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APPLICATION-ORIENTED RELATIVISTIC ELECTRODYNAMICS

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4.1 Introduction and Rationale

The intimate relationship between Maxwell's theory for the electromagnetic field and Einstein's special relativity theory [1] is generally recognized nowadays. Throughout the present century many educators found it necessary to include a chapter on special relativity in textbooks devoted to electromagnetic field theory, e.g., in the book by Becker, edited by Sauter [2] (a book that has its roots in the last century and appeared practically in sixteen editions!), see also Stratton [3], Fano Chu and Adler [4], Sommerfeld [5], Jordan and Balmain [6], Panofsky and Phillips [7], Shadowitz [8], Jackson [9], Portis [10], Lorrain and

Corson [11], Wangness [12], Griffiths [13], Frankl [14], Chen [15], Kong [16], Plonus [17]. This list is representative, rather than exhaustive. By examining these and other textbooks, it becomes apparent that a direct approach suitable for educating applied physicists and electromagnetic radiation engineers is lacking. Some authors introduce special relativity theory in the traditional "Gedanken experiment" approach, and by the time the reader finishes with the moving trains, flashing torchlights, and rods and clocks, the relevance to practical electromagnetic problems is obscured. Others move along more formalistic lines and derive the field tensors, mostly by using general coordinate systems and the heavy machinery of differential geometry, e.g., covariant and contravariant coordinate systems, but the mathematical elegance hardly inspires the engineering student who usually cannot find in it any motivation to move on in this field. On the other hand, we are nowadays aware of real life problems, e.g., design of satellite network supported global navigation systems, which involve special (and sometimes even general) relativistic considerations related to precision of time and frequency bases and errors incurred during propagation through complicated inhomogeneous and time varying media and in the presence of relative motion between objects. It is therefore mandatory to devise the methodological tools and suitable representations for teaching relativistic electrodynamics to applied physics and electrical engineering students. In the course of such a pedagogical experiment with Electrical Engineering graduates, it became clear that the rudiments of special relativity should be presented axiomatically, with as little phenomenological "explanation" as feasible, working on the assumption that this aspect has been covered, at least to some extent, in "Baby physics" courses. It also became clear that four-dimensional Fourier transforms should be introduced from the beginning, a novel approach, not indicated in the literature, as far as this author is aware. This facilitates the work in an algebraic, rather than differential equations environment, thus simplifying mathematical manipulations. It also became clear that four-vectors, which are easily handled, almost as easily as the classical three-vectors, should be extensively used. Most of the students met had a good grasp of linear algebra, and the introduction of tensor algebra was effected by moving from the familiar idea of matrices to their representation as dyadics and finally via the Einstein summation convention to tensors. However, only Euclidian systems are considered, and even in this context, the elegance of the field tensors and the associated representation of Maxwell's equations has been avoided. Within these limits, it is then the personal preference of the teacher that will guide him to emphasize certain classes of problems. From this point of view, the specific material described here serves merely as an example. Surprisingly, in the course of compiling and teaching this course, new ideas and representations emerged, which do not appear in the literature, as far as this author is aware. These innovative concepts do not alter special relativity theory, still it is a pleasant surprise that at the turn of the century anything new at all can be said about the now veteran special relativity theory. From this point of view, there are some novel ideas given here and the present study is not merely tutorial. For example, the section on the Fermat principle shows that the generalized principle, for inhomogeneous and time dependent media, acquires a new meaning that can only be stated in the context of special relativity: verbally stated, it says that the ray propagates along a path that minimizes (or in general extremizes) the proper time. It is also shown that the Fermat principle is equivalent to a simple mathematical condition on the smoothness of the phase function.

A pioneering attempt to compile many, more or less practical problems of relativistic electrodynamics to serve the needs of the engineering community has been made by Van Bladel [18]. Although written as a textbook with problems at the end of each chapter, it is too specialized (as some readers might also deem this article to be), although certain parts could certainly be incorporated in any syllabus for a course in application-oriented relativistic electrodynamics. It also follows too much of the traditional presentation of the fundamentals of the theory, instead of using an axiomatic approach and moving directly into the *materia*. It is hoped that experiments like Van Bladel's book and the present article contribute to clarify the question of how to present such a course.

The present article is organized as follows: after introducing notation and stating relativistic electrodynamics axiomatically, and exploring properties of relevant four-vectors, the technique of algebraization by using four-fold Fourier transforms is introduced. This already touches on the important question of relativistic transformations in this representation space. Further exploration of four-vectors follows. Next, four-potentials are introduced. This is followed by a discussion of the cross multiplication operation and the related rotor operation. It is mentioned, without going into too much detail, that we are dealing with tensor operations, and the general advice is to work with the various components (without mentioning the antisymmetric tensors and their properties, which would encumber the presentation without contributing to application-oriented problems). A section on the proper time and related concepts follows. What might appear as a melange of unrelated subjects is actually an attempt to lead the student gradually from the less complicated to the more sophisticated subjects. We now have enough tools for discussing specific problems. As an example, the Minkowski constitutive equations for moving media are derived for dispersive anisotropic media. Dispersion equations and their relativistic invariance are discussed. This provides the basis for discussing Hamiltonian ray propagation for inhomogeneous and time varying dispersive media. For pedagogical reasons, this section is separated from the discussion of the generalized Fermat principle, given in the following section. As a final application, which is of course biassed by the author's personal preferences, the question of propagation in lossy media is discussed, in the context of the ray equations, their generalization to lossy media and the questions of Lorentz transformations and mathematical complex analyticity involved.

4.2 Special Relativity

In this section relativistic electrodynamics is introduced. The formalism needed by the applied physicist and engineer is stipulated in an axiomatic manner. The introduction of the field tensors and the ensuing elegant representation of the field equations by means of operations on these tensors, a cornerstone of relativistic formalism, is obviated. Four-vectors and Minkowski space are introduced at the end as a notational and operational tool, rather than a conceptual generalization of the space-time manifold idea, as given in books specializing in relativity theory. In the following sense, it is the same methodology educators use now for years when teaching for example waves in metallic waveguides and resonators: the fact is that our students never get a comprehensive course in the theory of the special functions needed for comprehending this subject, but we realize that we do not have the time to plough this field, lest no time will remain to teach the pertinent engineering aspects, and therefore a short resume of the theory of Bessel functions, Legendre polynomials, etc. appears at the end of many textbooks for later reference, and we force our students to plunge into the main subject matter, hoping they will swim and not sink.

Maxwell's equations for the electromagnetic field are given by

$$\begin{aligned}\nabla \times \mathbf{E} &= -\,\partial_t \mathbf{B} - \mathbf{j}_m \\ \nabla \times \mathbf{H} &= \partial_t \mathbf{D} + \mathbf{j}_e \\ \nabla \cdot \mathbf{D} &= \rho_e \\ \nabla \cdot \mathbf{B} &= \rho_m \end{aligned} \tag{1}$$

where ∂_t denotes the partial time derivative, the fields are functions of the space and time coordinates, e.g. $\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$, and indices e, m denote the electric and magnetic densities, respectively, of current and charge. To date, the existence of the magnetic current and charge densities in (1) has not been empirically established. Therefore, at this time they should be considered as fictitious, in the sense that they are not intrinsic physical entities, however, they are amply used in boundary value problems to represent the associated fields, e.g., see Stratton [3], or Kong [16], as well as many other textbooks cited above. Even though the present approach claims to be practical, we should always be aware of the fact that physicists have not given up the quest for magnetic charges and currents, e.g., see Jackson [9].

The statement of Maxwell's equations (1) is incomplete in the sense that it is unrelated to other physical models. For example, we need a way of linking electrodynamics to familiar concepts like force and energy introduced in mechanics. One way of achieving this goal is by stating a force formula. Thus the presence of a conventional charge q_e can be detected through the forces exerted on it according to the Lorentz force formula

$$\mathbf{f}_e = q_e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{2}$$

Henceforth we shall suppress the index on $\mathbf{f}$ unless necessary.

The teaching of electromagnetic theory in a phenomenological-historical way, as evolving from crucial experiments and the subsequent "laws" that are added into the model, tends to obscure the fact that (2) is extrinsic and does not follow from Maxwell's equations. This important fact should be stressed at this point. Actually, (2) is an extension of the simple Coulomb force formula $\mathbf{f} = q_e\mathbf{E}$, which should be considered not as a "law" but as a link between mechanics and electrodynamics. Inasmuch as $\mathbf{f}$ is already known from mechanics, q_e in $\mathbf{f} = q_e\mathbf{E}$ can be considered as a proportionality coefficient. Once q_e is defined we have at our disposal the "rationalized" Giorgi MKSQ system of units. On introducing special relativity axiomatically, (2) can be derived from $\mathbf{f} = q_e\mathbf{E}$. The analog of (2) is given by

$$\mathbf{f}_m = q_m(\mathbf{H} - \mathbf{v} \times \mathbf{D}) \tag{3}$$

again an extension derived by means of relativistic transformation formulas from a magnetic formula $\mathbf{f} = q_m\mathbf{H}$. Although this example is far fetched, it demonstrates the symmetry introduced into Maxwell's equations by stipulating magnetic sources, and the stimulus it provides for looking at things in a new way. These derivations are left for an exercise stated below.

Special relativity deals with observations performed in inertial frames, i.e., frames of reference in relative uniform motion. The "unprimed" frame (1) is characterized by the space-time coordinates $\mathbf{x}, t$. The "primed" frame of reference moving at a velocity $\mathbf{v}$ as observed from the "unprimed" frame is characterized by $\mathbf{x}', t'$. One of the assumptions of special relativity is that corresponding to (1), in the primed frame Maxwell's equations assume the same functional structure (sometimes referred to as the "covariance of Maxwell's equations", or "invariance of Maxwell's equations"), i.e., we have

$$\begin{aligned} \nabla' \times \mathbf{E}' &= -\,\partial_{t'}\mathbf{B}' - \mathbf{j}'_m \\ \nabla' \times \mathbf{H}' &= \partial_{t'}\mathbf{D}' + \mathbf{j}'_e \\ \nabla' \cdot \mathbf{D}' &= \rho'_e \\ \nabla' \cdot \mathbf{B}' &= \rho'_m \end{aligned} \tag{4}$$

where now $\mathbf{E}' = \mathbf{E}'(\mathbf{x}', t')$ etc. The relation between the space-time coordinates in the two frames is given by the celebrated Lorentz transformation

$$\begin{aligned} \mathbf{x}' &= \tilde{\mathbf{U}} \cdot (\mathbf{x} - \mathbf{v}t) \\ t' &= \gamma(t - \mathbf{x} \cdot \mathbf{v}/c^2) \end{aligned} \tag{5}$$

where

$$\begin{aligned} \gamma &= (1 - \beta^2)^{-1/2}, \quad \tilde{\mathbf{U}} = \tilde{\mathbf{I}} + (\gamma - 1)\hat{\mathbf{v}}\hat{\mathbf{v}}, \\ \hat{\mathbf{v}} &= \mathbf{v}/v, \quad v = |\mathbf{v}|, \quad \beta = v/c \end{aligned} \tag{6}$$

and where $\tilde{I}$ is the idemfactor dyadic (same as unit matrix), c is the speed of light in free space (vacuum), which is a universal constant for all observers attached to inertial frames of reference; and e.g. $\hat{\mathbf{v}}\hat{\mathbf{v}}$, i.e., in general two juxtaposed vectors (or a linear combination of such pairs) without a dot or cross multiplication sign between them denotes a dyadic, same as a multiplication of a column vector times a row vector in matrix theory (or a linear combination of such pairs). For example, $\tilde{\mathbf{U}}$ is a dyadic too. It is easily verified that the inverse transformation is obtained from (5) by exchanging primed and unprimed symbols and inverting the sign of $\mathbf{v}$. This is also valid for other transformations given below. Use of the chain rule of calculus yields the transformation formulas for the differential operators

$$\begin{aligned} \nabla' &= \tilde{\mathbf{U}} \cdot (\nabla + \mathbf{v}\partial_t/c^2) \\ \partial_{t'} &= \gamma(\partial_t + \mathbf{v} \cdot \nabla) \end{aligned} \tag{7a}$$

Subsequently, it will be convenient to symbolically denote the del operator ∇ as a differentiation with respect to the space variable, thus (7a) becomes

$$\begin{aligned} \partial_{\mathbf{x}'} &= \tilde{\mathbf{U}} \cdot (\partial_{\mathbf{x}} + \mathbf{v}\partial_t/c^2) \\ \partial_{t'} &= \gamma(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}}) \end{aligned} \tag{7b}$$

The field variables are related by the following transformation formulas:

$$\begin{aligned} \mathbf{E}' &= \tilde{\mathbf{V}} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \\ \mathbf{B}' &= \tilde{\mathbf{V}} \cdot (\mathbf{B} - \mathbf{v} \times \mathbf{E}/c^2) \\ \mathbf{D}' &= \tilde{\mathbf{V}} \cdot (\mathbf{D} + \mathbf{v} \times \mathbf{H}/c^2) \\ \mathbf{H}' &= \tilde{\mathbf{V}} \cdot (\mathbf{H} - \mathbf{v} \times \mathbf{D}) \\ \mathbf{j}'_e &= \tilde{\mathbf{U}} \cdot (\mathbf{j}_e - \rho_e \mathbf{v}) \\ \rho'_e &= \gamma(\rho_e - \mathbf{j}_e \cdot \mathbf{v}/c^2) \\ \mathbf{j}'_m &= \tilde{\mathbf{U}} \cdot (\mathbf{j}_m - \rho_m \mathbf{v}) \\ \rho'_m &= \gamma(\rho_m - \mathbf{j}_m \cdot \mathbf{v}/c^2) \\ \tilde{\mathbf{V}} &= \gamma\tilde{\mathbf{I}} + (1 - \gamma)\hat{\mathbf{v}}\hat{\mathbf{v}} \end{aligned} \tag{8}$$

where $\mathbf{E} = \mathbf{E}(\mathbf{x}, t), \mathbf{E}' = \mathbf{E}'(\mathbf{x}', t')$ etc. and the space-time coordinates are related according to (5). It can be shown that subject to (5), (8), the set of Maxwell's equations (1) implies (4) and *vice-versa*. This in a nutshell is the basis of special relativistic electrodynamics [19], in fact, the set of relations given above is more than necessary to axiomatically define the theory with the absolute minimum assumptions, e.g., the set of the primed Maxwell equations (4) can be considered as a result which can be derived by exploiting the other relations given above.

Note that $\tilde{\mathbf{U}} \cdot \tilde{\mathbf{V}} = \gamma \tilde{\mathbf{I}}$. Also interesting are the roles of the dyadics $\tilde{\mathbf{U}}, \tilde{\mathbf{V}}$, sorting out the component of the three dimensional vectors, parallel, perpendicular to the velocity $\mathbf{v}$, respectively and multiplying them by γ. Of course, we know that the reason for a three dimensional vector to be associated with $\tilde{\mathbf{U}}$ or $\tilde{\mathbf{V}}$ depends whether it is a true vector like $\mathbf{x}$ or $\mathbf{j}$, or a component of an antisymmetrical tensor, like $\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H}$. Exactly, this is the part of the story that we should present axiomatically and avoid discussing with students novices to the subject of relativistic electrodynamics. The details can wait for a later encounter with this material.

Minkowski [20] introduced the four-vector concept which will enable us to compact our notation and simplify the algebraic and differential manipulations. To the three components x_j, $j = 1, 2, 3$, we add $x_4 = ict$, thus for real t we have an imaginary x_4. Henceforth four-vectors will be denoted by capital letters, e.g.,

$$\begin{aligned} \mathbf{X} &= (\mathbf{x}, ict) \\ \partial_{\mathbf{X}} &= \left(\partial_{\mathbf{x}}, -\frac{i}{c}\partial_t\right) \end{aligned} \tag{9}$$

where the second equation (9) is the four-dimensional Cartesian gradient operator. It is not necessary at this stage to introduce the geometrical concepts pertaining to the Minkowski space, i.e., to describe the Lorentz transformation as a rotation in this space. What is important for the student to know is the fact that the length of a four-vector is invariant with respect to the Lorentz transformation (5). It can be verified as an exercise that subject to (5)

$$\mathbf{X} \cdot \mathbf{X} = x^2 - c^2 t^2 = \mathbf{X}' \cdot \mathbf{X}' = x'^2 - c^2 t'^2 = \text{constant} \tag{10}$$

In the specific case (10), the value of the constant is chosen as zero. The reason is simple: consider a rotation of coordinates such that ξ is in the direction of $\mathbf{x}$. Then (10) amounts to $\xi - ct = a$, where a is a constant. Clearly setting the constant to any value is tantamount to choosing the time or space origin in a special way. The choice of $a = 0$ means that at $t = 0$ we have $\xi = 0$. Consequently, $\mathbf{X}$ is called the null vector. By inspection of (7b) compared to the Lorentz transformation (5), it becomes clear that $\partial_{\mathbf{X}}$ (9), is also a four-vector. Inasmuch as current and charge sources in (8) follow the same transformation formulas as the transformations for space-time coordinates, (5), we also identify as four vectors

$$\begin{aligned} \mathbf{J}_e &= (\mathbf{j}_e, ic\rho_e) \\ \mathbf{J}_m &= (\mathbf{j}_m, ic\rho_m) \end{aligned} \tag{11}$$

It then follows from Maxwell's equations (1) that

$$\begin{aligned} &(\nabla \times \mathbf{H} - \partial_t \mathbf{D}, ic\nabla \cdot \mathbf{D}) \\ &(-\nabla \times \mathbf{E} - \partial_t \mathbf{B}, ic\nabla \cdot \mathbf{B}) \end{aligned} \tag{12}$$

are also four-vectors, therefore their spatial parts (first expression in parentheses) transform like $\mathbf{x}$, and their temporal coordinates (second expression in parentheses) transform like ict, according to (5); these conclusions, although evidently true, have not been found explicitly in the literature, and provide an example to some new insight one gets when teaching these subjects. At the least, such results are useful for a class or home exercise.

Four-vectors in two reference frames can be related directly. Thus, (5) can be represented as a four-dimensional dyadic or a 4×4 matrix, e.g., $\mathbf{X}' = \tilde{\mathbf{W}} \cdot \mathbf{X}$. We can also use a mixed dyadic-matrix notation,

$$\begin{bmatrix} \mathbf{x}' \\ ict' \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{U}} & \frac{i\gamma \mathbf{v}}{c} \\ -\frac{i\gamma \mathbf{v}}{c} & \gamma \end{bmatrix} \cdot \begin{bmatrix} \mathbf{x} \\ ict \end{bmatrix} \tag{13}$$

or represent $\tilde{\mathbf{W}}$ in matrix forms with entries $W_{ij}, i, j = 1, \ldots, 4$. For example, take $\mathbf{v}$ in the x-direction, this yields for $\tilde{\mathbf{W}}$ the simple form

$$\begin{bmatrix} \gamma & 0 & 0 & i\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\gamma\beta & 0 & 0 & \gamma \end{bmatrix} \tag{14}$$

This matrix is Hermitian and it is easily verified that det $[W_{ij}] = 1$. It is interesting and useful for the sequel to show that det $[W_{ij}]$ is the Jacobian matrix. Working in Cartesian components in four-space, and using the Einstein summation convention, i.e., that an index appearing twice in one term of an equation is a dummy index on which summation is to be performed, we have

$$\frac{\partial X_i'}{\partial X_j} = \frac{\partial}{\partial X_j} W_{ik} X_k = W_{ik} \frac{\partial X_k}{\partial X_j} = W_{ik} \delta_{kj} = W_{ij} \tag{15a}$$

or in terms of a tensor product:

$$\partial_{\mathbf{X}} \mathbf{X}' = \partial_{\mathbf{X}} \tilde{\mathbf{W}} \cdot \mathbf{X}' = \tilde{\mathbf{W}} \cdot \partial_{\mathbf{X}} \mathbf{X} = \tilde{\mathbf{W}} \cdot \tilde{\mathbf{I}} = \tilde{\mathbf{W}} \tag{15b}$$

4.3 Fourier Transforms and the Doppler Effect

Consider the four-fold Fourier transformation (for brevity the writing of four integration signs and their limits from $-\infty$ to $+\infty$ are suppressed)

$$f(x, y, z, ict) = \int f(k_x, k_y, k_z, i\omega/c) e^{i(k_x x + k_y y + k_z z + (i\omega/c)ict)} \\ dk_x dk_y dk_z d i\omega/c \tag{16a}$$

Note that we use the same notation f for the function and its transform. To avoid ambiguity, the arguments are shown too. For brevity, (16a) will now be denoted as

$$f(\mathbf{X}) = \int f(\mathbf{K}) e^{i\mathbf{K}\cdot\mathbf{X}} d^4\mathbf{K} \tag{16b}$$

where $\mathbf{X}$ is given in (9) and $\mathbf{K} = (\mathbf{k}, i\omega/c)$ is written like a four vector, although at this stage we still need prove that it actually is a four-vector. The definition of $d^4\mathbf{K}$ follows from the comparison of (16a) and (16b). Now apply the four-dimensional gradient operation to (16a) and (16b), preferably one component at a time to avoid confusion. We then obtain

$$\partial_{\mathbf{X}} f(\mathbf{X}) = \int i\mathbf{K} f(\mathbf{K}) e^{i\mathbf{K}\cdot\mathbf{X}} d^4\mathbf{K} \tag{17}$$

We start in (16b) with a scalar function f, leading to a four-vector on the left in (17). It therefore follows that $\mathbf{K} = (\mathbf{k}, i\omega/c)$ is indeed a four-vector. By inspection of (5), the transformation formulas for $\mathbf{k}, \omega$ are seen to be

$$\begin{aligned} \mathbf{k}' &= \tilde{\mathbf{U}} \cdot (\mathbf{k} - \mathbf{v}\omega/c^2) \\ \omega' &= \gamma(\omega - \mathbf{v} \cdot \mathbf{k}) \end{aligned} \tag{18}$$

but this is the relativistic Doppler effect first announced by Einstein! [1] From the fact that $\mathbf{K}$ is a four-vector it also follows that the phase $\mathbf{K}\cdot\mathbf{X}$ is an invariant. How many rivers of ink have flown in order to "explain" the relativistic Doppler effect and the concept of "Phase invariance"! All this becomes superfluous when the present systematic approach is adopted. From the associated inverse Fourier transformation

$$f(\mathbf{K}) = \frac{1}{(2\pi)^4}\int f(\mathbf{X})e^{-i\mathbf{K}\cdot\mathbf{X}}d^4\mathbf{X} \tag{19}$$

one is led to construct the analog of (17), thus obtaining another four-vector differential operator

$$\begin{aligned}\partial_{\mathbf{K}} &=(\partial_{\mathbf{k}}, -ic\partial_\omega)\\ \partial_{\mathbf{k}'} &=\tilde{\mathbf{U}}\cdot(\partial_{\mathbf{k}}+\mathbf{v}\partial_\omega)\\ \partial_{\omega'} &=\gamma\left(\partial_\omega+\frac{1}{c^2}\mathbf{v}\cdot\partial_{\mathbf{k}}\right)\end{aligned} \tag{20}$$

which we could of course derive from $\mathbf{K}$ directly, by using the chain rule of calculus. Exploiting the duality of expressions in $\mathbf{X}$ and $\mathbf{K}$ spaces is a device that will be furthermore used subsequently.

Maxwell's equations (1), (4) subjected to the Fourier transformation (16b) yields algebraic equations which are often easier to manipulate. Thus in (1) $\partial_{\mathbf{x}}, \partial_t$ are replaced by $i\mathbf{k}, -i\omega$, respectively, and $\mathbf{E}=\mathbf{E}(\mathbf{k},\omega)$ etc. is understood. Similarly in (4) the primed differential operators are replaced by $i\mathbf{k}', -i\omega'$ and $\mathbf{E}'=\mathbf{E}'(\mathbf{k}',\omega')$ etc. are understood. The Fourier transformation of the field transformation formulas (8) is not a trivial concept: consider for example $\mathbf{E}'(\mathbf{X}') = \tilde{\mathbf{V}}\cdot[\mathbf{E}(\mathbf{X})+\mathbf{v}\times\mathbf{B}(\mathbf{X})]$ which is Fourier transformed according to

$$\int \mathbf{E}'(\mathbf{K}')e^{i\mathbf{K}'\cdot\mathbf{X}'}d^4\mathbf{K}' = \int \tilde{\mathbf{V}}\cdot[\mathbf{E}(\mathbf{K})+\mathbf{v}\times\mathbf{B}(\mathbf{K})]e^{i\mathbf{K}\cdot\mathbf{X}}d^4\mathbf{K} \tag{21}$$

If we identify $\mathbf{K}', \mathbf{K}$ as the four-vectors related according to $\mathbf{K}'=\tilde{\mathbf{W}}\cdot\mathbf{K}$ then the exponentials in (21) are identical since the exponents involve a scalar product of two four-vectors, which is invariant. Furthermore, on the right side of (21) a change of variables is effected by properly using the Jacobian determinant according to

$$d^4\mathbf{K}' = |\partial_{\mathbf{K}}\mathbf{K}'|d^4\mathbf{K} = |\tilde{\mathbf{W}}|d^4\mathbf{K} = d^4\mathbf{K} \tag{22}$$

and the determinant of the transformation matrix equals one, as discussed above. It is now clear that

$$\mathbf{E}'(\mathbf{K}') = \tilde{\mathbf{V}} \cdot [\mathbf{E}(\mathbf{K}) + \mathbf{v} \times \mathbf{B}(\mathbf{K})] \tag{23}$$

etc. Similarly to (22) we also have

$$d^4\mathbf{X} = d^4\mathbf{X}' \tag{24}$$

The results (22), (24) are usually phrased in the special relativity jargon as saying that "the four-dimensional volume element is a relativistic invariant". This is of course true for **K** space too, according to (22), in fact, for any four-vector, e.g., $\mathbf{J}_e, \mathbf{J}_m$, a representation space can be assigned and a volume element be defined, which will also be a relativistic invariant in this sense. All this is of course well known, e.g., see Pauli [21], the difficulty is in explaining it to our application-oriented students in a simple and coherent manner. Once Maxwell's equations and the field transformation formulas are available in algebraic form, it becomes much easier to manipulate the expression, e.g., to show that by substitution of the field transformation formulas into the unprimed set of Maxwell's equation, the primed set is derived.

4.4 Invariants Galore

In a sense, all physical laws and models are declarations about the invariance of certain quantities. Conservation laws are obviously in this category, but many other properties, e.g., symmetry in whatever sense, is also a declaration that something is unaffected, or conserved, or invariant, subject to some operation. Even writing a mathematical (algebraic, differential, integral etc.) equation for a physical law, such that everything appears on the left and is equal to zero on the right, is a declaration that "something" (the expression on the left) is immutable, i.e., equal to zero.

The scalar product of two four-vectors is one way of deriving Lorentz invariants, some of them have been recognized as fundamental laws, others are less important, but stand by for whenever they might be used. Thus $\mathbf{X} \cdot \mathbf{X} = 0$ is a cornerstone of relativity theory. Not less important is the fact that the D'Alembert operator

$$\partial_{\mathbf{x}} \cdot \partial_{\mathbf{x}} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{25}$$

is a Lorentz invariant. Another invariant that has been elevated to the status of "law" is the equation of continuity

$$\partial_{\mathbf{x}} \cdot \mathbf{J}_e = \nabla \cdot \mathbf{j}_e + \partial_t \rho_e = 0 \tag{26}$$

and the corresponding expression for the magnetic current density four-vector $\mathbf{J}_m$. Although the following invariant (note that it is not zero as in (26))

$$\partial_K \cdot \mathbf{J}_e(\mathbf{K}) = \partial_{\mathbf{k}} \cdot \mathbf{j}_e(\mathbf{k}, \omega) + c^2 \partial_\omega \rho_e(\mathbf{k}, \omega) = \text{constant} \tag{27}$$

is not recognized as a "law", I would like my students to be able to see that (27) follows from (16b) by identifying f with $\mathbf{J}_e$ and multiplying both sides by $\mathbf{X}\cdot$. Or, to see that the Fourier transformation of (26) prescribes for $\mathbf{J}_e$ and for the dual $\mathbf{J}_m$ too,

$$\mathbf{K} \cdot \mathbf{J}_e(\mathbf{K}) = \mathbf{K} \cdot \mathbf{J}_m(\mathbf{K}) = 0 \tag{28}$$

We have already introduced many four-vectors, e.g.,

$$\mathbf{X}, \partial_{\mathbf{X}}, \mathbf{J}_e, \partial_{\mathbf{J}_e}, \mathbf{J}_m, \partial_{\mathbf{J}_m}, \mathbf{K}, \partial_{\mathbf{K}} \tag{29}$$

including (12) and many more that are introduced below or elsewhere. Needless to say that linear combinations of invariants, invariant operations like (25) acting on invariants, and so on, also yield invariants, hence we are dealing with an infinite group. Another way of deriving invariants is through the field transformation formulas in (8). Of course, this is related to the properties of the field tensors, but can easily be verified directly. Thus, we have [3] the following expressions:

$$\begin{aligned}
c^2 B^2 - E^2 &= \text{constant}, \\
H^2 - c^2 D^2 &= \text{constant}, \\
\mathbf{B} \cdot \mathbf{E} &= \text{constant}, \\
\mathbf{H} \cdot \mathbf{D} &= \text{constant}, \\
\mathbf{B} \cdot \mathbf{H} - \mathbf{E} \cdot \mathbf{D} &= \text{constant}, \\
c^2 \mathbf{B} \cdot \mathbf{D} + \mathbf{E} \cdot \mathbf{H} &= \text{constant}
\end{aligned} \tag{30}$$

(of course in (30) "constant" is generic, it is not the same constant for all expressions). Still another way for deriving invariants through the Jacobian determinant is shown above (22), (24).

4.5 Potentials

As a variation of the theme, the potentials will be discussed in the context of the Fourier transformed algebraic Maxwell equations. The original equations are split into two sets of fields; one driven by $\mathbf{j}_e, \rho_e$, the other by $\mathbf{j}_m, \rho_m$. This yields

$$\begin{aligned}
i\mathbf{k} \times \mathbf{E}_e &= i\omega\mathbf{B}_e &,\quad i\mathbf{k} \times \mathbf{E}_m &= i\omega\mathbf{B}_m - \mathbf{j}_m \\
i\mathbf{k} \times \mathbf{H}_e &= -i\omega\mathbf{D}_e + \mathbf{j}_e &,\quad i\mathbf{k} \times \mathbf{H}_m &= -i\omega\mathbf{D}_m \\
i\mathbf{k} \cdot \mathbf{D}_e &= \rho_e &,\quad i\mathbf{k} \cdot \mathbf{D}_m &= 0 \\
i\mathbf{k} \cdot \mathbf{B}_e &= 0 &,\quad i\mathbf{k} \cdot \mathbf{B}_m &= \rho_m
\end{aligned} \tag{31}$$

where $\mathbf{E}_e = \mathbf{E}_e(\mathbf{k}, \omega)$ etc. Corresponding to (31), there exists in the primed frame of reference a set of Maxwell's equations with primed symbols. The transformation formulas relating $\mathbf{K}$ and the fields in both frames are given above. The students are more acquainted with the e-indexed set in (31). The relation between the two sets follows from the formal similarity and leads to the following "dictionary": by substitution according to this dictionary we obtain the e-indexed set of Maxwell's equations from the m-indexed one, and *vice-versa*:

$$\begin{aligned}
\mathbf{j}_e &\leftrightarrow -\mathbf{j}_m \\
\rho_e &\leftrightarrow -\rho_m \\
\mathbf{E}_e &\leftrightarrow \mathbf{H}_m \\
\mathbf{H}_e &\leftrightarrow \mathbf{E}_m \\
\mathbf{B}_e &\leftrightarrow -\mathbf{D}_m \\
\mathbf{D}_e &\leftrightarrow -\mathbf{B}_m \\
\mathbf{A}_e &\leftrightarrow -\mathbf{A}_m \\
\phi_e &\leftrightarrow -\phi_m \\
\Phi_e &\leftrightarrow -\Phi_m
\end{aligned} \tag{32}$$

In (32), the potentials have been included, defined according to

$$\begin{aligned}
\mathbf{B}_e &= i\mathbf{k} \times \mathbf{A}_e &,\quad \mathbf{D}_m &= i\mathbf{k} \times \mathbf{A}_m \\
\mathbf{E}_e &= -i\mathbf{k}\phi_e + i\omega\mathbf{A}_e &,\quad \mathbf{H}_m &= i\mathbf{k}\phi_m - i\omega\mathbf{A}_m \\
\Phi_e &= \left(\mathbf{A}_e, \tfrac{i}{c}\phi_e\right) &,\quad \Phi_m &= \left(\mathbf{A}_m, \tfrac{i}{c}\phi_m\right)
\end{aligned} \tag{33}$$

In (33), the potentials have been formally grouped into two four-vectors, essentially having the same structure as $\mathbf{K} = (\mathbf{k}, i\omega/c)$. Note that dimensionally $\mathbf{A} = \phi/c$ hence there exists no other alternative for grouping these terms. It therefore follows from (18) that the associated transformation formulas should be

$$\begin{array}{ll} \mathbf{A}'_e = \tilde{\mathbf{U}} \cdot (\mathbf{A}_e - \mathbf{v}\phi_e/c^2) \quad , & \mathbf{A}'_m = \tilde{\mathbf{U}} \cdot (\mathbf{A}_m - \mathbf{v}\phi_m/c^2) \\ \phi'_e = \gamma(\phi_e\omega - \mathbf{v} \cdot \mathbf{A}_e) \quad , & \phi'_m = \gamma(\phi_m\omega - \mathbf{v} \cdot \mathbf{A}_m) \end{array} \tag{34}$$

The definitions of the relativistic transformation formulas (34) guarantee that Φ_e, Φ_m are indeed four-vectors. Therefore $\Phi_e \cdot \Phi_e$, $\Phi_m \cdot \Phi_m$ and $\Phi_m \cdot \Phi_e$ and other combinations of Φ_e, Φ_m with four-vectors are new Lorentz invariants. As before, some are more interesting, others do not seem to have an immediate application. Noteworthy is the invariant

$$\mathbf{K} \cdot \Phi_e = \mathbf{k} \cdot \mathbf{A}_e - \frac{\omega}{c^2}\phi_e \tag{35}$$

and the m-indexed analog. In free space, $1/c^2 = \mu_0\varepsilon_0$ and (35) becomes the well-known Lorentz condition. However, in material media (35) ceases to be the Lorentz condition. This is a point that might cause some confusion, especially in view of the fact that the Lorentz condition is a gauge transformation invariant. This subject is well covered in many textbooks and need not be elaborated here.

4.6 The Cross Multiplication and Curl Operators

Teachers of a first course in electromagnetic field theory at sophomore or junior level are aware of the fact that vector analysis, in particular the Curl operation, are a major stumbling block for most students. Witness the long introductory chapters or detailed appendices in most textbooks. Suddenly, after some assimilation of the new concepts took place, they are told in the context of relativistic electrodynamics that the Curl operation is "not really a vector operation", actually an antisymmetric tensor with certain properties. In a short and condensed course, it was found expedient to keep tensor analysis and the formal details to the absolutely necessary minimum. Thus, we already know that **AB** is a tensor operation creating a dyadic, or a matrix with components A_iB_j. It is easy to see that a construct $A_iB_j - A_jB_i$ is an antisymmetric matrix. This in general defines the Curl operation where we now have $A_i = \partial/\partial x_i$. For $i, j = 1, 2, 3$ there are only three independent entries in the matrix, therefore the Curl operation in three dimensional space could be disguised as a vector operation, on the other hand in four dimensional space $i, j = 1, 2, 3, 4$, there are six independent entries; therefore, there is no way that such an entity could be represented as a four-vector. This discussion is considered sufficient for a first course in applied relativistic electrodynamics.

There are many cases where the six equations $\partial A_i/\partial x_j - \partial A_j/\partial x_i = 0$, $i, j = 1, 2, 3, 4$ must be satisfied. There is no harm in symbolically writing $\mathbf{A} \times \mathbf{B} = 0$, or $\Box \times \mathbf{A} = 0$ as long as we know what we are doing. This facilitates an association to already known concepts, such as $\nabla \times \nabla a = 0$, where a is a scalar field. Similarly $\Box \times \Box a = 0$ will be understood as

$$\frac{\partial}{\partial x_i}\frac{\partial a}{\partial x_j} - \frac{\partial}{\partial x_j}\frac{\partial a}{\partial x_i} = 0, \quad i, j = 1, 2, 3, 4 \tag{36}$$

and it is seen that for smooth a, such that the order of differentiation may be interchanged, (36) is identically satisfied. The analogy cannot be taken too far, for example the analog of $\nabla \cdot \nabla \times \mathbf{A} = 0$ does not exist. Simple examples are $\mathbf{X} \times \mathbf{X} = 0$, $\partial_{\mathbf{X}} \times \mathbf{X} = 0$ are easily verified. We can also show that $\partial_{\mathbf{X}} \times \Phi_e$ and $\partial_{\mathbf{X}} \times \Phi_m$ yield Maxwell's equations.

4.7 Proper Time and Related Concepts

In a subsequent section, ray equations are considered. The concept of a ray is intimately associated with wave packets and their motion in space. For that and other subjects, we have to include a short section on the concept of proper time and related concepts of velocity and acceleration. Actually, it is also warranted on ground of intrinsically being an ingenious idea: the creation of new four-vectors by associating four-vectors with invariants, e.g., differentiating $\mathbf{X}$ with respect to the proper time to derive the four-velocity, as done below.

In analogy with a three-dimensional space, we define the four-dimensional arc length element as

$$dS = \sqrt{d\mathbf{X} \cdot d\mathbf{X}} = \sqrt{(d\mathbf{x})^2 - c^2(dt)^2} \tag{37}$$

and this is an invariant. Using (37), we further define

$$d\tau = dS/ic = dt\sqrt{1 - \frac{(d\mathbf{x})^2}{c^2(dt)^2}} = dt\sqrt{1 - \frac{v^2}{c^2}} = dt/\gamma \tag{38}$$

where we now introduce the proper time τ. In (38), τ is the time in an arbitrary inertial frame of reference moving with velocity $\mathbf{v}$ and speed v relative to the proper frame in which $v = 0$, for which the time τ is now labelled as the proper time. From (38) we have $dt \geq d\tau$, the celebrated relativistic time dilation phenomenon. The four-velocity is defined as

$$\mathbf{V} = \frac{d\mathbf{X}}{d\tau} = \gamma(\mathbf{v}, ic) \tag{39}$$

and it follows that the relativistic transformation formula for $\mathbf{v}$ is

$$\mathbf{v}' = \frac{\tilde{\mathbf{U}} \cdot (\mathbf{v} - \mathbf{v}_0)}{\gamma(1 - \mathbf{v} \cdot \mathbf{v}_0/c^2)}, \quad \beta = \mathbf{v}_0/c \tag{40}$$

where a distinction is made between the velocity to be transformed and $\mathbf{v}_0$, the relative velocity between the frames of reference. Note that $\mathbf{V} \cdot \mathbf{V} = -c^2$, i.e., the length of the four-velocity four-vector is an imaginary constant. This is even easier to see from (39) when $\mathbf{v}_0 = 0$ i.e. $\gamma = 1$, and $\mathbf{v}_0 = 0$. The process of creating such new four-vectors can be continued. We define the four-acceleration as

$$\mathbf{W} = \frac{d\mathbf{V}}{d\tau} \tag{41}$$

and it is an interesting result that

$$\mathbf{V} \cdot \mathbf{W} = \mathbf{V} \cdot \frac{d\mathbf{V}}{d\tau} = \frac{1}{2}\frac{d}{d\tau}|\mathbf{V}|^2 = -\frac{1}{2}\frac{d}{d\tau}c^2 = 0 \tag{42}$$

i.e., the two four-vectors are always perpendicular, in a formal sense.

Now is a good time to pick up the subject of the Coulomb and Lorentz force formulas started in equation (2). It is not our intention to discuss in detail relativistic mechanics because this will again divert us from the main theme. It is, however, straightforward to associate with the four-velocity the momentum-energy four-vector

$$\mathbf{P} = m\mathbf{V} \tag{43}$$

where the proportionality factor m is the rest mass of a particle, measured by an observer at rest with respect to the object. Newton's law in four-vector form follows as

$$\mathbf{F} = m\mathbf{W} \tag{44}$$

Obviously (42) and (44) prescribe

$$\mathbf{V} \cdot \mathbf{F} = 0 \tag{45}$$

We shall state the Lorentz force formula in four-vector form and check our stipulation:

$$\mathbf{F} = (\gamma \mathbf{f}, i\gamma q_e \mathbf{v} \cdot \mathbf{E}/c) \tag{46}$$

where $\mathbf{f}$ is given in (2). For a point charge at rest in the primed frame of reference substitute $\mathbf{v} = 0, \gamma = 1$ and apply primes, thus (46) becomes $\mathbf{F}' = (q_e \mathbf{E}', 0)$. Therefore, if (46) defines a four-vector, we must have

$$\mathbf{F} \cdot \mathbf{F} = \mathbf{F}' \cdot \mathbf{F}' = \mathbf{f}' \cdot \mathbf{f}' = q_e^2 \mathbf{E}' \cdot \mathbf{E}' \tag{47}$$

Note that the right hand side of (47) expresses the Coulomb force formula (squared), hence dimensionally we already deal with an expression describing force. Using the definition of $\mathbf{F}$ in (46), the definition of the constant for the scalar product (47) and the transformation formula for $\mathbf{E}'$ given in (8) it can be shown (a good exercise!) that (46) indeed defines a four-vector. Finally, it is easy to verify that (46) satisfies (45), hence it is a properly defined four-force. The relation of (2) and (3) to the respective force formulas for $\mathbf{v} = 0$ is now clear.

4.8 The Minkowski Constitutive Relations

Sommerfeld [5] discusses the Minkowski constitutive relations for moving media. The question is an old one, and can be asked in various ways. If you ask "how does a moving medium behave, for example, does it appear to be a different medium with different constitutive parameters?", then the answer to the question is given in terms of the transformation formulas for the constitutive parameters. This has been amply discussed in the literature, e.g., see Post [22], see also Hebenstreit [23], [24], and Hebenstreit and Suchy [25], but this author's opinion is that this manner of asking the question does not contribute to any problem of application-oriented relativistic electrodynamics. The question should be asked in the way Minkowski asked it: what are the relations between the fields in a moving medium? A general discussion of bianisotropic media is given by Kong [16], who also cites previous studies. Even this definition is not as practical as the direct derivation of the dispersion equations, discussed below. Sommerfeld [5] considers the simple case of a medium which is linear, isotropic nondispersive and homogeneous in its rest frame, i.e., the comoving frame of reference. The treatment is not much more complicated when anisotropic dispersive media are assumed. A bonus of this approach is that we can now mention, through the subject of dispersive systems, the problem of generally non-local and non-instantaneous processes, and its relation to the light cone and causality.

In the comoving frame of reference, in the Fourier transformation representation space the constitutive relations

$$\begin{aligned} \mathbf{D}'(\mathbf{K}') &= \tilde{\boldsymbol{\varepsilon}}'(\mathbf{K}') \cdot \mathbf{E}'(\mathbf{K}') \\ \mathbf{B}'(\mathbf{K}') &= \tilde{\boldsymbol{\mu}}'(\mathbf{K}') \cdot \mathbf{H}'(\mathbf{K}') \end{aligned} \tag{48}$$

are assumed to hold, where the constitutive parameters here are dyadics (or call them matrices, or second rank tensors). The frequency dependent dispersive medium is very common and familiar, e.g., $\mathbf{D}'(\omega) = \varepsilon'(\omega) \cdot \mathbf{E}'(\omega)$, pertinent to the dielectric medium at rest within a capacitor, say. See for example Jackson [9]. It follows that in the time domain the constitutive relation becomes the convolution integral

$$\mathbf{D}'(t') = \int_{-\infty}^{t'} \tilde{\boldsymbol{\varepsilon}}'(\tau') \cdot \mathbf{E}'(t' - \tau')d\tau' \tag{49}$$

where the upper limit is taken as t' in order to have effects at time t' only from retarded (previous) causes occurring before t'. In view of (48), the ω dependent case of dispersion is termed temporal dispersion. It provides an example for processes observed at time t', caused by effects initiated previously, i.e., not simultaneously. This is a simple but important case, it has nothing intrinsic to do with relativistic considerations. However, the introduction of a general dependence on $\mathbf{K}$, i.e., $\mathbf{k}$ and ω as in (48), ties the problem of causality to special relativity. Thus in $\mathbf{X}$ space (48) becomes a four-dimensional integral

$$\mathbf{D}'(\mathbf{X}') = \int_{\Xi_1}^{\Xi_2} \tilde{\boldsymbol{\varepsilon}}'(\Xi') \cdot \mathbf{E}'(\mathbf{X}' - \Xi')d^4\Xi' \tag{50}$$

The choice of the integration limits in (50) is subject to (10), the so called cone of light which is explained in practically every book on special relativity. Inasmuch as c is the maximum speed for signals we must have $|\mathbf{x}|^2 \leq c^2t^2$, i.e., $|\mathbf{X}' - \Xi'| \geq 0$ must be chosen. This prescribes the choice of the limits in (50). In the special relativity jargon, if $\mathbf{X}'$ is chosen at the apex of the light cone, i.e., the present, then Ξ' must be within the light cone in the region corresponding to the past.

The **K**-space field transformation formulas, i.e., (8) with the argument changed according to (21) are now substituted into (48), and both sides are premultiplied by $\tilde{\mathbf{V}}^{-1}\cdot$ yielding

$$\mathbf{D} + \mathbf{v}\times\mathbf{H}/c^2 = \tilde{\boldsymbol{\varepsilon}}_{\mathbf{v}}\cdot(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

$$\mathbf{B} - \mathbf{v}\times\mathbf{E}/c^2 = \tilde{\boldsymbol{\mu}}_{\mathbf{v}}\cdot(\mathbf{H}-\mathbf{v}\times\mathbf{D})$$

$$\tilde{\boldsymbol{\varepsilon}}_{\mathbf{v}} = \tilde{\mathbf{V}}^{-1}\cdot\tilde{\boldsymbol{\varepsilon}}'\cdot\tilde{\mathbf{V}},\quad \tilde{\boldsymbol{\mu}}_{\mathbf{v}} = \tilde{\mathbf{V}}^{-1}\cdot\tilde{\boldsymbol{\mu}}'\cdot\tilde{\mathbf{V}} \tag{51}$$

where $\mathbf{D} = \mathbf{D}(\mathbf{K})$ etc. Now multiply the second line of (51) by $\mathbf{v}\times$ and substitute $\mathbf{v}\times\mathbf{B}$ into the first line. After some additional manipulation, we obtain the Minkowski constitutive relations for the present case

$$\begin{aligned}\mathbf{D} =& [\tilde{\mathbf{I}} + \tilde{\boldsymbol{\varepsilon}}_{\mathbf{v}}\cdot\mathbf{v}\times\tilde{\boldsymbol{\mu}}_{\mathbf{v}}\cdot\mathbf{v}\times\tilde{\mathbf{I}}]^{-1}\\ &\cdot\left[\tilde{\boldsymbol{\varepsilon}}_{\mathbf{v}}\cdot\left(\tilde{\mathbf{I}}+\frac{\mathbf{v}\times\mathbf{v}\times\tilde{\mathbf{I}}}{c^2}\right)\cdot\mathbf{E}+\left(\tilde{\boldsymbol{\varepsilon}}_{\mathbf{v}}\cdot\mathbf{v}\times\tilde{\boldsymbol{\mu}}_{\mathbf{v}}-\frac{\mathbf{v}\times\tilde{\mathbf{I}}}{c^2}\right)\cdot\mathbf{H}\right]\\ \mathbf{B} =& [\tilde{\mathbf{I}} + \tilde{\boldsymbol{\mu}}_{\mathbf{v}}\cdot\mathbf{v}\times\tilde{\boldsymbol{\varepsilon}}_{\mathbf{v}}\cdot\mathbf{v}\times\tilde{\mathbf{I}}]^{-1}\\ &\cdot\left[\tilde{\boldsymbol{\mu}}_{\mathbf{v}}\cdot\left(\tilde{\mathbf{I}}+\frac{\mathbf{v}\times\mathbf{v}\times\tilde{\mathbf{I}}}{c^2}\right)\cdot\mathbf{H}+\left(\tilde{\boldsymbol{\mu}}_{\mathbf{v}}\cdot\mathbf{v}\times\tilde{\boldsymbol{\varepsilon}}_{\mathbf{v}}-\frac{\mathbf{v}\times\tilde{\mathbf{I}}}{c^2}\right)\cdot\mathbf{E}\right]\end{aligned} \tag{52}$$

The result (52) reduces to the simple form given, by Sommerfeld [5] for example. For special cases of bianisotropic media in motion see Kong [16]. In conclusion, it is noted that the present discussion is based on the existence of (48) and the validity of (8) only. This remark is important for the case where one attempts to incorporate losses into the definition of the constitutive parameters.

4.9 Dispersion Equations in Moving Media

The concept of a dispersion equation is central to wave propagation in general, and especially in connection with ray propagation in dispersive media, discussed subsequently. It is therefore essential for engineers and applied physicists to cover these subjects in the course of discussing relativistic electrodynamics.

Consider Maxwell's equations in the comoving frame, given by

$$\begin{aligned} i\mathbf{k}' \times \mathbf{E}' &= i\omega'\mathbf{B}' - \mathbf{j}'_m \\ i\mathbf{k}' \times \mathbf{H}' &= -\, i\omega'\mathbf{D}' + \mathbf{j}'_e \\ i\mathbf{k}' \cdot \mathbf{D}' &= 0 \\ i\mathbf{k}' \cdot \mathbf{B}' &= 0 \end{aligned} \tag{53a}$$

i.e., with zero charge densities within the region of interest, and substitute the constitutive coefficients from (48). Furthermore, "Ohm's law" is assumed, i.e., the currents are not free source currents prescribed as constraints, but depend on the fields in the form

$$\begin{aligned} \mathbf{j}'_e &= \tilde{\boldsymbol{\sigma}}'_e \cdot \mathbf{E}' \\ \mathbf{j}'_m &= \tilde{\boldsymbol{\sigma}}'_m \cdot \mathbf{H}' \end{aligned} \tag{54}$$

and are also substituted into (53a). Consequently, it is possible to define new parameters and rewrite (53a) in the form

$$\begin{aligned} \mathbf{k}' \times \mathbf{E}' - \omega'\tilde{\boldsymbol{\mu}}'^{\dagger} \cdot \mathbf{H}' &= 0 \\ \mathbf{k}' \times \mathbf{H}' + \omega'\tilde{\boldsymbol{\varepsilon}}'^{\dagger} \cdot \mathbf{E}' &= 0 \\ \mathbf{k}' \cdot \mathbf{D}' &= 0 \\ \mathbf{k}' \cdot \mathbf{B}' &= 0 \end{aligned} \tag{53b}$$

The last two equations merely state that $\mathbf{D}'$ and $\mathbf{B}'$ are perpendicular to $\mathbf{k}'$. The first two equations in (53b) and their solution provides wave modes which are of interest. Mathematically they provide a system of six scalar homogeneous equations, for which the condition for nontrivial solutions is that the determinant of the system must vanish. This condition prescribes a scalar relation between ω' and $\mathbf{k}'$, the so called dispersion equation, which can be written in the form

$$F'(\mathbf{K}') = 0 \tag{55a}$$

Inasmuch as (55a) is a scalar condition, it is very suggestive to assume that the mere substitution of (18) to obtain

$$F'(\mathbf{K}'[\mathbf{K}]) = 0 = F(\mathbf{K}) \tag{55b}$$

provides the dispersion equation for the unprimed frame of reference. What we have done in the transition from (55a) to (55b) is merely to express F' in terms of $\mathbf{K}$. This does not mean that F is the dispersion equation in the unprimed frame of reference. The confusion is compounded by the fact that indeed $F(\mathbf{K}) = 0$ is Lorentz invariant and is the dispersion equation in the unprimed frame of reference, but this must be shown! One must start with the first two vector equations of (53b). The first can be rewritten as

$$\mathbf{H}' = \frac{1}{\omega'}\tilde{\boldsymbol{\mu}}'^{\dagger-1} \cdot \mathbf{k}' \times \mathbf{E} \tag{56a}$$

and substituted into the second, yields

$$(\mathbf{k}' \times \tilde{\boldsymbol{\mu}}'^{\dagger-1} \cdot \mathbf{k}' \times \tilde{\mathbf{I}} + \omega'^2 \tilde{\boldsymbol{\varepsilon}}'^{\dagger}) \cdot \mathbf{E}' = 0 \tag{57a}$$

Or, isolating $\mathbf{E}'$ first

$$\mathbf{E}' = -\frac{1}{\omega'}\tilde{\boldsymbol{\varepsilon}}'^{\dagger-1} \cdot \mathbf{k}' \times \mathbf{H}' \tag{56b}$$

and substituting into the first, yields

$$(\mathbf{k}' \times \tilde{\boldsymbol{\varepsilon}}'^{\dagger-1} \cdot \mathbf{k}' \times \tilde{\mathbf{I}} + \omega'^2 \tilde{\boldsymbol{\mu}}'^{\dagger}) \cdot \mathbf{H}' = 0 \tag{57b}$$

In the primed reference frame the dispersion equations are therefore

$$\det[\mathbf{k}' \times \tilde{\boldsymbol{\mu}}'^{\dagger-1} \cdot \mathbf{k}' \times \tilde{\mathbf{I}} + \omega'^2 \tilde{\boldsymbol{\varepsilon}}'^{\dagger}] = 0 \tag{58a}$$

$$\det[\mathbf{k}' \times \tilde{\boldsymbol{\varepsilon}}'^{\dagger-1} \cdot \mathbf{k}' \times \tilde{\mathbf{I}} + \omega'^2 \tilde{\boldsymbol{\mu}}'^{\dagger}] = 0 \tag{58b}$$

It is easy to show that the two conditions are identical (as they should be, because for a given wave mode there exists only one dispersion equation governing both the $\mathbf{E}'$ and $\mathbf{H}'$ fields). Consider multiplying (57a) from the left by $\mathbf{k}' \times \tilde{\boldsymbol{\varepsilon}}'^{\dagger-1}$. Using the rule that in a product of matrices, the product of determinants is equal to the determinant of the product, this yields

$$\begin{aligned} &\det[\mathbf{k}' \times \tilde{\boldsymbol{\varepsilon}}'^{\dagger-1}]\det[\mathbf{k}' \times \tilde{\boldsymbol{\mu}}'^{\dagger-1} \cdot \mathbf{k}' \times \tilde{\mathbf{I}} + \omega'^2 \tilde{\boldsymbol{\varepsilon}}'^{\dagger}] \\ &\quad = \det[\mathbf{k}' \times \tilde{\boldsymbol{\varepsilon}}'^{\dagger-1} \cdot \mathbf{k}' \times \tilde{\boldsymbol{\mu}}'^{\dagger-1} + \omega'^2 \tilde{\mathbf{I}}]\det[\mathbf{k}' \times \tilde{\mathbf{I}}] = 0 \end{aligned} \tag{59}$$

and because in the second line of (59) the second determinant is nonzero, we have $\det[\mathbf{k}' \times \tilde{\boldsymbol{\varepsilon}}'^{\dagger-1} \cdot \mathbf{k}' \times \tilde{\boldsymbol{\mu}}'^{\dagger-1} + \omega'^2 \tilde{\mathbf{I}}] = 0$. This is manipulated to yield

$$\begin{aligned} &\det[\mathbf{k}' \times \tilde{\boldsymbol{\varepsilon}}'^{\dagger-1} \cdot \mathbf{k}' \times \tilde{\boldsymbol{\mu}}'^{\dagger-1} + \omega'^2 \tilde{\mathbf{I}}] \\ &\quad = \det[\mathbf{k}' \times \tilde{\boldsymbol{\varepsilon}}'^{\dagger-1} \cdot \mathbf{k}' \times \tilde{\boldsymbol{\mu}}'^{\dagger-1} + \omega'^2 \tilde{\boldsymbol{\mu}}'^{\dagger} \cdot \tilde{\boldsymbol{\mu}}'^{\dagger-1}] \\ &\quad = \det[\mathbf{k}' \times \tilde{\boldsymbol{\varepsilon}}'^{\dagger-1} \cdot \mathbf{k}' \times \tilde{\mathbf{I}} + \omega'^2 \tilde{\boldsymbol{\mu}}'^{\dagger}] \det[\tilde{\boldsymbol{\mu}}'^{\dagger-1}] = 0 \end{aligned} \tag{60}$$

and since it is assumed that $\det[\tilde{\boldsymbol{\mu}}'^{\dagger-1}] \neq 0$, we obtain the second representation (58b).

We are now ready to explore the question of the corresponding dispersion equations for an observer attached to the unprimed frame of reference. Consider first the case where there are no magnetic currents, $\mathbf{j}_m = 0$. For this case we substitute from (8) into (57a) and use the fact that in the unprimed frame we have $\mathbf{k} \times \mathbf{E} = \omega \mathbf{B}$, obtaining

$$(\mathbf{k}' \times \tilde{\boldsymbol{\mu}}'^{\dagger-1} \cdot \mathbf{k}' \times \tilde{\mathbf{I}} + \omega'^2 \tilde{\boldsymbol{\varepsilon}}'^{\dagger}) \cdot [\tilde{\mathbf{V}} \cdot (\tilde{\mathbf{I}} + \mathbf{v} \times \mathbf{k} \times \tilde{\mathbf{I}}/\omega)] \cdot \mathbf{E} = 0 \tag{61a}$$

The determinant of the dyadic (matrix) in brackets (61a) is nonzero, hence the dispersion equation is again given by (58a) or (58b). Note that for $\omega' =$ constant the roots of the dispersion equation define wave modes. Leaving the dispersion equation in terms of $\mathbf{K}'$ as in (58a) or (58b) defines certain roots. In the unprimed frame for a choice of $\omega =$ constant, the dispersion equations are expressed in terms of the intrinsic $\mathbf{K}$ and the new roots and their different number from those encountered in the primed frame can give rise to new velocity induced wave modes. If $\mathbf{j}_m \neq 0$ then $\mathbf{k} \times \mathbf{E} = \omega \mathbf{B}$ does not apply and the argument leading to (61a) is not valid in this form. The best we can say for the general case is that in the unprimed frame we have

$$(\mathbf{k}' \times \tilde{\boldsymbol{\mu}}'^{\dagger-1} \cdot \mathbf{k}' \times \tilde{\mathbf{I}} + \omega'^2 \tilde{\boldsymbol{\varepsilon}}'^{\dagger}) \cdot [\tilde{\mathbf{V}} \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B})] = 0 \tag{61b}$$

and is satisfied by (58a) or (58b) for arbitrary $\mathbf{E}$ as long as the determinant of the expression in brackets in (61b) is nonvanishing. The discussion of the various pertinent modes is a complicated matter which will not be covered here (and is not recommended for the syllabus of a course based on the present article). See for example Chawla and Unz [26].

4.10 Application to Hamiltonian Ray Propagation

The subject of ray propagation in dispersive media is important for applied physicists and electromagnetic radiation engineers. It serves to compute field problems in inhomogeneous time-varying media. Usually, it is presented in the literature as a consequence of the Fermat principle, which is mathematically stated in terms of a variational principle. See for example Kelso [27], Van Bladel [28], Ghatak [29], Sommerfeld [30]. The subject is presented here in a simplified, although concise manner, which obviates the necessity of introducing the Fermat principle as a variational principle. This was found as a pedagogically preferable approach for the author's students. The Fermat principle (discussed here in the following section), is then presented when the student is already familiar with the Hamilton ray equations and possesses a basis for comparison. Ray propagation also serves here as an example for using four-vectors, for extending the **K** space beyond the Fourier transform, and it clarifies the role of the group velocity in ray theory.

In order to introduce the subject and relevant concepts, we start with the transition from general wave functions to wave packets in homogeneous media. This development is an extension of Stratton's [3] one-dimensional argument. Consider an arbitrary function as in (16b). In order for this function to be a solution of a wave system (e.g., Maxwell's equations rendered determinate by supplementing them by constitutive equations), it must satisfy the pertinent dispersion equation $F(\mathbf{K}) = 0$. This can be built into (16b) by rewriting it in the form

$$f(\mathbf{X}) = \int \delta(F) f(\mathbf{K}) e^{i\mathbf{K}\cdot\mathbf{X}} d^4\mathbf{K} = \int f(\mathbf{k}, \omega[\mathbf{k}]) e^{i\mathbf{k}\cdot\mathbf{x} - i\omega(\mathbf{k})t} d^3\mathbf{k} \quad (62)$$

where δ denotes the Heaviside unit impulse function which is zero for all values of the argument except $\delta(0)$, where it becomes singular, and $\omega = \omega(\mathbf{k})$ is the dispersion equation which, provided we can solve for ω, can be written as $F = \omega - \omega(\mathbf{k}) = 0$. Thus, the four-dimensional integral collapses into a three-dimensional integral, and of course we lose the identity of $f(\mathbf{X})$ as a four-dimensional Fourier transform integral. The closest we can come to a Fourier inverse transformation is to perceive t as a parameter and write

$$f(\mathbf{k}, \omega[\mathbf{k}]) e^{-i\omega(\mathbf{k})t} = \frac{1}{(2\pi)^3} \int f(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} d^3\mathbf{x} \quad (63)$$

Inasmuch as t is a parameter, (63) is valid for any value of t. Usually, we will find little use for (63), but the mathematical phenomenon is interesting. (62) is a general wave function for the wave system in question. The transition to a wave packet is facilitated by considering a narrow-band spectrum in $\mathbf{k}$, such that only the leading terms in the following Taylor expansion need to be retained:

$$\omega(\mathbf{k}) = \omega(\mathbf{k}_0) + \partial_{\mathbf{k}}\omega(\mathbf{k})|_{\mathbf{k}=\mathbf{k}_0} \cdot (\mathbf{k} - \mathbf{k}_0) = \omega_0 + \mathbf{v}_g \cdot (\mathbf{k} - \mathbf{k}_0) \quad (64a)$$

where $\mathbf{k}_0$ is the center value of the spectrum in $\mathbf{k}$, the vector derivative symbolizes the gradient operation in the representation space $\mathbf{k}$, and $\mathbf{v}_g$ will be identified below as the group velocity. Substituting (64a) into (62) yields, after some manipulation

$$f(\mathbf{x}, t, \mathbf{k}_0) = e^{i\mathbf{k}_0\cdot\mathbf{x}-i\omega_0 t} \int f(\mathbf{k}, \omega[\mathbf{k}]) e^{i(\mathbf{k}-\mathbf{k}_0)\cdot[\mathbf{x}-\mathbf{v}_g(\mathbf{k}_0)t]} d^3\mathbf{k} \quad (65a)$$

which is interpreted as a wave packet consisting of a carrier wave times an envelope (or modulation), the latter is a constant on the trajectory $\mathbf{x} - \mathbf{v}_g t$ = constant, i.e., defines the group velocity $d\mathbf{x}/dt = \mathbf{v}_g$. Obviously, (63), (64a), (65a) are easier to handle in terms of the three-velocity $\mathbf{v}_g$. However, just as an exercise, let us see that the same can be handled in four-vector notation too. Thus, instead of (64a) we write

$$F(\mathbf{K}) = F(\mathbf{K}_0) + \frac{\partial F}{\partial \mathbf{K}_0} \cdot (\mathbf{K} - \mathbf{K}_0) = 0 \quad (64b)$$

where the differentiation with respect to $\mathbf{K}_0$ means that this value is substituted into the derivative after differentiation. Inasmuch as $F(\mathbf{K}_0) = 0$ too, we conclude that within the approximation where (64b) holds the term in (64b) involving the derivative vanishes too. Adding this zero term in the exponent in (62) yields

$$\begin{aligned} f(\mathbf{X}) &= \int \delta(F) f(\mathbf{K}) e^{i\mathbf{K}\cdot\mathbf{X}} d^4\mathbf{K} \\ &= e^{i\mathbf{K}_0\cdot\mathbf{X}} \int \delta(F) f(\mathbf{K}) e^{i(\mathbf{K}-\mathbf{K}_0)\cdot(\mathbf{X}-\partial F/\partial \mathbf{K}_0)} d^4\mathbf{K} \end{aligned} \quad (65b)$$

which again displays the wave packet structure of a carrier wave multiplying the envelope function, and the envelope is constant on a trajectory defined by

$$\mathbf{X} - \partial F/\partial \mathbf{K}_0 = \text{constant} \quad (66)$$

where the constant stands for a four-vector. From the components of (66), by differentiation, follows the definition of the group velocity $d\mathbf{x}/dt = \mathbf{v}_g$ given above. It appears that in this case the four-vector treatment is somewhat more cumbersome, although still feasible.

The definition of wave packets in inhomogeneous, time dependent media is impossible in general. However, for "slow variation" such that the variation of the properties of the medium over distances and time intervals commensurate with the wavelength and the period, respectively, of the signal, an approximate procedure can be defined. This is usually referred to as working in the high frequency limit. Clearly, spatial and temporal changes in the constitutive parameters do not fit into our formalism for homogeneous media. Such changes cannot be included in the comoving frame, e.g., see (48) or in the unprimed laboratory frame, (52), they are not consistent with a Fourier transform representation as in (53b), and therefore (58a), (58b) are not valid, nor is a representation of a wave function in terms of a superposition of plane waves, as in (62) a legitimate solution. In order to overcome this difficulty, we introduce the so called iconal approximation (this is usually called in the mathematicians jargon "the WKB approximation"). For further explanation and previous literature citations see for example Censor [31] and Molcho and Censor[32]. In time-invariant, homogeneous media the basic solution is a plane wave $Ae^{i\theta}$, A =const., $\theta = \mathbf{K}\cdot\mathbf{X}$. In slowly varying time-varying inhomogeneous media the fundamental solution is chosen as

$$A(\mathbf{X})e^{i\theta(\mathbf{x})}, \partial_{\mathbf{X}}\theta(\mathbf{X}) = \mathbf{K}(\mathbf{X}) \tag{67}$$

Therefore $\mathbf{K}$ is obtained as the four-gradient of the phase, as in the simple case. This is the iconal approximation. The existence and the representation of the new function θ is yet an open question and will be discussed shortly. The idea of slow variation is mathematically stated by assuming that derivatives of the amplitude in (67) are negligibly small compared to the derivatives of the exponential, e.g.

$$\begin{aligned}\partial_t[A(\mathbf{X})e^{i\theta(\mathbf{x})}] &= [\partial_t A(\mathbf{X})]e^{i\theta(\mathbf{x})} - i\omega A(\mathbf{X})e^{i\theta(\mathbf{x})} \\ &\approx -i\omega A(\mathbf{X})e^{i\theta(\mathbf{x})}, |\partial_t A(\mathbf{X})/A(\mathbf{X})| \ll |i\omega|\end{aligned} \tag{68}$$

Therefore the iconal approximation has the same property as the Fourier transformation in (17), namely that the differential operation $\partial_{\mathbf{X}}$ is equivalent to algebraization, by producing a factor $i\mathbf{K}$.

The simplest way to introduce the representation of θ, which is also very appealing to students familiar with electrostatics, is the following: $\mathbf{K} = \partial_{\mathbf{X}}\theta$ is reminiscent of the way the electrostatic potential and $\mathbf{E} = -\partial_{\mathbf{x}}\phi$ was derived. If we write

$$\phi(\mathbf{x}) = \int_{\phi(\mathbf{x}_0)}^{\phi(\mathbf{x})} d\phi, \quad \phi(\mathbf{x}_0) = 0 \tag{69}$$

then we choose the lower limit, the so called reference potential as zero, and the integral depends on the limits only, hence in the mathematician's language $d\phi$ is a total differential. Using the chain rule of calculus, we write $d\phi = \partial_{\mathbf{x}}\phi \cdot d\mathbf{x} = -\mathbf{E} \cdot d\mathbf{x}$ and (69) becomes

$$\phi(\mathbf{x}) = -\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{E}(\boldsymbol{\xi}) \cdot d\boldsymbol{\xi} \tag{70}$$

where $\boldsymbol{\xi}$ denotes the integration (dummy) variable, but henceforth we shall write $\mathbf{x}$ also under the integration symbol, except in cases where confusion might arise. Recall that $\mathbf{E}$ was dubbed as a conservative field which satisfies $\nabla \times \mathbf{E} = \partial_{\mathbf{x}} \times \mathbf{E}(\mathbf{x}) = 0$. The last condition stems from the Stokes theorem, and it amounts to $\frac{\partial}{\partial x_i}\frac{\partial \phi}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial \phi}{\partial x_i}$ i.e., it is a statement on the smoothness of the function ϕ. Applying all this to ray theory, we now use the four-dimensional analogs and write

$$\theta(\mathbf{X}) = \int_{\theta(\mathbf{X}_0)}^{\theta(\mathbf{X})} d\theta = \int_{\mathbf{X}_0}^{\mathbf{X}} \partial_{\mathbf{X}}\theta(\mathbf{X}) \cdot d\mathbf{X} = \int_{\mathbf{X}_0}^{\mathbf{X}} \mathbf{K}(\mathbf{X}) \cdot d\mathbf{X} \tag{71}$$

where the reference phase is chosen as zero. The last expression in (71) is a line integral in four-space. Without proof here, we recall that the Stokes theorem is valid for higher dimensionality too; therefore, in analogy to electrostatics, for the integral to depend on the limits only, i.e., for $d\theta$ to be a total differential, we must impose

$$\Box \times \mathbf{K} = \partial_{\mathbf{X}} \times \mathbf{K}(\mathbf{X}) = 0 \tag{72a}$$

i.e.

$$\frac{\partial}{\partial X_i}K_j - \frac{\partial}{\partial X_j}K_i = \frac{\partial}{\partial X_i}\frac{\partial \theta}{\partial X_j} - \frac{\partial}{\partial X_j}\frac{\partial \theta}{\partial X_i} = 0 \tag{72b}$$

which can also be written in terms of three-vectors as

$$\begin{aligned} \partial_{\mathbf{x}} \times \mathbf{k}(\mathbf{x}, t) &= 0 \\ \partial_t \mathbf{k}(\mathbf{x}, t) + \partial_{\mathbf{x}}\omega(\mathbf{x}, t) &= 0 \end{aligned} \tag{72c}$$

see Poeverlein [33]. The first line (72c) is Snell's law in disguise and is referred to as the Sommerfeld-Runge law of refraction. Recall that in electrostatics $\nabla \times \mathbf{E} = \partial_{\mathbf{x}} \times \mathbf{E}(\mathbf{x}) = 0$ implied the continuity of the tangential component of $\mathbf{E}$ at the interface between media with different constitutive parameters. In analogy, the first line (72c) prescribes the continuity of the tangential component of $\mathbf{k}$ at the interface between media with different constitutive parameters. But this is exactly what Snell's law states!

Using the iconal approximation in the $\mathbf{X}$-space Maxwell equations, (1), and including slowly varying constitutive equations

$$\begin{aligned} \mathbf{D}(\mathbf{K},\mathbf{X}) &= \tilde{\boldsymbol{\varepsilon}}(\mathbf{K},\mathbf{X}) \cdot \mathbf{E}(\mathbf{K},\mathbf{X}) \\ \mathbf{B}(\mathbf{K},\mathbf{X}) &= \tilde{\boldsymbol{\mu}}(\mathbf{K},\mathbf{X}) \cdot \mathbf{H}(\mathbf{K},\mathbf{X}) \end{aligned} \tag{73}$$

where (73) assumes that $\mathbf{X}$-space is the comoving frame, i.e., the frame where the medium is at rest, or instead of (73) we could use the corresponding Minkowski constitutive equations (52), in which it is assumed that $\mathbf{X}'$-space is the comoving frame, we are led to a space and time dispersion equation

$$F(\mathbf{K},\mathbf{X}) = 0 \tag{74a}$$

which can also be written as

$$F(\partial_{\mathbf{X}}\theta,\mathbf{X}) = 0 \tag{74b}$$

The last form is a differential equation on θ, referred to as the iconal differential equation. It is usually nonlinear and difficult to solve. The idea of deriving ray equations is to replace (74a), hence also (74b) by a set of coupled first order ordinary differential equations (this is usually called in the mathematicians jargon "the method of characteristics" and electrical engineers sometimes refer to "state space equations"). In the next section, it is shown how to derive the ray equations using the generalized Fermat principle due to Synge [34]. However, it must be realized that electrical engineering and applied physics students, even if they have been exposed to variational analysis, say if they had a course in analytical mechanics, can hardly cope with the subject. It was found advantageous to obviate this approach by using the following methodology. To satisfy (74a), it suffices to satisfy $dF = 0$, which implies $F =$ constant, and provided this constant is set to zero at least for one set of values of $\mathbf{K}_0, \mathbf{X}_0$ we have $F = 0$ everywhere. The last condition is taken care of by the initial and boundary conditions, so all we have to worry about is the solution of $dF = 0$. Choosing a real monotonous parameter τ, we now write

$$\frac{dF}{d\tau} = \frac{\partial F}{\partial \mathbf{K}} \cdot \frac{d\mathbf{K}}{d\tau} + \frac{\partial F}{\partial \mathbf{X}} \cdot \frac{d\mathbf{X}}{d\tau} = 0 \tag{75}$$

for which we "guess" a solution

$$\begin{aligned} \frac{d\mathbf{X}}{d\tau} &= \lambda(\tau)\frac{\partial F}{\partial \mathbf{K}} \\ \frac{d\mathbf{K}}{d\tau} &= -\lambda(\tau)\frac{\partial F}{\partial \mathbf{X}} \end{aligned} \tag{76}$$

where $\lambda(\tau)$ is an arbitrary Lagrange multiplier function, and τ is now understood as the proper time, which is a relativistic invariant and therefore serves to preserve the four-vector nature of $d\mathbf{X}/d\tau, d\mathbf{K}/d\tau$ in (76). Moreover, this defines $d\mathbf{X}/d\tau$ as a four-velocity as in (39) and the associated $d\mathbf{x}/dt$ as a conventional three-dimensional velocity which transforms from one reference frame to another according to the special relativistic formula for the transformation of velocity (40). The role of the various dependent and independent variables in (76) must be amplified. A solution of (76) (if we know how to solve it) yields a trajectory $\mathbf{X}(\tau)$ in four-space; the field $\mathbf{K}(\mathbf{X})$ is found as $\mathbf{K}(\mathbf{X}[\tau])$ on this trajectory. Note that we have defined $\mathbf{X}(\tau)$, i.e., $\mathbf{X}$ as a function of τ, but not τ as a function of $\mathbf{X}$. If sufficient rays are computed in a certain region, we have, in principle, at our disposal the field $\mathbf{K}(\mathbf{X})$ everywhere in this region. Inasmuch as the integration (71) is independent of the specific path of integration, the phase θ can be computed according to the definition (71), whether we integrate along a specific ray path or use the field $\mathbf{K}(\mathbf{X})$. Note that ray theory in its simplest form enables us to compute the phase, or wave fronts, but is mute as to the amplitude and polarization of the wave. The intensity (absolute value of the amplitude) can sometimes be determined by heuristically applying energy flux considerations. Information regarding polarization is almost always lost in a ray computation procedure. Obviously, (76) satisfies (75), hence subject to initial condition also (74a), (74b). However, (76) is not a unique choice. As an example for a different choice, see the subsequent discussion on ray propagation in lossy media.

What makes the choice (76) special is the fact that it also satisfies the uniqueness conditions (72a), (72b), (72c). Thus applying the $\partial_{\mathbf{X}} \times$

operation to (76) leads to

$$
\begin{aligned}
\frac{\partial}{\partial X_i}\frac{dK_j}{d\tau} - \frac{\partial}{\partial X_j}\frac{dK_j}{d\tau} &= \frac{d}{d\tau}\left(\frac{\partial K_j}{\partial X_i} - \frac{\partial K_i}{\partial X_j}\right) \\
&= \frac{\partial}{\partial X_i}\left[-\lambda(\tau)\frac{\partial F}{\partial X_j}\right] - \frac{\partial}{\partial X_j}\left[-\lambda(\tau)\frac{\partial F}{\partial X_i}\right] \\
&= \lambda(\tau)\frac{\partial}{\partial X_j}\frac{\partial F}{\partial X_i} - \lambda(\tau)\frac{\partial}{\partial X_i}\frac{\partial F}{\partial X_j} = 0
\end{aligned}
\tag{77}
$$

and consequently $d_\tau[\partial_{\mathbf{X}} \times \mathbf{K}] = 0$, i.e., $\partial_{\mathbf{X}} \times \mathbf{K} =$ constant and if this constant is zero for any value of τ, at the initial point of the ray say, then it is always zero. In performing the operations indicated in (77), it is assumed that we have at our disposal a ray and also neighboring rays in its vicinity, otherwise the operations $\partial/\partial X_i$ cannot be performed. We conclude that the set of equations (76) uniquely determines the phase, and therefore can be considered as equivalent to a direct solution of the iconal equation (74b). In performing (77), it is assumed that $F(\mathbf{K}, \mathbf{X}) = 0$ and therefore also derivatives of F are available as algebraic expressions in terms of $\mathbf{K}, \mathbf{X}$. Consequently, expressions for $\partial K_i/\partial X_j$ are obtained. Equation (77) does *not* mean that the field $\mathbf{K}(\mathbf{X})$ is known and differentation according to $d_\tau[\partial_{\mathbf{X}} \times \mathbf{K}]$ is actually performed on this function.

Inasmuch as $d\mathbf{X}/d\tau$ (76) is a four-velocity, $\mathbf{V} \cdot \mathbf{V} = -c^2$ applies, hence

$$
\lambda(\tau) = ic \Big/ \sqrt{\frac{\partial F}{\partial \mathbf{K}} \cdot \frac{\partial F}{\partial \mathbf{K}}}
\tag{78}
$$

This result appears rather strange at a first glance: on one hand we announced that $\lambda(\tau)$ is a function of τ, while on the other hand (78) declares λ as a function of the derivatives. What (78) means is that here $\partial F(\mathbf{K}[\tau], \mathbf{X}[\tau])/\partial \mathbf{K}$ *is a function of* τ *along the ray.* This is a very delicate point that will be again mentioned below in connection with rays in lossy media.

The ray theory developed above involves four-vectors and therefore applies to any frame of reference, provided the dispersion equation is available. Dividing all the equations (76) by $dt/d\tau$, a set of equations is obtained in which t is the parameter along the ray. This has the advantage of eliminating $\lambda(\tau)$:

$$\mathbf{v}_g = \frac{d\mathbf{x}}{dt} = -\frac{\partial F/\partial \mathbf{k}}{\partial F/\partial \omega}$$
$$\frac{d\mathbf{k}}{dt} = \frac{\partial F/\partial \mathbf{x}}{\partial F/\partial \omega} \tag{79a}$$
$$\frac{d\omega}{dt} = -\frac{\partial F/\partial t}{\partial F/\partial \omega}$$

Note that the relativistic nature of the variables is thus obscured, hence a transformation of trajectories and associated group velocities becomes complicated. Furthermore, in the present form (79a), the application of (77) is invalid. It is easy to see that in (79a) we actually deal with the group velocity as defined in (64a), (64b), this is a direct result of the chain rule of calculus:

$$\frac{\partial F(\mathbf{K},\mathbf{X})}{\partial \mathbf{k}} = \left.\frac{\partial F(\mathbf{K},\mathbf{X})}{\partial \mathbf{k}}\right|_{\omega=\mathrm{const}} + \left.\frac{\partial F(\mathbf{K},\mathbf{X})}{\partial \omega}\right|_{K=\mathrm{const}} \frac{\partial \omega(\mathbf{k},\mathbf{X})}{\partial \mathbf{k}} \tag{80}$$

In all the operations in (80), $\mathbf{X}$ is held constant. For the special case of a medium not varying in time the third equation in (79a) reduces to ω = constant. If we furthermore represent F as $F = \omega - \omega(\mathbf{k},\mathbf{x}) = 0$, then we obtain

$$\mathbf{v}_g = \frac{d\mathbf{x}}{dt} = -\frac{\partial F}{\partial \mathbf{k}}$$
$$\frac{d\mathbf{k}}{dt} = \frac{\partial F}{\partial \mathbf{x}} \tag{79b}$$

4.11 The Fermat Principle and its Relativistic Connotations

The Fermat principle is usually stated as saying that the ray will traverse the distance between two points in extremal (usually minimal) time. For media not varying in time, after integration with respect to time, the phase becomes a line integral in $\mathbf{x}$-space and has the form

$$\theta(\mathbf{x},t) = \left[\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{k}\cdot d\mathbf{x}\right] - \omega t \tag{81}$$

where the brackets emphasize that ωt is not included in the integral. Taking θ at $t = 0$, say, taking the upper limit in (81) as a fixed value $\mathbf{x}_1$ and dividing (81) by the constant ω yield a function $T(\mathbf{x}_0, \mathbf{x}_1)$ whose dimension is time, and it depends on the fixed endpoints $\mathbf{x}_0, \mathbf{x}_1$:

$$T(\mathbf{x}_0, \mathbf{x}_1) = \frac{1}{\omega} \int_{\mathbf{x}_0}^{\mathbf{x}_1} \mathbf{k}(\mathbf{x}) \cdot d\mathbf{x} \tag{82}$$

The statement of the Fermat principle is that $\delta T(\mathbf{x}_0, \mathbf{x}_1) = 0$, where δ denotes the variation operation. Obviously, dealing with a definite integral we cannot find the extremum by differentiating T and equating the result to zero as done for functions in calculus. The variation operator, which for all other purposes acts as the conventional differentiation operator, operates on the functional, i.e., operates on the functions in the integrand, (82). Exchanging order of integration and variation, $\delta T(\mathbf{x}_0, \mathbf{x}_1)$ becomes

$$\begin{aligned} 0 =& \delta T(\mathbf{x}_0, \mathbf{x}_1) = \frac{1}{\omega} \int_{\mathbf{x}_0}^{\mathbf{x}_1} \{\delta[\mathbf{k}(\mathbf{x})] \cdot d\mathbf{x} + \mathbf{k}(\mathbf{x}) \cdot d\delta\mathbf{x}\} \\ =& \frac{1}{\omega} \int_{\mathbf{x}_0}^{\mathbf{x}_1} \{\delta\mathbf{k} \cdot d\mathbf{x} - d\mathbf{k} \cdot \delta\mathbf{x}\} + \frac{1}{\omega} \int_{\mathbf{x}_0}^{\mathbf{x}_1} d[\mathbf{k} \cdot \delta\mathbf{x}] \end{aligned} \tag{83}$$

The last integral in (83) is directly integrable, and since at the fixed endpoints the variation vanishes (that is what is meant by fixed endpoints), this integral vanishes. For arbitrary $\delta\mathbf{k}$, $\delta\mathbf{x}$ the integrand $\delta\mathbf{k} \cdot d\mathbf{x} - d\mathbf{k} \cdot \delta\mathbf{x}$ in (83) must vanish. Another constraint that must be met is the dispersion equation $F = \omega - \omega(\mathbf{k}, \mathbf{x}) = 0$, i.e., its variation δF must vanish too. This yields a second equation and after slightly modifying $\delta\mathbf{k} \cdot d\mathbf{x} - d\mathbf{k} \cdot \delta\mathbf{x}$ by introducing an arbitrary parameter w and exchanging derivatives for the differentials, we have

$$\begin{aligned} \frac{d\mathbf{x}}{d\mathrm{w}} \cdot \delta\mathbf{k} - \frac{d\mathbf{k}}{d\mathrm{w}} \cdot \delta\mathbf{x} =& 0 \\ \frac{\partial F}{\partial \mathbf{k}} \cdot \delta\mathbf{k} + \frac{\partial F}{\partial \mathbf{x}} \cdot \delta\mathbf{x} =& 0 \end{aligned} \tag{84}$$

consistent with (79b) when w is arbitrarily identified with t. We could also include F in the integrand in (83), because $F = 0$ and thus does not change the value of the integral. This will be included as an illustration in the derivation of the generalized Fermat principle. The equations (79b) resulting from the variational principle are called the Euler-Lagrange equations. The generalization of the Fermat principle to include time-varying media is given by Synge [34]. Here the notation is simplified by the use of four-vectors. The Fermat principle is represented in the form (again one must keep in mind that $\mathbf{X}$ in the integrand is the dummy variable)

$$0 = \delta\theta(\mathbf{X}) = \delta \int_{\mathbf{X}_0}^{\mathbf{X}_1} d\theta(\mathbf{X}) = \delta \int_{\mathbf{X}_0}^{\mathbf{X}_1} \mathbf{K}(\mathbf{X}) \cdot d\mathbf{X} \tag{85a}$$

where we have a line integral in four-space between two fixed so called world points $\mathbf{X}_0, \mathbf{X}_1$. Equation (85a) expresses the idea that the integral path has to be chosen in such a way that the sum of the increments $d\theta$ along the path will be minimal (or extremal, in general). Inasmuch as the points $\mathbf{X}_0, \mathbf{X}_1$ already define a fixed time interval $t_1 - t_0$, the question arises as to what can be minimized (or in general extremized) in this process. The answer is fascinating, and can only be given in the context of special relativity theory: the quantity to be minimized is $d\theta = \mathbf{K} \cdot d\mathbf{X} = \mathbf{K} \cdot (d\mathbf{X}/d\tau)d\tau$ where τ is the proper time. The components of the $\mathbf{K}$ vector, as well as the components of the four-velocity $d\mathbf{X}/d\tau$ are slowly varying functions and may be considered as constant for an incremental $d\theta$, i.e., when $\mathbf{X}_0, \mathbf{X}_1$ are close world points. Therefore, the integral (85a) becomes $\delta \int_{\mathbf{X}_0}^{\mathbf{X}_1} d\tau = 0$. Another way of looking at it is to exploit the invariance of $d\theta = \mathbf{K} \cdot d\mathbf{X} = \mathbf{K}' \cdot d\mathbf{X}'$ which in the proper frame where $\mathbf{v}_g = 0$ and $\gamma = 1$ becomes $d\theta = \omega' d\tau$. If ω', which is a slowly varying function, is considered to be constant over the distance and time of $d\theta$, then the same conclusion is reached, i.e., that $d\theta \propto d\tau$, i.e., minimizing θ means that the proper time along the ray is minimized! This interpretation has been previously proposed by Censor [31,35]. The variation integral (85a) is now rewritten as

$$0 = \delta\theta(\mathbf{X}) = \delta \int_{\mathbf{X}_0}^{\mathbf{X}_1} \left\{ \mathbf{K}(\mathbf{X}[\tau]) \cdot \frac{d\mathbf{X}[\tau]}{d\tau} - \lambda(\tau) F(\mathbf{K}[\tau], \mathbf{X}[\tau]) \right\} d\tau \tag{85b}$$

and the variation operation performed. In (85b), to illustrate the technique, the constraint $F = 0$ is included in the integral by adding a term $-\lambda F$, where $\lambda(\tau)$ is an arbitrary Lagrange multiplier function. Using the same technique as in (83) we now obtain

$$0 = \delta\theta(\mathbf{X})$$

$$= \int_{x_0}^{x_1} \left\{ \delta\mathbf{K} \cdot \frac{d\mathbf{X}}{d\tau} - \delta\mathbf{X} \cdot \frac{d\mathbf{K}}{d\tau} \right.$$

$$\left. - \lambda(\tau) \left[\frac{\partial F}{\partial \mathbf{K}} \cdot \delta\mathbf{K} + \frac{\partial F}{\partial \mathbf{X}} \cdot \delta\mathbf{X} \right] \right\} d\tau \qquad (85c)$$

and for arbitrary $d\mathbf{X}$, $d\mathbf{K}$ the expression in braces in 85c) yields once again the ray equations (76). Note that in the present development the step of proving that the ray equations define **K** which satisfies (72a)-(72c) is not necessary. The two methods (i.e., "guessing" the result and verifying its validity using (72a)-(72c), and finding the ray equations by deriving the Euler-Lagrange equations of the Fermat-Synge variational principle, which by the way must also be viewed as an ingenious guess because it is stated axiomatically!) lead to the same ray equations. The logical conclusion is that the Fermat principle, an edifice of physics, is equivalent to (72b), a modest statement on the smoothness of the function $\theta(\mathbf{X})$. It would be a good thing for our students to know this and to shed some of the mystery involved in the attempts to explain the Fermat principle.

4.12 Application to Ray Propagation in Lossy Media

Another application which invokes questions of Lorentz invariance and relativistic transformations, coupled with analyticity of functions, is the question of ray propagation in lossy media. At a first glance, this appears as a very unlikely candidate for this role, but there are some fundamental questions involved, tied in with relativistic problems. In lossy media as discussed above, when the losses are introduced through currents, say, as in (53a), (53b), (54), the ensuing dispersion equations as in (74a) are complex. Consequently, the group velocity according to (79) will be complex too, in general, in turn implying complex space and time. A recent study by Censor and Gavan [36] cites earlier work in this area. The main problem is that numerous models are feasible for extending the group velocity to the present case. All the models are mathematically self-consistent, all define group velocities which reduce to the conventional definition in lossless media, but the physical consequences vary from one definition to another. There are essentially two main schools of thought: some researchers do not experience any difficulty in continuing the concepts of a real group velocity, and real space and time, into the complex domain. The trouble is that complex group velocities are mathematically possible for real dispersion equations in lossless media. A ray which starts in a lossless region in real space-time is propagated into a lossy region. According to the complex ray tracing method the group velocity and the trajectory will become complex. Upon reentering a lossless region, the group velocity will not automatically revert to real values. Thus, in addition to the conceptual difficulties of dealing with complex space and time, and having to come to terms with a complex group velocity for which no physical explanation can be found, advocates of this approach are also confronted with complex group velocities in lossless media, completely losing the physical appeal of the group velocity concept. The other group of researchers advocates using real group velocities even in lossy media. The present model belongs to this class. The difficulty with many of these models is that they do not maintain analyticity, consequently differential operators as appearing in (79a), if the functions do not satisfy the Cauchy-Riemann conditions for analyticity, become meaningless. Furthermore, transformation formulas such as (40) apply to complex functions only if they are analytic, otherwise we get different transformations for real and imaginary components. We also mention in passing that analyticity has a lot to do with causality, e.g., *via* the Kramers-Kronig relations, see for example Jackson [9], and Kong [16], and also due to the fact that the zeroes of the dispersion equation are the poles determining the free space Green function for the medium at hand, see for example Felsen and Marcuvitz [37]. The following model offers a definition for the phase and group velocity which keeps the group velocity simultaneously real (i.e., confined to the real axes of the relevant complex variables complex planes) and analytic, therefore commensurate with the pertinent relativistic transformation formulas. To achieve this goal Censor and Suchy [38] modified (79a), by introducing a new degree of freedom expressed by including a new vector $\boldsymbol{\beta}$ in the ray equations, as explained below. To achieve this goal it is assumed that in addition to (75) there also exists the constraint that everywhere along the path the product $\mathbf{\Lambda}(\tau)^\dagger \cdot \mathbf{V}(\tau)$ vanishes. The new four-vector $\mathbf{\Lambda}(\tau)^\dagger$ will be defined in such a way that the group velocity along the ray will be real. For simplicity, we assume that $\mathbf{\Lambda}(\tau)^\dagger$ already absorbs the Lagrange multiplier vector function. We now have instead of (75)

$$\frac{dF}{d\tau} = \frac{\partial F}{\partial \mathbf{K}} \cdot \frac{d\mathbf{K}}{d\tau} + \frac{\partial F}{\partial \mathbf{X}} \cdot \frac{d\mathbf{X}}{d\tau} + \mathbf{\Lambda}(\tau)^\dagger \cdot \frac{d\mathbf{X}}{d\tau} = 0 \qquad (86)$$

Instead of (76), we now have

$$
\begin{aligned}
\frac{d\mathbf{X}}{d\tau} &= \lambda(\tau)\frac{\partial F}{\partial \mathbf{K}} \\
\frac{d\mathbf{K}}{d\tau} &= -\lambda(\tau)\frac{\partial F}{\partial \mathbf{X}} - \mathbf{\Lambda}(\tau) \\
\mathbf{\Lambda}(\tau) &= \lambda(\tau)\mathbf{\Lambda}(\tau)^{\dagger}
\end{aligned} \tag{87a}
$$

and by substitution it is verified that (87a) satisfies (86). It is noted that (87a) like (76) satisfies (77), hence the phase is uniquely determined independently of the ray path, according to (72b). Finally, we have to define the four-vector $\mathbf{\Lambda}(\tau)$. If we take it as a four-vector $\mathbf{\Lambda} = \Re\mathbf{\Lambda} + \Im\mathbf{\Lambda}$, where the symbols $\Re, \Im$ indicate that the spatial components are real and the temporal component is imaginary, the spatial components are imaginary and the temporal component is real, respectively. The four-vector $\mathbf{\Lambda}$ is chosen such that $\Re\mathbf{\Lambda} = 0$. Consequently, $\mathbf{V} = d\mathbf{X}/d\tau$ involves real $\mathbf{x}$, t everywhere along the ray path. To find $\mathbf{\Lambda}$, expand

$$
\Im\frac{d}{d\tau}\frac{\partial F}{\partial \mathbf{K}} = 0 = \Im\left(\frac{d\mathbf{K}}{d\tau}\cdot\frac{\partial}{\partial \mathbf{K}}\frac{d\mathbf{K}}{d\tau} + \frac{d\mathbf{K}}{d\tau}\cdot\frac{\partial}{\partial \mathbf{X}}\frac{\partial F}{\partial \mathbf{K}}\right) \tag{88}
$$

and substitute from (87a), this finally yields the value of $\mathbf{\Lambda}$ on the ray in terms of known functions. Rewriting (88) in three-dimensional quantities and using the same technique yields instead of (87a) the analog of (79a)

$$
\begin{aligned}
\mathbf{v}_g &= \frac{d\mathbf{x}}{dt} = -\frac{\partial F/\partial \mathbf{k}}{\partial F/\partial \omega}, \\
\frac{d\mathbf{k}}{dt} &= \frac{\partial F/\partial \mathbf{x}}{\partial F/\partial \omega} + i\boldsymbol{\beta}, \\
\frac{d\omega}{dt} &= -\frac{\partial F/\partial t}{\partial F/\partial \omega} + i\mathbf{v}_g\cdot\boldsymbol{\beta}, \\
\boldsymbol{\beta} &= -\,\mathrm{Re}\left[\left(\frac{\partial \mathbf{v}_g}{\partial \mathbf{k}} + \frac{\partial \mathbf{v}_g}{\partial \omega}\mathbf{v}_g\right)\right]^{-1} \\
&\quad \cdot \mathrm{Im}\left(\frac{\partial \mathbf{v}_g}{\partial \mathbf{k}}\cdot\frac{\partial F/\partial \mathbf{x}}{\partial F/\partial \omega} - \frac{\partial \mathbf{v}_g}{\partial \omega}\frac{\partial F/\partial t}{\partial F/\partial \omega} + \frac{\partial \mathbf{v}_g}{\partial \mathbf{x}}\cdot\mathbf{v}_g + \frac{\partial \mathbf{v}_g}{\partial t}\right)
\end{aligned} \tag{87b}
$$

The ray tracing model (87a), (87b) for lossy media guarantees that the group velocity remains real, also that the ray paths are confined to real space and time and the appropriate dispersion equation is satisfied. This is achieved by choosing both **k** and ω as complex quantities. Inasmuch as the group velocity is analytic, it obeys the conventional relativistic transformation for velocities (40). It should be noticed that if the dispersion equation $F(\mathbf{K},\mathbf{X}) = 0$ and $\mathbf{v}_g(\mathbf{K},\mathbf{X})$ are analytic in all the components of $\mathbf{K},\mathbf{X}$, then $\boldsymbol{\beta}$ involves only derivatives of analytic functions, although by its definition $\boldsymbol{\beta}$ itself is nonanalytic (a real or imaginary part of an analytic function is not analytic, as we know). Given the definition (71) for the phase, we ask ourselves if the integration performed by using $\mathbf{K},\mathbf{X}$ obtained as a solution of (87a), (87b) yields an analytic result for $\theta(\mathbf{X})$. If not, then the prescription for $\mathbf{K} = \partial_{\mathbf{X}}\theta(\mathbf{X})$ involves differentiations of a non-analytic function and becomes useless, and therefore raises the question whether the model (87a), (87b) is valid at all. To answer this question, we start with the dispersion equation (74a) which presumably is analytic. It follows that its derivatives appearing in (75) are analytic too. Inasmuch as $dF/d\tau$, $\partial F/\partial\mathbf{X}$ and $d\mathbf{X}/d\tau$ are analytic, it follows that $\partial F/\partial\mathbf{K} \cdot d\mathbf{K}/d\tau$ is analytic too. Using the first equation (76), which applies also to the model (87a), (87b) implies that $d\mathbf{X}/d\tau \cdot d\mathbf{K}/d\tau$ is analytic along the ray. The increment $d\mathbf{K}$ is arbitrary hence $\mathbf{K} \cdot d\mathbf{X}/d\tau$ is analytic, and from the definition of the phase (71) it follows that the phase is analytic.

4.13 Concluding Remarks

Relativistic electrodynamics is now a tangibly needed subject in the education of electromagnetic radiation engineers, and persons that will reach this subject who started their education in the physics department, discovered that they are more application-oriented and drifted towards modern electromagnetic theory and applications. The experience of the present author is dictating a pedagogical approach which is very unorthodox from the point of view of physicists, whose way of presenting the subject also percolated into the electrical engineering electromagnetic theory textbooks. It is suggested that the rudiments be stipulated axiomatically, and the implications and conclusions of relativistic electrodynamics for applications be introduced by minimizing the mathematical machinery to the absolutely necessary minimum. It has been found that four-vectors and dyadics (i.e., matrices) is practically all the mathematical equipment needed (of course, previous courses in electromagnetic field theory are assumed). The various applications and examples given here are of course optional. It is expected that educators will be biassed by their own interest in relevant subjects.

Acknowledgments

The ideas expressed here, regarding Hamiltonian ray propagation, especially in lossy media, have been profoundly influenced by two distinguished scientists: Julius J. Brandstatter, author of the excellent book "An introduction to waves, rays and radiation in plasma media", McGraw-Hill, 1963, with whom I had the privilege to work in Tel Aviv University in the years 1973-1975, and Kurt Suchy, Institute of Physics of the University of Düsseldorf, with whom I visited in 1975.

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