frac{Y_\theta^2 - 1}{Y^2 - 1} - \left(1 + \frac{X}{Y - 1}\right)\left(1 - \hat{K}_\theta^2\right)\right]E_\theta\end{aligned}$$

$$+\left[-i\frac{XY_r}{Y^2-1}+1+\left(1+\frac{X}{Y-1}\right)\hat{K}_\theta\hat{K}_\phi\right]E_\phi=0$$

$$\left[-i\frac{XY_\theta}{Y^2-1}+\left(1+\frac{X}{Y-1}\right)\hat{K}_\phi\hat{K}_r\right]E_r$$
$$+\left[i\frac{XY_r}{Y^2-1}+\left(1+\frac{X}{Y-1}\right)\hat{K}_\theta\hat{K}_\phi\right]E_\theta$$
$$-\frac{XY}{Y^2-1}E_\phi=0 \tag{57}$$

Solving for E_r and E_θ in terms of E_ϕ , we get to the first order approximation,

$$\left.\begin{aligned} E_r &= \tfrac{1}{(X-1)Y}\left\{i\left[(X-1)Y_\theta-\left(1+\tfrac{X}{Y-1}\right)(Y_r\hat{K}_\theta-Y_\theta\hat{K}_r)\hat{K}_r\right]\right.\\ &\quad\left.+\left(1+\tfrac{X}{Y-1}\right)Y\hat{K}_\phi\hat{K}_r\right\}E_\phi\\ E_\theta &= \tfrac{1}{(X-1)Y}\left\{-i\left[(X-1)Y_r+\left(1+\tfrac{X}{Y-1}\right)(Y_r\hat{K}_\theta-Y_\theta\hat{K}_r)\hat{K}_\theta\right]\right.\\ &\quad\left.+\left(1+\tfrac{X}{Y-1}\right)Y\hat{K}_\theta\hat{K}_\phi\right\}E_\phi \end{aligned}\right\} \tag{58}$$

By the first equation of (37) and (43), also to the first-order approximation, we have

$$\left.\begin{aligned} H_\theta &= \sqrt{\frac{\epsilon_0}{\mu_0}}\left(1+\frac{X}{Y-1}\right)^{\frac{1}{2}}\left(i\frac{Y_\theta}{Y}\hat{K}_\phi-\hat{K}_r\right)E_\phi\\ H_\phi &= \mp i\sqrt{\frac{\epsilon_0}{\mu_0}}\left(1+\frac{X}{Y-1}\right)^{\frac{1}{2}}E_\phi \end{aligned}\right\} \tag{59}$$

Where the upper and lower signs apply respectively to the case of propagation from the southern to the northern hemisphere and to the case of propagation from the northern to the southern hemisphere.

6.4 The Penetration Problem

Let us examine the conditions for penetration from the terrestrial waveguide into the ionosphere at the point (b,θ,ϕ) on the boundary surface, b being the radius of the boundary surface. Equating the expressions for $E_\theta, E_\phi, H_\theta$, and H_ϕ on the two sides of the surface, we get to the first order approximation.

$$E^t_\phi\Big|_{r=b^-} = \frac{1}{b\sin\theta}\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}(r\Gamma^+)\Big|_{r=b^-} = E_\phi\Big|_{r=b^+} \tag{60a}$$

$$\begin{aligned} E^\dagger_\theta\Big|_{r=b^-} &= \frac{1}{b}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}(r\Gamma^+)\Big|_{r=b^-} = E_\theta\Big|_{r=b^+} \\ &= \frac{1}{(X-1)Y}\{[-i(X-1)Y_r \\ &+ \left(1+\frac{X}{Y-1}\right)(Y_r\hat{k}_\theta - Y_\theta\hat{K}_r)\hat{K}_\theta\Big] \\ &+ \left(1+\frac{X}{Y-1}\right)Y\hat{K}_\theta\hat{K}_\phi\Big\} E_\phi\Big|_{r=b^+} \end{aligned} \tag{60b}$$

$$\begin{aligned} H^\dagger_\theta\Big|_{r=b^-} &= -\frac{i\omega\epsilon_0}{\sin\theta}\frac{\partial\Gamma^\dagger}{\partial\phi}\Big|_{r=b^-} = H_\theta\Big|_{r=b^+} \\ &= \sqrt{\frac{\epsilon_0}{\mu_0}}\left(1+\frac{X}{Y-1}\right)^{\frac{1}{2}}\left(i\frac{Y_\theta}{Y}\hat{K}_\phi - \hat{K}_r\right)E_\phi\Big|_{r=b^+} \end{aligned} \tag{60c}$$

$$\begin{aligned} H^\dagger_\phi\Big|_{r=b^-} &= i\omega\epsilon_0\frac{\partial\Gamma^t}{\partial\theta}\Big|_{r=b^-} = H_\phi\Big|_{r=b^+} \\ &= \mp i\sqrt{\frac{\epsilon_0}{\mu_0}}\left(1+\frac{X}{Y-1}\right)^{\frac{1}{2}} E_\phi\Big|_{r=b^+} \end{aligned} \tag{60d}$$

From (60a) and (60b),

$$\begin{aligned} \frac{\partial}{\partial\theta}\frac{\partial}{\partial r}(r\Gamma^\dagger)\Big|_{r=b} &= \frac{1}{X-1}Y\Big\{-i\Big[(X-1)Y_r \\ &+ \left(1+\frac{X}{Y-1}\right)(Y_r\hat{K}_\theta - Y_\theta\hat{K}_r)\hat{K}_\theta\Big] \\ &+ \left(1+\frac{X}{Y-1}\right)Y\hat{K}_\theta\hat{K}_\phi\Big\}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}(r\Gamma^\dagger)\Big|_{r=b} \end{aligned} \tag{61}$$

and from (60c) and (60d),

$$\frac{\partial\Gamma^\dagger}{\partial\phi}\Big|_{r=b} = \pm\left(i\hat{K}_r + \frac{Y_\theta}{Y}\hat{K}_\phi\right)\sin\theta\frac{\partial\Gamma^\dagger}{\partial\theta}\Big|_{r=b} \tag{62}$$

Since $E_\theta, E_\phi, H_\theta$ and H_ϕ are continuous for $b \geq r \geq a$, so are $\Gamma^\dagger$ and $\partial\Gamma^\dagger/\partial r$, and therefore (62) can also be written as

$$\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}(r\Gamma^\dagger)\bigg|_{r=b} = \pm\left(i\hat{K}_r + \frac{Y_\theta}{Y}\hat{K}_\phi\right)\sin\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial r}(r\Gamma^\dagger)\bigg|_{r=b} \tag{63}$$

From (63) and (61),

$$\left\{-i\left[(X-1)\frac{Y_r}{Y} + \left(1+\frac{X}{Y-1}\right)\frac{1}{Y}(Y_r\hat{K}_\theta + Y_\theta\hat{K}_r)\hat{K}_\theta\right]\right.$$
$$\left. + \left(1+\frac{X}{Y-1}\right)\hat{K}_\theta\hat{K}_\phi\right\}\left(i\hat{K}_r + \frac{Y_\theta}{Y}\hat{K}_\phi\right)$$
$$= \pm(X-1) \tag{64}$$

By equating the real and imaginary parts respectively, we get

$$(X-1)\frac{Y_r}{Y}\hat{K}_r + \left(1+\frac{X}{Y-1}\right)\frac{1}{Y}\left(Y_r\hat{K}_\theta - Y_\theta\hat{K}_r\right)\hat{K}_r\hat{K}_\theta = \pm(X-1)$$
$$(X-1)\frac{Y_rY_\theta}{Y^2} + \left(1+\frac{X}{Y-1}\right)\frac{Y_\theta}{Y^2}\left(Y_r\hat{K}_\theta - Y_\theta\hat{K}_r\right)\hat{K}_\theta$$
$$= \left(1+\frac{X}{Y-1}\right)\hat{K}_r\hat{K}_\theta \quad \text{or} \quad \hat{K}_\phi = 0 \tag{65}$$

In the first equation of (65), the upper and lower signs apply to the case of propagation from southern to the northern hemisphere and to the case of propagation from the northern to the southern hemisphere respectively. In the former case,

$$\left.\begin{array}{ll} \dfrac{Y_r}{Y} = -\dfrac{2\cos\theta}{\sqrt{1+3\cos^2\theta}} \gtrless 0, & \text{in the } {\text{southern} \atop \text{northern}} \text{ hemisphere} \\ \dfrac{Y_\theta}{Y} = -\dfrac{\sin\theta}{\sqrt{1+3\cos^2\theta}} < 0 & \\ \hat{K}_r \gtrless 0, & \text{in the } {\text{southern} \atop \text{northern}} \text{ hemisphere} \\ \hat{K}_\theta < 0 & \end{array}\right\} \tag{66}$$

and in the latter case,

$$\left.\begin{array}{ll} \dfrac{Y_r}{Y} = -\dfrac{2\cos\theta}{\sqrt{1+3\cos^2\theta}} \gtrless 0, & \text{in the } {\text{southern} \atop \text{northern}} \text{ hemisphere} \\ \dfrac{Y_\theta}{Y} = -\dfrac{\sin\theta}{\sqrt{1+3\cos^2\theta}} < 0 & \\ \hat{K}_r \gtrless 0, & \text{in the } {\text{northern} \atop \text{southern}} \text{ hemisphere} \\ \hat{K}_\theta > 0 & \end{array}\right\} \tag{67}$$

When $\hat{K}_\phi \neq 0$ but $\hat{K}_\phi \cong 0$ (65) can also be transformed to the following form approximately

$$\left.\begin{aligned} &\hat{K}_\theta^3 - \hat{K}_\theta \pm \frac{X-1}{X+Y-1}\frac{Y-1}{Y}Y_\theta = 0 \\ &\hat{K}_r^3 - \frac{XY}{X+Y-1}\hat{K}_r \pm \frac{X-1}{X+Y-1}\frac{Y-1}{Y}\frac{Y^2+Y_\theta^2}{Y_r} = 0 \\ &\hat{K}_\theta \cong \mp(1-\hat{K}_r^2)^{\frac{1}{2}} \end{aligned}\right\} \tag{68}$$

Many conclusions can be drawn from the examination of the first equation of (65). Since at the bottom boundary of the ionosphere, X is much smaller than Y so that $1+\frac{X}{Y-1}$ is essentially equal to 1, the first equation of (65) can be written approximately

$$\begin{aligned} &\pm(X-1)\left(1 \pm \frac{2\cos\theta}{\sqrt{1+3\cos^2\theta}}\hat{K}_r\right) \\ &\quad = \left(-\frac{2\cos\theta}{\sqrt{1+3\cos^2\theta}}\hat{K}_\theta + \frac{\sin\theta}{\sqrt{1+3\cos^2\theta}}\hat{K}_r\right)\hat{K}_r\hat{K}_\theta \end{aligned} \tag{69}$$

i.e.,

$$\begin{aligned} &\left[(X-1)^2 - \hat{K}_r^4\hat{K}_\theta^2\right]\tan^2\theta + 4\hat{K}_r^3\hat{K}_\theta(X-1+\hat{K}_\theta^2)\tan\theta \\ &\quad + 4\left[(X-1)^2 - \hat{K}_r^2(X-1+\hat{K}_\theta^2)^2\right] = 0 \end{aligned} \tag{70}$$

Hence,

$$\begin{aligned} \tan\theta &= -2\frac{\hat{K}_r^3\hat{K}_\theta(X-1+\hat{K}_\theta^2)}{(X-1)^2-\hat{K}_r^4\hat{K}_\theta^2} \mp 2\left\{\frac{\hat{K}_r^6\hat{K}_\theta^2(X-1+\hat{K}_\theta^2)^2}{\left[(X-1)^2-\hat{K}_r^4\hat{K}_\theta^2\right]^2}\right. \\ &\qquad \left. - \frac{(X-1)^2-\hat{K}_r^2(X-1+\hat{K}_\theta^2)^2}{(X-1)^2-\hat{K}_r^4\hat{K}_\theta^2}\right\}^{1/2} \\ &= -2\frac{\hat{K}_r^3\hat{K}_\theta(X-1+\hat{K}_\theta^2)}{(X-1)^2-\hat{K}_r^4\hat{K}_\theta^2} + 2\frac{\hat{K}_\theta\,|X-1|}{(X-1)^2-\hat{K}_r^4\hat{K}_\theta^2} \\ &\qquad \left[\hat{K}_r^2(1+\hat{K}_r^2) - (X-1-\hat{K}_r^2)^2\right]^{\frac{1}{2}} \end{aligned} \tag{71}$$

If $\hat{K}_\theta = 0$, then $\theta = 0$ or π and $\hat{L}_r = 1$. Hence the magnetic poles are always possible points of penetration. If $\hat{K}_\theta \neq 0$, in order that $|\tan\theta|$ be real and positive, we must have

$$\left[(X-1)^2 - \hat{K}_r^4\hat{K}_\theta^2\right]\left[(X-1-\hat{K}_r^2)^2 - \hat{K}_r^2\right] \leq 0 \tag{72}$$

i.e.,

$$1 + \hat{K}_r(1+\hat{K}_r) > X > 0 \tag{73}$$

From the above derivations, we see that X has a maximum value of about 3, the higher the value of X, the higher the value of $\hat{K}_r$, and that penetration occurs in narrow ranges of magnetic latitude (except at the isolated magnetic poles).

In order to determine the constants $A^\dagger_{\ell m}$ in the series expansions of $\Gamma^\dagger$ and of the field components, we make use of the fact that whistlers are pulse signals and that penetration occurs in narrow ranges of magnetic latitude, one in each hemisphere, besides at the magnetic poles. We know that

$$\begin{aligned}
E^\dagger_\phi\Big|_{r=b^-} &= \frac{1}{b}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\frac{\partial}{\partial r}(r\Gamma^\dagger)\Big|_{r=b^-} \\
&= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(F_{\ell m} + G_{\ell m}A^\dagger_{\ell m})P_\ell^m(\cos\theta)e^{im(\phi-\phi)} = E_\theta\Big|_{r=b^+} \\
H^\dagger_\theta\Big|_{r=b^-} &= -\frac{i\omega\epsilon_0}{\sin\theta}\frac{\partial\Gamma^+}{\partial\phi}\Big|_{r=b^-} \\
&= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(K_{\ell m} + L_{\ell m}A^\dagger_{\ell m})P_\ell^m(\cos\theta)e^{im(\phi-\phi_0)} \\
&= -\sqrt{\frac{\epsilon_0}{\mu_0}}\left(1+\frac{X}{Y-1}\right)\hat{K}_r E_\phi\Big|_{r=b^+}
\end{aligned} \tag{74}$$

where from (31) and (27)

$$F_{\ell m} = -\frac{p}{4\pi\epsilon_0}\frac{k_0^2}{r_0\sin\theta}(-1)^m(2\ell+1)m$$
$$\left\{j_\ell(k_0r_0)\left[h_\ell^{(1)\prime}(k_0b) + \frac{1}{k_0b}h_\ell^{(1)}(k_0b)\right]\right.$$

$$
- \frac{j_\ell'(k_0a) + \frac{1}{k_0a} j_\ell(k_0a)}{h_\ell^{(2)'}(k_0a) + \frac{1}{k_0a} h_\ell^{(2)}(k_0a)}
\cdot h_\ell^{(1)}(k_0r_0)\left[h_\ell^{(2)'}(k_0b) + \frac{1}{k_0b} h_\ell^{(2)}(k_0b)\right]\Bigg\} P_\ell^{-m}(\cos\theta_0)
$$

$$
G_{\ell m} = -\frac{p}{4\pi\epsilon_0}\frac{k_0^2}{r_0\sin\theta}(-1)^m(2\ell+1)m\Bigg\{ h_\ell^{(1)'}(k_0b) + \frac{1}{k_0b} h_\ell^{(1)}(k_0b)
- \frac{h_\ell^{(1)}(k_0a) + \frac{1}{k_0a} h_\ell^{(1)}(k_0a)}{h_\ell^{(2)'}(k_0a) + \frac{1}{k_0a} h_\ell^{(2)}(k_0a)}
\left[h_\ell^{(2)'}(k_0b) + \frac{1}{k_0b} h_\ell^{(2)}(k_0b)\right]\Bigg\} P_\ell^{-m}(\cos\theta_0)
$$

$$
K_{\ell m} = i\frac{p}{4\pi}\frac{\omega k_0}{r_0}\frac{1}{\sin\theta}(-1)^m(2\ell+1)m\Bigg\{ j_\ell(k_0r_0)h_\ell^{(1)}(k_0b)
- \frac{j_\ell'(k_0a) + \frac{1}{k_0a} j_\ell(k_0a)}{h_\ell^{(2)}(k_0a) + \frac{1}{k_0a} h_\ell^{(2)} k_0a} h_\ell^{(1)}(k_0r_0)h_\ell^{(2)}(k_0b)\Bigg\} P_\ell^{-m}(\cos\theta_0)
$$

$$
L_{\ell m} = -\frac{p}{4\pi}\frac{\omega k_0}{r_0}\frac{1}{\sin\theta}(-1)^m(2\ell+1)m\Bigg\{ h_\ell^{(1)}(k_0b)
- \frac{h_\ell^{(1)'}(k_0a) + \frac{1}{k_0a} h_\ell^{(1)}(k_0a)}{h_\ell^{(2)'}(k_0a) + \frac{1}{k_0a} h_\ell^{(2)}(k_0a)} h_\ell^{(2)}(k_0b)\Bigg\} P_\ell^{-m}(\cos\theta_0) \tag{75}
$$

Multiplying the both sides of the equations of (74) by $\frac{1}{2\pi}e^{-im(\phi-\phi_0)}$ and integrating with respect to ϕ from, 0 to 2π and again multiplying the both sides of the equations of (74) by $(1+\frac{1}{2})\frac{(l-m)!}{(l+m)!}P_\ell^m(\cos\theta)\sin^2\theta$

and integrating with respect to θ from 0 to π, we have

$$F_{\ell m} + G_{\ell m} A^{\dagger}_{\ell m} = \left(\ell + \frac{1}{2}\right) \frac{(\ell - m)!}{(\ell + m)!} E_{\phi}(b, \theta) P^{m}_{\ell}(\cos\theta) \sin^2\theta \, \triangle\theta$$

$$K_{\ell m} + L_{\ell m} A^{\dagger}_{\ell m} = -\left(\ell + \frac{1}{2}\right) \frac{(\ell - m)!}{(\ell + m)!} \sqrt{\frac{\epsilon_0}{\mu_0}} \left(1 + \frac{X}{Y - 1}\right)^{\frac{1}{2}} \hat{K}_r E_{\phi}(b, \theta) \cdot P^{m}_{\ell}(\cos\theta) \sin^2\theta \, \triangle\theta \tag{76}$$

where θ is the magnetic co-latitude of the central line of the range of magnetic latitude suitable for penetration. Hence,

$$\frac{K_{\ell m} + L_{\ell m} A^{\dagger}_{\ell m}}{F_{\ell m} + G_{\ell m} A^{\dagger}_{\ell m}} = -\sqrt{\frac{\epsilon_0}{\mu_0}} \left(1 + \frac{X}{Y - 1}\right)^{\frac{1}{2}} \hat{K}_r \equiv -Q(b, \theta) \tag{77}$$

and

$$A^{\dagger}_{\ell m} = -\frac{K_{\ell m} + F_{\ell m} Q(b, \theta)}{L_{\ell m} + G_{\ell m} Q(b, \theta)} \tag{78}$$

6.5 Ray Tracing

Following Haselegrove [2] and Budden [3], we define

$$G(\overline{r}, \overline{p}) = \frac{(\overline{p} \cdot \overline{p})^{\frac{1}{2}}}{n(\overline{r}, \hat{p})} \tag{79}$$

where

$$\overline{p} = \frac{\overline{K}}{k_0} = \hat{K} n = \hat{p} p = \hat{p} n \tag{80}$$

Then, the propagation of the ray is governed by the following equations:

$$\left.\begin{aligned} \frac{\dot{\overline{r}}}{c} &= \frac{\partial G}{\partial \overline{p}} \\ \frac{\dot{\overline{p}}}{c} &= -\frac{\partial G}{\partial \overline{r}} \end{aligned}\right\} \tag{81}$$

where, in Cartesian coordinates,

$$\frac{\partial G}{\partial p_i} = \frac{p_i}{n^2} - \frac{1}{n} \sum_k \frac{\partial n}{\partial \hat{p}_k} \frac{\partial \hat{p}_k}{\partial p_i} \qquad (i, k = 1, 2, 3) \tag{82}$$

In spherical coordinates, letting

$$\left.\begin{aligned}\bar{r} &= (r, \theta, \phi)\\ \bar{p} &= (p, \gamma, \psi)\end{aligned}\right\} \tag{83}$$

we have

$$\bar{p} = p(\hat{x}\sin\gamma\cos\psi + \hat{y}\sin\gamma\sin\psi + \hat{z}\cos\gamma) = \hat{r}p_r + \hat{\theta}p_\theta + \hat{\phi}p_\phi \tag{84}$$

It is easy to show that

$$\left.\begin{aligned}p &= n\\ p_r &= p[\sin\theta\sin\gamma\cos(\phi-\psi) + \cos\theta\cos\gamma]\\ p_\theta &= p[\cos\theta\sin\gamma\cos(\phi-\psi) - \sin\theta\cos\gamma]\\ p_\phi &= -p\sin\gamma\sin(\phi-\psi)\end{aligned}\right\} \tag{85}$$

The temporal variations of θ and ϕ, $\dot{\theta}$ and $\dot{\phi}$, constitute a rotational motion with angular velocity vector

$$\bar{\Omega} = \hat{\phi}\dot{\theta} + \hat{z}\dot{\phi} = \hat{\phi}\dot{\theta} + (\hat{r}\cos\theta - \hat{\theta}\sin\theta)\dot{\phi} \tag{86}$$

Then,

$$\left.\begin{aligned}\dot{\hat{r}} &= \left[\hat{\phi}\dot{\theta} + (\hat{r}\cos\theta - \hat{\theta}\sin\theta)\dot{\phi}\right] \times \hat{r} = \hat{\theta}\dot{\theta} + \hat{\phi}\dot{\phi}\sin\theta\\ \dot{\hat{\theta}} &= \left[\hat{\phi}\dot{\theta} + (\hat{r}\cos\theta - \hat{\theta}\sin\theta)\dot{\phi}\right] \times \hat{\theta} = -\hat{r}\dot{\theta} + \hat{\phi}\dot{\phi}\cos\theta\\ \dot{\hat{\phi}} &= \left[\hat{\phi}\dot{\theta} + (\hat{r}\cos\theta - \hat{\theta}\sin\theta)\dot{\phi}\right] \times \hat{\phi} = -\hat{\theta}\dot{\phi}\cos\theta - \hat{r}\dot{\phi}\sin\theta\end{aligned}\right\} \tag{87}$$

$$\left.\begin{aligned}\dot{\bar{r}} &= \hat{r}\dot{r} + \hat{\theta}r\dot{\theta} + \hat{\phi}r\dot{\phi}\sin\theta\\ \dot{\bar{p}} &= \hat{r}\dot{p}_r + p_r(\hat{\theta}\dot{\theta} + \hat{\phi}\dot{\phi}\sin\theta) + \hat{\theta}\dot{p}_\theta + p_\theta(-\hat{r}\dot{\theta} + \hat{\phi}\dot{\phi}\cos\theta)\\ &\quad + \hat{\phi}\dot{p}_\phi - p_\phi(\hat{\theta}\dot{\phi}\cos\theta + \hat{r}\dot{\phi}\sin\theta)\\ &= \hat{r}(\dot{p}_r - p_\theta\dot{\theta} - p_\phi\dot{\phi}\sin\theta) + \hat{\theta}(\dot{p}_\theta + p_r\dot{\theta} - p_\phi\dot{\phi}\cos\theta)\\ &\quad + \hat{\phi}(\dot{p}_\phi + p_r\dot{\phi}\sin\theta + p_\theta\dot{\phi}\cos\theta)\end{aligned}\right\} \tag{88}$$

Hence,

$$\frac{1}{c}\left(\hat{r}\dot{r} + \hat{\theta}r\dot{\theta} + \hat{\phi}r\dot{\phi}\sin\theta\right) = \hat{r}\frac{\partial G}{\partial p_r} + \hat{\theta}\frac{\partial G}{\partial p_\theta} + \hat{\phi}\frac{\partial G}{\partial p_\phi} \tag{89}$$

i.e.,

$$\left.\begin{aligned}\frac{\dot{r}}{c} &= \frac{\partial G}{\partial p_r}\\ \frac{\dot{\theta}}{c} &= \frac{1}{r}\frac{\partial G}{\partial p_\theta}\\ \frac{\dot{\phi}}{c} &= \frac{1}{r\sin\theta}\frac{\partial G}{\partial p_\phi}\end{aligned}\right\} \tag{90}$$

Also,

$$\begin{aligned}\frac{1}{c}\Big[\hat{r}(\dot{p}_r - p_\theta\dot{\theta} - p_\phi\dot{\phi}\sin\theta) + \hat{\theta}(\dot{p}_\theta + p_r\dot{\theta} - p_\phi\dot{\phi}\cos\theta)\\ + \hat{\phi}(\dot{p}_\phi + p_r\dot{\phi}\sin\theta + p_\theta\dot{\phi}\cos\theta)\\ = \hat{r}\frac{1}{n}\frac{\partial n}{\partial r} + \hat{\theta}\frac{1}{rn}\frac{\partial n}{\partial\theta} + \hat{\phi}\frac{1}{rn\sin\theta}\frac{\partial n}{\partial\phi}\end{aligned} \tag{91}$$

i.e.,

$$\begin{aligned}\frac{\dot{p}_r}{c} &= \frac{1}{n}\frac{\partial n}{\partial r} + p_\theta\frac{\dot{\theta}}{c} + p_\phi\sin\theta\frac{\dot{\phi}}{c} = \frac{1}{n}\frac{\partial n}{\partial r} + \frac{p_\theta}{r}\frac{\partial G}{\partial p_\theta} + \frac{p_\phi}{r}\frac{\partial G}{\partial p_\phi}\\ \frac{\dot{p}_\theta}{c} &= \frac{1}{rn}\frac{\partial n}{\partial\theta} - p_r\frac{\dot{\theta}}{c} + p_\phi\cos\theta\frac{\dot{\phi}}{c} = \frac{1}{rn}\frac{\partial n}{\partial\theta} - \frac{p_r}{r}\frac{\partial G}{\partial p_\theta} + \frac{p_\phi\cot\theta}{r}\frac{\partial G}{\partial p_\phi}\\ \frac{\dot{p}_\phi}{c} &= \frac{1}{rn\sin\theta}\frac{\partial n}{\partial\phi} - p_r\sin\theta\frac{\dot{\phi}}{c} - p_\theta\cos\theta\frac{\dot{\phi}}{c}\\ &= \frac{1}{rn\sin\theta}\frac{\partial n}{\partial\phi} - \frac{p_r + p_\theta\cot\theta}{r}\frac{\partial G}{\partial p_\phi}\end{aligned} \tag{92}$$

For whistler ray, assuming quasi-longitudinal propagation, we have approximately,

$$G = \frac{(p_r^2 + p_\theta^2 + p_\phi^2)^{\frac{1}{2}}}{\left[1 + \dfrac{X}{\dfrac{|Y_r p_r + Y_\theta p_\theta + Y_\phi p_\phi|}{(p_r^2 + p_\theta^2 + p_\phi^2)^{\frac{1}{2}}} - 1}\right]^{\frac{1}{2}}} \tag{93}$$

After some manipulations, we can show that

$$\frac{\partial G}{\partial p_r} = \frac{p_r}{n^2} + \frac{1}{2n^5}\frac{X}{(|Y_L| - 1)^2}\frac{Y_L}{|Y_L|}\left[(p_\theta^2 + p_\phi^2)Y_r - \Big(Y_\theta p_\theta + Y_\phi p_\phi\Big)p_r\right]$$

$$\frac{\partial G}{\partial p_\theta} = \frac{p_\theta}{n^2} + \frac{1}{2n^5}\frac{X}{(\mid Y_L \mid -1)^2}\frac{Y_L}{\mid Y_L \mid}\left[(p_\phi^2 + p_r^2)Y_\theta - (Y_\phi p_\phi + Y_r p_r)p_\theta\right]$$

$$\frac{\partial G}{\partial p_\phi} = \frac{p_\phi}{n^2} + \frac{1}{2n^5}\frac{X}{(\mid Y_L \mid -1)^2}\frac{Y_L}{\mid Y_L \mid}$$
$$\left[(p_r^2 + p_\theta^2)Y_\phi - (Y_r p_r + Y_\theta p_\theta)p_\phi\right] \tag{94}$$

If X, Y and $\overline{p}$ as functions of $\overline{r}$ are known, then by using (90), (92) and (94), we can perform ray tracing in the ionized atmosphere by numerical methods, considering the ionized atmosphere as locally homogeneous. This can be done comparatively easily in the ionosphere, but in the magnetosphere, the Earth's magnetic field is much distorted by the solar wind, the configuration of the Earth's magnetic field and the electron-density distribution are rather complicated, quite variable and not well known. To calculate the change of intensity of the whistler beam as it travels along the path, we need to know the attenuation caused by the collisions of electrons with other particles and to know the divergence of the beam and its change during the first half of the path and the convergence of the beam and its change during the second half of the path. In this chapter, the effect of collisions is neglected. To determine the change of intensity due to the divergence and convergence of the beam, we can calculate the paths of four or more neighboring rays by ray tracing to find the change of the cross-sectional area of the beam, the intensity of the beam being inversely proportional to the square root of the cross-sectional area if absorption is neglected.

6.6 Re-Entry Problem

Suppose that there is a whistler ray which penetrates at the point (b, θ, ϕ) in a certain narrow range of magnetic latitude on the boundary surface and after propagating through the ionized atmosphere as predicted by the method of ray tracing arrives at the point (b_1, θ_1, ϕ_1) on the boundary surface which is approximately the magnetic conjugate point of (b, θ, ϕ). After re-entry, the field in the terrestrial waveguide is given by equations similar to (14), namely,

$$\overline{H} = -i\omega\epsilon_0 \nabla \times (\overline{r}\Gamma) = -\hat{\theta}\,\frac{i\omega\epsilon_0}{\sin\theta}\,\frac{\partial \Gamma}{\partial \phi} + \hat{\phi}\omega\epsilon_0\,\frac{\partial \Gamma}{\partial \theta}$$

$$\overline{E} = \nabla \times \nabla \times (\bar{r}\Gamma) = \hat{\Gamma}\left[\frac{\partial^2}{\partial r^2}(r\Gamma) + k_0^2 r\Gamma\right] + \hat{\theta}\,\frac{1}{r}\,\frac{\partial}{\partial \theta}\,\frac{\partial}{\partial r}(r\Gamma)$$
$$+ \hat{\phi}\,\frac{1}{r\sin\theta}\,\frac{\partial}{\partial \phi}\,\frac{\partial}{\partial r}(r\Gamma) \tag{95}$$

where

$$\Gamma = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[C_{\ell m}h_\ell^{(1)}(k_0 r) + D_{\ell m}h_\ell^{(2)}(k_0 r)\right]P_\ell^m(\cos\theta)e^{im\phi} \tag{96}$$

$C_{\ell m}$ and $D_{\ell m}$ being constants to be determined. On the Earth's surface which is perfectly conducting, $\frac{\partial}{\partial r}(r\Gamma)\Big|_{r=a} = 0$, we have

$$C_{\ell m}\left[h_\ell^{(1)'}(k_0 a) + \frac{1}{k_0 a}h_\ell^{(1)}(k_0 a)\right] + D_{\ell m}\left[h_\ell^{(2)'}(k_0 a) + \frac{1}{k_0 a}h_\ell^{(2)}(k_0 a)\right] = 0 \tag{97}$$

so that

$$\Gamma = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} C_{\ell m}\left[h_\ell^{(1)}(k_0 r) - \frac{h_\ell^{(1)'}(k_0 a) + \frac{1}{k_0 a}h_\ell^{(1)}(k_0 a)}{h_\ell^{(2)'}(k_0 a) + \frac{1}{k_0 a}h_\ell^{(2)}(k_0 a)}h_\ell^{(2)}(k_0 r)\right]$$
$$P_\ell^m(\cos\theta)e^{im\phi} \tag{98}$$

and

$$H_r = 0$$
$$H_\theta = -\frac{i\omega\epsilon_0}{\sin\theta}\frac{\partial\Gamma}{\partial\phi}$$
$$= \frac{\omega\epsilon_0}{\sin\theta}\sum_{l=0}^{\infty}\sum_{m-1}^{l} C_{\ell m}$$
$$\left[h_\ell^{(1)}(k_0 r) - \frac{h_\ell^{(1)'}(k_0 a) + \frac{1}{k_0 a}h_\ell^{(1)}(k_0 a)}{h_\ell^{(2)'}(k_0 a) + \frac{1}{k_0 a}h_\ell^{(2)}(k_0 a)}h_\ell^{(2)}(k_0 r)\right]$$
$$\cdot P_\ell^m(\cos\theta)e^{im\phi}$$

$$H_\phi = i\omega\epsilon_0 \frac{\partial\Gamma}{\partial\theta}$$

$$= -i\omega\epsilon_0 \sin\theta \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} C_{\ell m}$$

$$\left[h_\ell^{(1)}(k_0 r) - \frac{h_\ell^{(1)'}(k_0 a) + \frac{1}{k_0 a} h_\ell^{(1)}(k_0 a)}{h_\ell^{(2)'}(k_0 a) + \frac{1}{k_0 a} h_\ell^{(2)}(K_0 a)} h_\ell^{(2)}(k_0 r) \right]$$

$$\cdot P_\ell^{m'}(\cos\theta) e^{im\phi}$$

$$E_r = \frac{\partial^2}{\partial r^2}(r\Gamma) + k_0^2(r\Gamma)$$

$$= \frac{1}{r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} C_{\ell m} \ell(\ell+1)$$

$$\left[h_\ell^{(1)}(k_0 r) - \frac{h_\ell^{(1)'}(k_0 a) + \frac{1}{k_0 a} h_\ell^{(1)}(k_0 a)}{h_\ell^{(2)'}(k_0 a) + \frac{1}{k_0 a} h_\ell^{(2)}(k_0 a)} h_\ell^{(2)}(k_0 r) \right]$$

$$\cdot P_\ell^{m}(\cos\theta) e^{im\phi}$$

$$E_\theta = \frac{1}{r} \frac{\partial}{\partial\theta} \frac{\partial}{\partial r}(r\Gamma)$$

$$= -k_0 \sin\theta \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} C_{\ell m} \left\{ h_\ell^{(1)'}(k_0 r) + \frac{1}{k_0 r} h_\ell^{(1)}(k_0 r) \right.$$

$$- \frac{h_\ell^{(1)'}(k_o a) + \frac{1}{k_0 a} h_\ell^{(1)}(k_0 a)}{h_\ell^{(2)'}(k_0 a) + \frac{1}{k_0 a} h_\ell^{(2)}(k_0 a)} \cdot$$

$$\left. \left[h_\ell^{(2)'}(k_0 r) + \frac{1}{k_0 r} h_\ell^{(2)}(k_0 r) \right] \right\} \cdot P_\ell^{m'}(\cos\theta) e^{im\phi}$$

$$E_\phi = \frac{1}{r \sin\theta} \frac{\partial}{\partial\phi} \frac{\partial}{\partial r}(r\Gamma)$$

$$= \frac{ik_0}{\sin\theta} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} C_{\ell m} m \left\{ h_\ell^{(1)'}(k_0 r) + \frac{1}{k_0 r} h_\ell^{(1)}(k_0 r) \right.$$

$$-\frac{h_\ell^{(1)\prime}(k_0a)+\frac{1}{k_0a}h_\ell^{(1)}(k_oa)}{h_\ell^{(2)}(k_0a)+\frac{1}{k_0a}h_\ell^{(2)}(k_oa)}$$

$$\cdot\left[h_\ell^{(2)\prime}(k_0r)+\frac{1}{k_0r}h_\ell^{(2)}(k_0r)\right]\Bigg\}P_\ell^m(\cos\theta)e^{im\phi} \tag{99}$$

In all the above equations, we put $r=b$, $\theta=\theta_1$, and $\phi=\phi_1$.

Similar to penetration, re-entry occurs in narrow ranges of magnetic latitude. Equations of the same forms as (60), (61), (62), (63), (64), (65), (68), (71) and (73) hold. If the point (b_1,θ_1,ϕ_1) is situated within the range of magnetic latitude suitable for the re-entry, then the wave constitutes a whistler signal which can be received at any point in the terrestrial waveguide if the point is not too distant from the point of re-entry to cause the signal becoming too weak in the course of propagation in the terrestrial waveguide. The constant $C_{\ell m}$ can be determined in a similar way as $A_{\ell m}^{\dagger}$.

6.7 Modifications Due to the Consideration of Electron Density Profile in the Lower Ionosphere

The effect on penetration and re-entry of the electron-density profile of the lower ionosphere is not negligible. In the following, we shall give a more accurate treatment of the problem. For the convenience of mathematical analysis, let us introduce a rectangular coordinate system (ξ,η,ζ) such that the origin is at the point (b,θ,ϕ) in the coordinate system S, the ζ-axis is parallel to $\hat{r}$. the ξ-axis parallel to $\hat{\theta}$, the η-axis parallel to $\hat{\phi}$ and $\overline{p}=\hat{\zeta}\zeta+\hat{\xi}\xi+\hat{\eta}\eta$. Obviously, $E_\zeta\leftrightarrow E_r, E_\xi\leftrightarrow E_\theta, \frac{\partial}{\partial\zeta}\leftrightarrow\frac{\partial}{\partial r}, \frac{\partial}{\partial\xi}\leftrightarrow\frac{1}{b}\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial\eta}\leftrightarrow\frac{1}{b\sin\theta}\frac{\partial}{\partial\phi}$. Similar to (40). $\overline{\overline{\epsilon}}$ of the ionosphere can be written as

$$\overline{\overline{\epsilon}}(\overline{p})=\begin{bmatrix}1-X\frac{Y_\zeta^2-1}{Y^2-1} & -X\frac{Y_\zeta Y_\xi}{Y^2-1} & i\frac{XY_\xi}{Y^2-1}\\ -X\frac{Y_\zeta Y_\xi}{Y^2-1} & 1-\frac{Y_\xi^2-1}{Y^2-1} & -i\frac{XY_\zeta}{Y^2-1}\\ -i\frac{XY_\xi}{Y^2-1} & i\frac{XY_\zeta}{Y^2-1} & 1+\frac{X}{Y^2-1}\end{bmatrix} \tag{100}$$

In the lower ionosphere, while the Earth's magnetic field is regarded as locally homogeneous, the spatial distribution of the electron density as a function of ζ is considered. We assume that $X(\zeta)$ obeys a form of Epstein distribution [4].

$$X(\zeta) = G \tanh \frac{\alpha\zeta}{2} \tag{101}$$

where G and α are positive constants which can be chosen to fit approximately the observational data of the lower ionosphere. Here, we are only interested in the part of the field pertaining to the ordinary wave that can penetrate into the lower ionosphere. In the following, it is implied that all the field components written are meant to be those pertaining to this part of the wave. By making use of the equation $K = K_0 n$, $\hat{K} = -i\nabla$, it is easy to show that this part of the field satisfies the following equation approximately

$$\frac{\partial^2 E_\eta}{\partial \xi^2} + \frac{\partial^2 E_\eta}{\partial \zeta^2} + k_0^2 \left[1 + \frac{X(\zeta)}{Y-1}\right] E_\eta = 0 \tag{102}$$

By the method of separating of variables, letting

$$E_\eta = E_1(\xi) E_2(\zeta) \tag{103}$$

we have

$$\frac{1}{E_1}\frac{d^2 E_1}{d\xi^2} + \frac{1}{E_2}\frac{d^2 E_2}{d\zeta^2} + k_0^2 \left(1 + \frac{G}{Y-1}\tanh\frac{\alpha\zeta}{2}\right) = 0 \tag{104}$$

We can let

$$\frac{d^2 E_1}{d\xi^2} + k_0^2 \sin^2\chi E_1 = 0 \tag{105}$$

where $\sin\chi$ is approximately equal to $\sin\delta = \sin\theta/\sqrt{1+3\cos^2\theta}$. Therefore

$$E_1 = e^{\mp i k_0 \xi \sin\chi} \tag{106}$$

where the upper and lower signs apply to the case of propagation from the southern to the northern hemisphere and to the case of propagation from the northern to the southern hemisphere respectively. Also,

$$\frac{d^2 E_2}{d\zeta^2} + k_0^2 \left(\cos^2\chi + \frac{G}{Y-1}\tanh\frac{\alpha\zeta}{2}\right) E_2 = 0 \tag{107}$$

Let us introduce

$$u = -e^{\alpha\zeta} \tag{108}$$

and

$$v = \frac{E_2}{f(u)} \tag{109}$$

where $f(u)$ is a function to be determined. We have

$$\begin{aligned}
\frac{du}{d\zeta} &= -\alpha u \\
\frac{dE_2}{d\zeta} &= \frac{dE_2}{du}\frac{du}{d\zeta} = -\alpha u\left[f(u)\frac{dv}{du} + f'(u)v\right] \\
\frac{d^2E_2}{d\zeta^2} &= \alpha^2u^2f(u)\frac{d^2v}{du^2} + \left[2\alpha^2u^2f'(u) + \alpha^2uf(u)\right]\frac{dv}{du} \\
&\quad + \left[\alpha^2u^2f''(u) + \alpha^2uf'(u)\right]v
\end{aligned} \tag{110}$$

Hence,

$$\begin{aligned}
&\frac{d^2v}{du^2} + \left[2\frac{f'(u)}{f(u)} + \frac{1}{u}\right]\frac{dv}{du} + \left[\frac{f''(u)}{f(u)} + \frac{1}{u}\frac{f'(u)}{f(u)}\right. \\
&\qquad \left. + \frac{k_0^2}{\alpha^2u^2}\left(\cos^2\chi - \frac{G}{Y-1}\frac{1+u}{1-u}\right)\right]v = 0
\end{aligned} \tag{111}$$

Equation (111) can be transformed into the hypergeometric differential equation of the form

$$\frac{d^2v}{du^2} + \frac{C-(A+B+1)u}{u(1-u)}\frac{dv}{du} - \frac{AB}{u(1-u)}v = 0 \tag{112}$$

where A, B, C are constants. Comparing (111) and (112), we have

$$\frac{f'(u)}{f(u)} = \frac{C-1}{2}\frac{1}{u} - \frac{A+B-C+1}{2}\frac{1}{1-u} \tag{113}$$

whence,

$$f(u) = f_0u^{\frac{C-1}{2}}(1-u)^{\frac{A+B-C+1}{2}} \tag{114}$$

Hence

$$\frac{f''(u)}{f(u)} + \frac{1}{u}\frac{f'(u)}{f(u)} + \frac{k_0^2}{\alpha^2 u^2}\left(\cos^2\chi - \frac{G}{Y-1}\frac{1+u}{1-u}\right)$$
$$= \frac{1}{4}\left(\frac{C-1}{u} - \frac{A+B-C+1}{1-u}\right)^2 - \frac{A+B-C+1}{2}\frac{1}{u(1-u)}$$
$$- \frac{A+B-C+1}{2}\frac{1}{(1-u)^2}$$
$$+ \frac{k_0^2}{\alpha^2 u^2}\left(\cos^2\chi - \frac{G}{Y-1}\frac{1+u}{1-u}\right)$$
$$= -\frac{AB}{u(1-u)} \tag{115}$$

By reducing to the common denominator and equating the coefficients of u^0, u^1 and u^2 in the numerator on the both sides of the equal sign, we have

$$\left.\begin{aligned}
&(C-1)^2 + 4\frac{k_0^2}{\alpha^2}\left(\cos^2\chi - \frac{G}{Y-1}\right) = 0\\
&(A-B)^2 + 4\frac{k_0^2}{\alpha^2}\left(\cos^2\chi + \frac{G}{Y-1}\right) = 0\\
&(A+B)^2 - 2C(A+B) + 2(C-1) - 4\frac{k_0^2}{\alpha_2}\left(\cos^2\chi - \frac{G}{Y-1}\right) = 0
\end{aligned}\right\} \tag{116}$$

Solving we get

$$\left.\begin{aligned}
C &= 1 + i2\frac{k_0}{\alpha}\left(\cos^2\chi - \frac{G}{Y-1}\right)^{\frac{1}{2}}\\
A &= 1 + i\frac{k_0}{\alpha}\left[\left(\cos^2\chi - \frac{G}{Y-1}\right)^{\frac{1}{2}} - \left(\cos^2\chi + \frac{G}{Y-1}\right)^{\frac{1}{2}}\right]\\
B &= 1 + i\frac{k_0}{\alpha}\left[\left(\cos^2\chi - \frac{G}{Y-1}\right)^{\frac{1}{2}} + \left(\cos^2\chi + \frac{G}{Y-1}\right)^{\frac{1}{2}}\right]
\end{aligned}\right\} \tag{117}$$

Hence,

$$E_2 = f_0 u^{\frac{C-1}{2}}(1-u)^{\frac{A+B-C+1}{2}} v \tag{118}$$

where v is the hypergeometric function. The hypergeometric differential equation has three singular points 0, 1, and ∞. For each of these points there are two fundamental series expansions convergent in its neighborhood, namely,

about $u = 0$

$$v_1 = C_1 F(A, B, C, u)$$
$$v_2 = C_2 u^{1-C} F(A - C + 1,\ B - C + 1,\ 2 - C,\ u)$$

about $u = 1$

$$v_3 = C_3 F(A, B, A + B - C + 1, 1 - u)$$
$$v_4 = C_4 (1 - u)^{(C-A-B)}$$
$$F(C - A, C - B, C - A - B + 1, 1 - u)$$

about $u = \infty$

$$v_5 = C_5 u^{-A} F(A, A - C + 1, A - B + 1, u^{-1})$$
$$v_6 = C_6 u^{-B} F(B, B - C + 1, B - A + 1, u^{-1}) \tag{119}$$

where $C1$, $C2$, $C3$, $C4$, $C5$ and $C6$ are constants and

$$F(A, B, C, u) = \frac{\Gamma(C)}{\Gamma(A)\Gamma(B)} \sum_{k=0}^{\infty} \frac{\Gamma(A + k)\Gamma(B + k)}{k!\,\Gamma(C + k)} u^k \tag{120}$$

Although the functions are given in the form of series expansions which are convergent only within their own appropriate definite intervals, each of these series defines an analytic function by analytic continuation which is a solution of the hypergeometric differential equation extending beyond the convergent interval of the series. Since any three solutions of a linear differential equation of the second order always are linearly dependent, there is between these three functions a linear relation, this relation holds for the analytic function obtained by analytic continuation and therefore for all values of u which can be assigned to the three functions. Therefore it is legitimate and for our purpose more convenient to write the u's in (114) and (118) as $-u$ then $v_1 - v_6$ are expressed as

about $u = 0$

$$v_1 = C_1 F(A, B, C, -u)$$
$$v_2 = C_2 (-u)^{1-C} F(A - C + 1, B - C + 1, 2 - C, -u)$$

about $u = -1$

$$
\begin{aligned}
v_3 &= C_3 F(A, B, A+B-C+1, 1+u) \\
v_4 &= C_4 (1+u)^{C-A-B} \\
&\qquad F(C-A, C-B, C-A-B+1, 1+u)
\end{aligned}
$$

about $u = -\infty$

$$
\begin{aligned}
v_5 &= C_5 (-u)^{-A} F(A, A-C+1, A-B+1, -u^{-1}) \\
v_6 &= C_6 (-u)^{-B} F(B, B-C+1, B-A+1, -u^{-1})
\end{aligned}
\tag{119'}
$$

where

$$
F(A, B, C, -u) = \frac{\Gamma(C)}{\Gamma(A)\Gamma(B)} \sum_{k=0}^{\infty} \frac{\Gamma(A+k)\Gamma(B+k)}{k!\,\Gamma(C+k)} (-u)^k \tag{120)'}
$$

We are interested in knowing the relation between v_3, v_4 and v_5. It can be proved that [5]

$$
\begin{aligned}
F(A, A-C+1, A-B+1, -u^{-1}) &= \frac{\Gamma(A-B+1)\Gamma(C-A-B)}{\Gamma(1-B)\Gamma(C-B)} \\
&(-u)^A F(A, B, A+B-C+1, 1+u) \\
&+ \frac{\Gamma(A-B+1)\Gamma(A+B-C)}{\Gamma(A)\Gamma(A-C+1)} (-u)^{A-C+1} (-1-u)^{C-A-B} \\
&\cdot F(1-B, 1-A, C-A-B+1, 1+u)
\end{aligned}
\tag{121}
$$

When ζ becomes very large, $u \to \infty$ and

$$
\begin{aligned}
E_2 &\to C_5 f_0 (-1)^{\frac{C-1}{2}} (-u)^{\frac{B-A}{2}} F(A, A-C+1, A-B+1, -u^{-1}) \\
&\to C_5' f_0 (-u)^{\frac{B-A}{2}} F(A, A-C+1, A-B+1, -u^{-1}) \\
&\to C_5' f_0 e^{ik_0 \zeta (\cos^2 \chi + \frac{G}{Y-1})^{\frac{1}{2}}}
\end{aligned}
\tag{122}
$$

where $C_5' = C_5 (-1)^{\frac{C-1}{2}}$. From (106) and (122), we have

$$
E_\eta \to C_5' f_0 e^{ik_0 \left[\mp \xi \sin\chi + 5(\cos^2 \chi + \frac{G}{Y-1})^{\frac{1}{2}}\right]} = C_5' f_0 e^{iK(\mp \xi \sin\chi' + \zeta \cos\chi')} \tag{123}
$$

where

$$K = k_0 \left(1 + \frac{G}{Y-1}\right)^{\frac{1}{2}}$$

$$\sin\chi' = \frac{\sin\chi}{\left(1 + \frac{G}{Y-1}\right)^{\frac{1}{2}}}$$

$$\cos\chi' = \frac{\left(\cos^2\chi + \frac{G}{Y-1}\right)^{\frac{1}{2}}}{\left(1 + \frac{G}{Y-1}\right)^{\frac{1}{2}}} \tag{124}$$

From (58) and (59), letting $X \to G$, we have

$$\begin{aligned}
E_\zeta \to & \frac{1}{(G-1)Y}\left\{ i\left[(G-1)Y_\xi - \left(1 + \frac{G}{Y-1}\right)(Y_\zeta \hat{K}_\xi - Y_\xi \hat{K}_\zeta)\hat{K}_\zeta\right]\right. \\
& \left. + \left(1 + \frac{G}{Y-1}\right) Y\hat{K}_\eta \hat{K}_\zeta \right\} E_\eta \\
E_\xi \to & \frac{1}{(G-1)Y}\left\{ -i\left[(G-1)Y_\zeta + \left(1 + \frac{G}{Y-1}\right)(Y_\zeta \hat{K}_\xi - Y_\xi \hat{K}_\zeta)\hat{K}_\xi\right]\right. \\
& \left. + \left(1 + \frac{G}{Y-1}\right) Y\hat{K}_\xi \hat{K}_\zeta \right\} E_\eta \\
H_\xi \to & \sqrt{\frac{\epsilon_0}{\mu_0}}\left(1 + \frac{G}{Y-1}\right)^{\frac{1}{2}} \left(i\frac{Y_\xi}{Y}\hat{K}\eta + \hat{K}_\zeta\right) E_\eta \\
H_\eta \to & \mp\sqrt{\frac{\epsilon_0}{\mu_0}}\left(1 + \frac{G}{Y-1}\right)^{\frac{1}{2}} E_\eta
\end{aligned} \tag{125}$$

where $\zeta = 0,\ u = -1$, from (121)

$$F(A, A-C+1, A-B+1, -u^{-1}) \to \frac{\Gamma(A-B+1)\Gamma(C-A-B)}{\Gamma(1-B)\Gamma(C-B)} \tag{126}$$

From (122), (125) and (126), we have on the bottom boundary of the ionosphere, writing Y_r, Y_θ, $\hat{K}_r$, $\hat{K}_\theta$, $\hat{K}_\phi$, E_r, E_θ, E_ϕ, H_θ and H_ϕ for Y_ζ, Y_ξ, $\hat{K}_\zeta$, $\hat{K}_\xi$, $\hat{K}_\eta$, E_ζ, E_ξ, E_η, H_ξ and H_η respectively,

$$
\begin{aligned}
E_\phi &= C_5' f_0 \frac{\Gamma(A-B+1)\Gamma(C-A-B)}{\Gamma(1-B)\Gamma(C-B)} \\
E_r &= \frac{1}{(G-1)Y}\left\{i\left[(G-1)Y_\theta - \left(1+\frac{G}{Y-1}\right)(Y_r\hat{K}_\theta - Y_\theta\hat{K}_r)\hat{K}_r\right]\right. \\
&\quad \left. + \left(1+\frac{G}{Y-1}\right) Y\hat{K}_\phi\hat{K}_r\right\} E_\phi \\
E_\theta &= \frac{1}{(G-1)Y}\left\{-i\left[(G-1)Y_r + \left(1+\frac{G}{Y-1}\right)(Y_r\hat{K}_\theta - Y_\theta\hat{K}_r)\hat{K}_\theta\right]\right. \\
&\quad \left. + \left(1+\frac{G}{Y-1}\right) Y\hat{K}_\theta\hat{K}_\phi\right\} E_\phi \\
H_\theta &= \sqrt{\frac{\epsilon_0}{\mu_0}}\left(1+\frac{G}{Y-1}\right)^{\frac{1}{2}}\left(i\frac{Y_\theta}{Y}\hat{K}_\phi - \hat{K}_r\right) E_\phi \\
H_\phi &= \mp i\sqrt{\frac{\epsilon_0}{\mu_0}}\left(1+\frac{G}{Y-1}\right)^{\frac{1}{2}} E_\phi \qquad (127)
\end{aligned}
$$

Comparing equations of (127) with equations of (58) and (59), the former equations can be obtained by replacing the X's in the latter equations by G. Similarly, we can get the equations corresponding to (61) to (73), especially, corresponding to (65), (71) and (73), we have

$$
\begin{aligned}
&(G-1)\frac{Y_r}{Y}\hat{K}_r + \left(1+\frac{G}{Y-1}\right)\frac{1}{Y}\left(Y_r\hat{K}_\theta - Y_\theta\hat{K}_r\right)\hat{K}_r\hat{K}_\theta = \pm(G-1) \\
&(G-1)\frac{Y_rY_\theta}{Y^2} + \left(1+\frac{G}{Y-1}\right)\frac{Y_\theta}{Y^2}\left(Y_r\hat{K}_\theta - Y_\theta\hat{K}_r\right)\hat{K}_\theta \\
&\quad = \left(1+\frac{G}{Y-1}\right)\hat{K}_r\hat{K}_\theta \qquad \text{or} \qquad \hat{K}_\phi = 0 \qquad (65)'
\end{aligned}
$$

$$
\begin{aligned}
\tan\theta &= -2\frac{\hat{K}_r^3\hat{K}_\theta(G-1+\hat{K}_\theta^2)}{(G-1)^2 - \hat{K}_r^4\hat{K}_\theta^2} \\
&\quad + 2\frac{\hat{K}_\theta\,|\,G-1\,|}{(G-1)^2 - \hat{K}_r^4\hat{K}_\theta^2}\left[\hat{K}_r^2(1+\hat{K}_r^2) - (G-1-\hat{K}_r^2)^2\right]^{\frac{1}{2}} \qquad (71)'
\end{aligned}
$$

$$1 + \hat{K}_r(1 + \hat{K}_r) > G > 0 \tag{73)'$$

It is interesting to note that our modified treatment leads to similar final results. Therefore, in our modified treatment, the penetration prolem, the re-entry problem and the determination of the constants $A^{\dagger}_{\ell m}$ and $C_{\ell m}$ can all be solved by following similar procedures as before. It is probably not necessary to point out that if our treatment is made only for the purpose of studying the penetration problem, the re-entry problem and the propagation in the lower ionosphere, then beyond the lower ionosphere, we can still use the expression $X(\zeta) = G \tanh \frac{\alpha\zeta}{2}$ to signify the electron density and still regard the Earth's magnetic field as the same as in the lower ionosphere near the bottom boundary, but beyond the lower ionosphere, the expressions for the field components given in (127) are not meant to give the true values there.

6.8 Conclusions and Discussions

By our theory of whistler propagation, the following interesting conclusions are obtained.

1) Since both the terrestrial waveguide propagation and the whistler mode propagation in the ionized atmosphere have to be considered, the common conviction that whistlers cannot be received below certain latitudes and that the receiving point and the source position must be mutually magnetic conjugate points are clearly not warranted.

2) Since our derivations show that whistler rays have a very small angular divergence and that both the penetration from the terrestrial waveguide into the ionosphere and the re-entry from the ionosphere into the terrestrial waveguide occurs in a narrow range of magnetic latitude (except at isolated points, the magnetic poles), one in each hemisphere for penetration and similarly for re-entry, these facts explain the apparent existence of ducts. The postulate that there are filamentary structures of electron density along the Earth's magnetic field lines is not only still lacking in direct experimental evidence, but is also not quite necessary for the explanation of the quasi-longitudinal propagation of whistlers.

3) Since penetration and re-entry require that the values of X and G on the bottom boundary of the ionosphere be less than 3, whistlers mostly occur before sunrise and to a lesser degree after sunset.

4) Since X and G are inversely proportional to the square of frequency and the higher the value of X or G, the higher the value of

$\hat{K}_r$, the penetration of the components of lower frequencies occur at higher magnetic latitudes than those of higher frequencies, and similarly for re-entry, so that the former components travel through longer paths and suffer longer delays than the latter components. This effect enhance the differential delay of arrival time caused by dispersion in the ionized atmosphere. For the same reason, whistlers with spectra of discrete frequencies may exhibit traces in fine successions.

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References

[1] Helliwell, R. A., *Whistlers and Related Ionospheric Phenomena*, Stanford University Press, Stanford, CA, 1965.

[2] Haselgrove, J., "Ray theory and a new method for ray tracing," in London Physical Society, *Report of Conference on the Physics of the Ionosphere*, 1954.

[3] Budden, K. G., *Radio Waves in the Ionosphere*, Cambridge, London, 1961.

[4] Epstein, P. S., "Reflection of waves in an inhomogeneous absorbing medium," *Proceedings of the National Academy of Science*, **16**, 1930.

[5] Erdelyi, A., *Higher Transcendental Functions*, **1**, McGraw-Hill, New York, 1953.