1

ON THE RADIATION EFFICIENCY AND THE ELECTROMAGNETIC FIELD OF A VERTICAL ELECTRIC DIPOLE IN THE AIR ABOVE A DIELECTRIC OR CONDUCTING HALF-SPACE

R. W. P. King

1.1 Introduction

When a vertical dipole is located in air at a height d over a dielectric or conducting earth (Fig. 1.1), the oscillating current in the dipole generates an electromagnetic field that travels outward in the air and across the boundary into the earth. The determination of the field and the power in each region as a function of the height d of the dipole involves not only direct, reflected, and refracted or transmitted waves, but also a surface wave that travels along the boundary in the air. When the dipole is in the earth, the surface wave is known as a

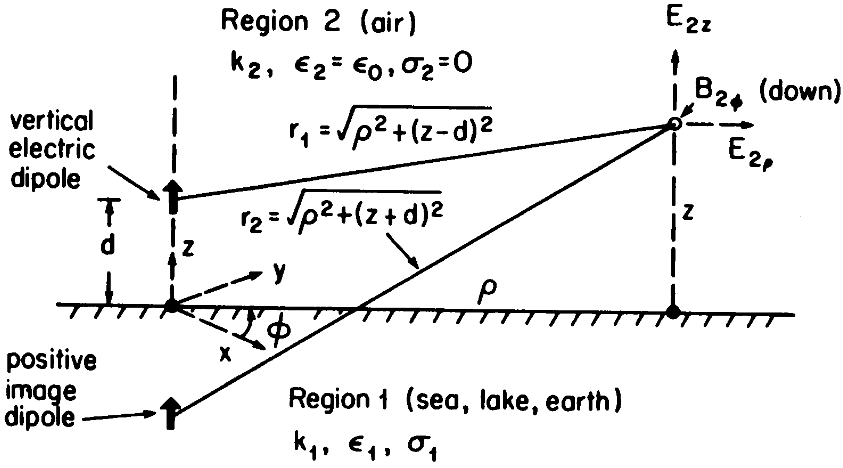


Figure 1.1 Vertical electric dipole at height d in Region 2. Field calculated at (ρ, z).

lateral wave; when the dipole is in the air, it is often called the Norton surface wave.

When the dipole is at the height d in air (Region 2, $z \geq 0$) over the earth (Region 1, $z \leq 0$), the components of the electromagnetic field are [1,2],

$$B_{2\phi}(\rho, z) = \frac{i\mu_0}{4\pi} \int_0^\infty \gamma_2^{-1} \left[e^{i\gamma_2|z-d|} + e^{i\gamma_2(z+d)} \left(1 - G_{21}\right) \right] \times J_1(\lambda\rho)\, \lambda^2\, d\lambda \tag{1}$$

$$E_{2\rho}(\rho, z) = \frac{i\omega\mu_0}{4\pi k_2^2} \int_0^\infty \left[\pm e^{i\gamma_2|z-d|} + e^{i\gamma_2(z+d)} \left(1 - G_{21}\right) \right] \times J_1(\lambda\rho)\, \lambda^2\, d\lambda; \quad \begin{cases} z > d \\ 0 \leq z \leq d \end{cases} \tag{2}$$

$$E_{2z}(\rho, z) = -\frac{\omega\mu_0}{4\pi k_2^2} \int_0^\infty \gamma_2^{-1} \left[e^{i\gamma_2|z-d|} + e^{i\gamma_2(z+d)}(1 - G_{21}) \right] \times J_0(\lambda\rho)\lambda^3\, d\lambda \tag{3}$$

where, with the time dependence $e^{-i\omega t}$,

$$k_1 = \beta_1 + i\alpha_1 = \omega\mu_0^{1/2}(\epsilon_1 + i\sigma_1/\omega)^{1/2}; \quad k_2 = \omega\mu_0^{1/2}\epsilon_0^{1/2} \tag{4}$$

$$G_{21} = 2k_2^2\gamma_1 N^{-1}; \quad N = k_1^2\gamma_2 + k_2^2\gamma_1;$$

$$\gamma_j = (k_j^2 - \lambda^2)^{1/2}, \quad j = 1, 2 \tag{5}$$

In each formula, the first integral with the factor $e^{i\gamma_2|z-d|}$ is the direct wave, the second integral with the factor $e^{i\gamma_2(z+d)}$ is the reflected or image field, and the third integral with the factor $G_{21}e^{i\gamma_2(z+d)}$ is the surface wave.

The power P radiated by the dipole was separated into two parts by Hansen [3]: a part P_a that remains in the upper half-space — air, and a part P_g that is transferred into the lower half-space — earth. These are defined in terms of the vertical component of the Poynting vector, viz.,

$$S_{2z}(\rho, z) = \frac{1}{2\mu_0} E_{2\rho}(\rho, z) B_{2\phi}^*(\rho, z) \tag{6}$$

where the asterisk denotes the complex conjugate. Thus,

$$\left.\begin{matrix} P_a \\ P_g \end{matrix}\right\} = \pm \operatorname{Re} 2\pi \int_0^\infty S_{2z}(\rho, z_\pm)\rho\, d\rho \tag{7}$$

In (7), z_+ means $z > d$, z_- means $0 \le z \le d$. The components of the electromagnetic field in (6) were evaluated by Hansen [3] with the help of the dyadic Green's function. They are more simply given by (1) and (2).

Hansen introduced the division of power given by (7) in order to define the *radiation efficiency* as follows:

$$\eta = \frac{P_a}{P_a + P_g} \tag{8}$$

This is the fraction of the power radiated by the dipole that remains in the upper half-space. Since it appears superficially obvious that radio transmission between antennas in the air depends on the power that remains in the upper half-space, it seems reasonable to assume that the most desirable antenna is one that has the highest radiation efficiency. Actually, this ignores the important fact that power ultimately transferred to the earth at large radial distances is available in the air at all shorter distances.

Hansen's numerical evaluation of the radiation efficiency of the vertical electric dipole (among others) indicated that P_g grows without limit as the height d of the dipole is reduced to zero. This means that P_a and the radiation efficiency both vanish as $d \to 0$. Hansen's calculations verify this conclusion for all values of $\sigma_1/\omega\epsilon_1$ in the wide range $0.0001 \leq \sigma_1/\omega\epsilon_1 \leq 1000$.

One purpose of this paper is to derive analytical formulas for P_a and P_g in order to study the limit as $d \to 0$ and determine what actually happens. Since for this purpose it is unimportant whether the lower half-space — earth — is a perfect dielectric or a conducting region, the analytically simpler case of a perfect dielectric is investigated.

1.2 Derivation of Formulas for P_a and P_g

When (1) and (2) are substituted in (6), the following result is obtained:

$$S_{2z}(\rho, z_\pm) = \frac{\omega\mu_0}{32\pi^2 k_2^2} \int_0^\infty f_\rho(\gamma_2, z_\pm) J_1(\lambda\rho)\lambda^2\, d\lambda \times \int_0^\infty f_\phi^*(\gamma_2^*, z_\pm) J_1(l\rho) l^2\, dl \tag{9}$$

For $z = z_+ > d$,

$$f_\rho(\gamma_2, z_+) = e^{i\gamma_2(z-d)} + (1 - G_{21})\, e^{i\gamma_2(z+d)} \tag{10}$$

$$f_\phi^*(\gamma_2^*, z_+) = \frac{1}{\gamma_2^*}\left[e^{-i\gamma_2^*(z-d)} + (1 - G_{21}^*)\, e^{-i\gamma_2^*(z+d)}\right] \tag{11}$$

For $z = z_-$, $0 \leq z \leq d$,

$$f_\rho(\gamma_2, z_-) = -e^{i\gamma_2(d-z)} + (1 - G_{21})\, e^{i\gamma_2(d+z)} \tag{12}$$

$$f_\phi^*(\gamma_2^*, z_-) = \frac{1}{\gamma_2^*}\left[e^{-i\gamma_2^*(d-z)} + (1 - G_{21}^*)\, e^{-i\gamma_2^*(d+z)}\right] \tag{13}$$

Note that $\gamma_2 = (k_2^2 - \lambda^2)^{1/2}$ when $0 \leq \lambda \leq k_2$, $\gamma_2 = i(\lambda^2 - k_2^2)^{1/2} = i\nu_2$ when $k_2 < \lambda$. In this latter range, $\gamma_2^* = -i\nu_2$, $G_{21}^* = 2k_2^2\gamma_1^*/N^*$, $\gamma_1^* = (k_1^{*2} - \lambda^2)^{1/2}$.

When (9) is substituted into (7), this becomes

$$\left.\begin{matrix} P_a \\ P_g \end{matrix}\right\} = \pm \operatorname{Re} \frac{\omega\mu_0}{16\pi k_2^2} \int_0^\infty d\lambda\, \lambda^2 \int_0^\infty dl\, l^2 \int_0^\infty d\rho\, \rho \\ \times \left[J_1(\lambda\rho)J_1(l\rho)\right] f_\rho(\gamma_2, z_\pm) f_\phi^*(\gamma_2^*, z_\pm) \tag{14}$$

Here the integration with respect to ρ can be carried out with the use of Formula (34) on p. 92 of Bateman [4], Vol. 2, with $\nu = 1$ and $\mu = 0$. The relation is

$$\int_0^\infty J_0(at)J_1(bt)\,dt = \frac{1}{b}\,U(b-a) = \begin{cases} 1/b, & b > a \\ 0, & b < a \end{cases} \tag{15}$$

where $U(b-a)$ is the step function. Differentiation of (15) with respect to a — noting that $(d/da)J_0(at) = -tJ_1(at)$ — gives

$$-\frac{d}{da}\int_0^\infty J_0(at)J_1(bt)\,dt = \int_0^\infty J_1(at)J_1(bt)t\,dt \\ = -\frac{1}{b}\frac{d}{da}U(b-a) \tag{16}$$

Since $-d/da = d/db$, (16) leads to the following orthogonality relation:*

$$\int_0^\infty J_1(at)J_1(bt)t\,dt = \frac{1}{b}\,\delta(b-a) \tag{17}$$

where $\delta(b-a)$ is the delta function.

With (17), (14) becomes

$$\left.\begin{matrix} P_a \\ P_g \end{matrix}\right\} = \pm \operatorname{Re} \frac{\omega\mu_0}{16\pi k_2^2} \int_0^\infty f_\rho(\gamma_2, z_\pm) f_\phi^*(\gamma_2^*, z_\pm)\lambda^3\,d\lambda \tag{18}$$

where $f_\rho(\gamma_2, z_\pm)$ and $f_\phi^*(\gamma_2^*, z_\pm)$ are defined in (10)–(13). The formula (18) is precisely (2) in Hansen [3] but with the dipole moment $m = Ih_e = 1$ Ampere · meter.

---

* A more rigorous derivation of this formula makes use of the Cartesian components of the field. The cylindrical components are used in order to parallel the work of Hansen [3].

The power radiated upward with $z = z_+$ involves the following product:

$$f_\rho(\gamma_2, z_+) f_\phi^*(\gamma_2^*, z_+) = \frac{1}{\gamma_2^*} \left[e^{i\gamma_2(z-d)} + (1 - G_{21})\, e^{i\gamma_2(z+d)} \right] \times \left[e^{-i\gamma_2^*(z-d)} + (1 - G_{21}^*)\, e^{-i\gamma_2^*(z+d)} \right] \quad (19)$$

In the range $0 \le \lambda \le k_2$, $\gamma_2^* = \gamma_2$ is real so that

$$\begin{aligned} f_\rho(\gamma_2, z_+) f_\phi^*(\gamma_2^*, z_+) &= \gamma_2^{-1} \Big[1 + (1 - G_{21})(1 - G_{21}^*) \\ &\qquad + (1 - G_{21}) e^{2i\gamma_2 d} + (1 - G_{21}^*) e^{-2i\gamma_2 d} \Big] \\ &= \gamma_2^{-1} \Big[2 - (G_{21} + G_{21}^*) + G_{21} G_{21}^* \\ &\qquad + 2\cos 2\gamma_2 d - \Big(G_{21} e^{2i\gamma_2 d} + G_{21}^* e^{-2i\gamma_2 d} \Big) \Big] \end{aligned} \quad (20)$$

In the range $\lambda > k_2$, $\gamma_2 = i\nu_2$, $\gamma_2^* = -i\nu_2$, so that

$$\begin{aligned} f_\rho(\gamma_2, z_+) f_\phi^*(\gamma_2^*, z_+) &= \frac{i}{\nu_2} \left[e^{-\nu_2(z-d)} + (1 - G_{21}) e^{-\nu_2(z+d)} \right] \\ &\qquad \times \left[e^{-\nu_2(z-d)} + (1 - G_{21}^*) e^{-\nu_2(z+d)} \right] \\ &= \frac{i}{\nu_2} \Big[e^{-2\nu_2(z-d)} + (1 - G_{21})(1 - G_{21}^*)\, e^{-2\nu_2(z+d)} \\ &\qquad + (1 - G_{21}) e^{-2\nu_2 z} + (1 - G_{21}^*)\, e^{-2\nu_2 z} \Big] \\ &= \frac{i e^{-2\nu_2 z}}{\nu_2} \Big[e^{2\nu_2 d} + (1 - G_{21})(1 - G_{21}^*) e^{-2\nu_2 d} \\ &\qquad + 2 - (G_{21} + G_{21}^*) \Big] \\ &= \frac{i e^{-2\nu_2 z}}{\nu_2} \Big[2(1 + \cosh 2\nu_2 d) \\ &\qquad + (G_{21} G_{21}^* - G_{21} - G_{21}^*) e^{-2\nu_2 d} - (G_{21} + G_{21}^*) \Big] \end{aligned} \quad (21)$$

When (20) and (21) are substituted into (18) with the upper signs, the result is

$$P_a = \operatorname{Re} \frac{\omega\mu_0}{16\pi k_2^2} \left\{ \int_0^{k_2} \gamma_2^{-1} \Big[2(1+\cos 2\gamma_2 d) + G_{21}G_{21}^* - G_{21} - G_{21}^* - G_{21}e^{2i\gamma_2 d} - G_{21}^* e^{-2i\gamma_2 d} \Big] \lambda^3 \, d\lambda + i \int_{k_2}^{\infty} \frac{e^{-2\nu_2 z}}{\nu_2} \Big[2(1+\cosh 2\nu_2 d) + (G_{21}G_{21}^* - G_{21} - G_{21}^*)\, e^{-2\nu_2 d} - (G_{21} + G_{21}^*) \Big] \lambda^3 \, d\lambda \right\} \tag{22}$$

In this expression, the two terms in square brackets are real so that the entire second integral is a pure imaginary and can be omitted. This leaves

$$P_a = \operatorname{Re} \frac{\omega\mu_0}{16\pi k_2^2} \int_0^{k_2} \gamma_2^{-1} [4\cos^2 \gamma_2 d + |G_{21}|^2 - 2G_{21} - 2G_{21}e^{2i\gamma_2 d}] \lambda^3 \, d\lambda \tag{23}$$

This follows since $\operatorname{Re} G_{21} = \operatorname{Re} G_{21}^*$, $\operatorname{Re} G_{21}e^{2i\gamma_2 d} = \operatorname{Re} G_{21}^* e^{-2i\gamma_2 d}$.

The power transferred into the lower half-space involves the following product:

$$f_\rho(\gamma_2, z_-) f_\phi^*(\gamma_2^*, z_-) = \frac{1}{\gamma_2^*} \Big[-e^{i\gamma_2(d-z)} + (1-G_{21})e^{i\gamma_2(d+z)} \Big] \times \Big[e^{-i\gamma_2^*(d-z)} + (1-G_{21}^*)e^{-i\gamma_2^*(d+z)} \Big] \tag{24}$$

In the range $0 \le \lambda \le k_2$, $\gamma_2^* = \gamma_2$ is real so that

$$\begin{aligned} f_\rho(\gamma_2, z_-) f_\phi^*(\gamma_2^*, z_-) &= \gamma_2^{-1} \Big[-1 + (1-G_{21})(1-G_{21}^*) + (1-G_{21})e^{2i\gamma_2 z} - (1-G_{21}^*)e^{-2i\gamma_2 z} \Big] \\ &= \gamma_2^{-1} \Big[G_{21}G_{21}^* - G_{21} - G_{21}^* - G_{21}e^{2i\gamma_2 z} + G_{21}^* e^{-2i\gamma_2 z} + 2i \sin 2\gamma_2 z \Big] \end{aligned} \tag{25}$$

In the range $\lambda > k_2$, $\gamma_2 = i\nu_2$, $\gamma_2^* = -i\nu_2$, and

$$\begin{aligned} f_\rho(\gamma_2, z_-) f_\phi^*(\gamma_2^*, z_-) &= \frac{i}{\nu_2}\left[-e^{-\nu_2(d-z)} + (1 - G_{21})e^{-\nu_2(d+z)}\right] \\ &\quad \times \left[e^{-\nu_2(d-z)} + (1 - G_{21}^*)e^{-\nu_2(d+z)}\right] \\ &= \frac{i}{\nu_2} e^{-2\nu_2 d}\Big[-e^{2\nu_2 z} + (1 - G_{21})(1 - G_{21}^*)e^{-2\nu_2 z} \\ &\qquad + (1 - G_{21}) - (1 - G_{21}^*)\Big] \\ &= \frac{i}{\nu_2} e^{-2\nu_2 d}\Big[-2\sinh 2\nu_2 z + e^{-2\nu_2 z}(G_{21}G_{21}^* \\ &\qquad - G_{21} - G_{21}^*) - G_{21} + G_{21}^*\Big] \end{aligned} \tag{26}$$

When (25) and (26) are substituted into (18) with the lower signs, the result is

$$\begin{aligned} P_g = -\operatorname{Re}\frac{\omega\mu_0}{16\pi k_2^2}\Bigg\{&\int_0^{k_2} \gamma_2^{-1}\Big[G_{21}G_{21}^* - G_{21} - G_{21}^* - G_{21}e^{2i\gamma_2 z} \\ &\quad + G_{21}^* e^{-2i\gamma_2 z} + 2i\sin 2\gamma_2 z\Big]\lambda^3\, d\lambda \\ &+ i\int_{k_2}^{\infty} \frac{e^{-2\nu_2 d}}{\nu_2}\Big[-2\sinh 2\nu_2 z + e^{-2\nu_2 z}(G_{21}G_{21}^* \\ &\quad - G_{21} - G_{21}^*) - G_{21} + G_{21}^*\Big]\lambda^3\, d\lambda\Bigg\} \end{aligned} \tag{27}$$

Here $G_{21}e^{2i\gamma_2 z} - G_{21}^* e^{-2i\gamma_2 z}$ and $G_{21} - G_{21}^*$ are pure imaginaries. Also, $G_{21} - G_{21}^* = 2i\operatorname{Im} G_{21}$. It follows that

$$\begin{aligned} P_g = -\frac{\omega\mu_0}{16\pi k_2^2}\Bigg\{&\operatorname{Re}\int_0^{k_2} \gamma_2^{-1}\Big[|G_{21}|^2 - 2G_{21}\Big]\lambda^3\, d\lambda \\ &+ 2\operatorname{Im}\int_{k_2}^{\infty} \nu_2^{-1} G_{21} e^{-2\nu_2 d}\lambda^3\, d\lambda\Bigg\} \end{aligned} \tag{28}$$

Note that both P_a and P_g depend on the height d of the dipole but are independent of the location z of the planes of integration within the ranges $0 \le z_- \le d$ and $z_+ > d$.

The total power radiated is

$$P_a + P_g = \frac{\omega\mu_0}{16\pi k_2^2}\left\{\mathrm{Re}\int_0^{k_2} \gamma_2^{-1}\left[4\cos^2\gamma_2 d - 2G_{21}e^{2i\gamma_2 d}\right]\lambda^3\,d\lambda \right.$$
$$\left. - 2\,\mathrm{Im}\int_{k_2}^{\infty} \nu_2^{-1} G_{21} e^{-2\nu_2 d}\lambda^3\,d\lambda\right\} \tag{29}$$

1.3 Special Cases: Homogenous Region; Perfectly Conducting Ground

Two special cases are of interest. The first is the homogeneous region with $k_1 = k_2$. The upward-directed power with $z > d$ now involves the functions

$$f_\rho(\gamma_2, z_+) = e^{i\gamma_2(z-d)}, \qquad f_\phi^*(\gamma_2^*, z_+) = \frac{1}{\gamma_2^*}\,e^{-i\gamma_2^*(z-d)} \tag{30}$$

so that

$$P_+ = \mathrm{Re}\,\frac{\omega\mu_0}{16\pi k_2^2}\int_0^\infty \frac{1}{\gamma_2^*}\,e^{i\gamma_2(z-d)}e^{-i\gamma_2^*(z-d)}\lambda^3\,d\lambda \tag{31}$$

In the range $0 \le \lambda \le k_2$, $\gamma_2^* = \gamma_2$ is real; in the range $\lambda > k_2$, $\gamma_2 = i(\lambda^2 - k_2^2)^{1/2} = i\nu_2$, $\gamma_2^* = -i\nu_2$. It follows that

$$\begin{aligned} P_+ &= \mathrm{Re}\,\frac{\omega\mu_0}{16\pi k_2^2}\left\{\int_0^{k_2}\frac{\lambda^3\,d\lambda}{(k_2^2-\lambda^2)^{1/2}} + ie^{-2\nu_2(z-d)}\int_{k_2}^\infty \frac{\lambda^3\,d\lambda}{(\lambda^2-k_2^2)^{1/2}}\right\} \\ &= \mathrm{Re}\,\frac{\omega\mu_0}{16\pi k_2^2}\left\{\frac{(k_2^2-\lambda^2)^{3/2}}{3} - k_2^2(k_2^2-\lambda^2)^{1/2}\right\}_0^{k_2} \\ &= \frac{\omega\mu_0 k_2}{24\pi} \end{aligned} \tag{32}$$

Note that the second integral above is real so that, when multiplied by i, it is a pure imaginary that contributes nothing to the real part.

The downward-directed power involves the functions

$$f_\rho(\gamma_2, z_-) = -e^{i\gamma_2(d-z)}, \qquad f_\phi^*(\gamma_2^*, z_-) = \frac{1}{\gamma_2^*}\,e^{-i\gamma_2^*(d-z)} \tag{33}$$

so that

$$P_{-} = \mathrm{Re}\,\frac{\omega\mu_0}{16\pi k_2^2}\int_0^\infty \frac{1}{\gamma_2^*}\, e^{i\gamma_2(d-z)}e^{-i\gamma_2^*(d-z)}\lambda^3\,d\lambda = \frac{\omega\mu_0 k_2}{24\pi} \tag{34}$$

The total radiated power for a dipole with the electric moment $2Ih_e$, where h_e is the effective half-length, is

$$P_0 = 10k_2^2 \times 4I^2h_e^2 = 40k_2^2h_e^2I^2 \tag{35}$$

The well-known formula for the radiation resistance is

$$R^r = \frac{2P_0}{I^2} = 80k_2^2h_e^2 \tag{36}$$

The second special case is the dipole over a perfectly conducting half-space or a dielectric half-space with infinite permittivity, so that $k_1 \to \infty$. This means that

$$G_{21} = \frac{2k_2^2\gamma_1}{k_1^2\gamma_2 + k_2^2\gamma_1} \to 0$$

so that, from (28), $P_g = 0$ and, from (23),

$$P_t = P_a = \frac{\omega\mu_0}{4\pi k_2^2}\,\mathrm{Re}\int_0^{k_2} \frac{\cos^2\gamma_2 d}{\gamma_2}\,\lambda^3\,d\lambda \tag{37}$$

This is readily integrated with the change of variable,

$$u = \gamma_2 d = d(k_2^2-\lambda^2)^{1/2}, \qquad u^2 = d^2(k_2^2-\lambda^2), \qquad u\,du = -d^2\lambda\,d\lambda$$
$$\lambda^2 = k_2^2 - \frac{u^2}{d^2}; \qquad \text{when } \lambda = 0,\ u = k_2 d; \qquad \text{when } \lambda = k_2,\ u = 0$$

Hence,

$$\begin{aligned} P_a &= \frac{\omega\mu_0}{4\pi k_2^2}\,\mathrm{Re}\int_0^{k_2 d} \frac{\cos^2 u}{(u/d)}\left(k_2^2 - \frac{u^2}{d^2}\right)\frac{u}{d^2}\,du \\ &= \frac{\omega\mu_0 k_2}{12\pi}\left[1 + 3\,\frac{\sin 2k_2 d - 2k_2 d\cos 2k_2 d}{(2k_2 d)^3}\right] \end{aligned} \tag{38}$$

When the dipole is on the boundary so that $d = 0$, it constitutes a monopole with the electric moment Ih_e on a perfectly conducting half-space. The power radiated by the monopole is

$$P_m = \frac{\omega\mu_0 k_2}{12\pi} I^2 h_e^2 \times 2 = \frac{\omega\mu_0 k_2}{6\pi} I^2 h_e^2 = 20 k_2^2 h_e^2 I^2 \tag{39}$$

[Note that as $k_2 d \to 0$, $\sin 2k_2 d \to 2k_2 d - (2k_2 d)^3/6$; $\cos 2k_2 d \to 1 - (2k_2 d)^2/2$.]

The well-known formula for the radiation resistance is

$$R^r = \frac{2P_m}{I^2} = 40 k_2^2 h_e^2 \tag{40}$$

1.4 Evaluation of the Normalized Powers in the Air and Earth

Paralleling the formulation of Hansen [3], the powers in the air and earth are normalized with respect to the power radiated by the isolated dipole with the same electric moment. The normalized powers are

$$p_a = \frac{P_a}{P_0} = \frac{3}{4k_2^3} \operatorname{Re} \int_0^{k_2} \gamma_2^{-1} \Big[4\cos^2 \gamma_2 d + |G_{21}|^2 - 2G_{21} - 2G_{21} e^{2i\gamma_2 d} \Big] \lambda^3 \, d\lambda \tag{41}$$

$$p_g = \frac{P_g}{P_0} = -\frac{3}{4k_2^3} \left\{ \operatorname{Re} \int_0^{k_2} \gamma_2^{-1} \Big[|G_{21}|^2 - 2G_{21} \Big] \lambda^3 \, d\lambda + 2 \operatorname{Im} \int_{k_2}^{\infty} \nu_2^{-1} G_{21} e^{-2\nu_2 d} \lambda^3 \, d\lambda \right\} \tag{42}$$

In these formulas, $\gamma_2 = (k_2^2 - \lambda^2)^{1/2} = i\nu_2 = i(\lambda^2 - k_2^2)^{1/2}$; $G_{21} = 2k_2^2 \gamma_1 N^{-1}$, $N = k_1^2 \gamma_2 + k_2^2 \gamma_1$. In the range, $k_2 \le \lambda \le \infty$, $N = ik_1^2 \nu_2 + k_2^2 \gamma_1$. These formulas can be expressed as follows:

$$p_a = 1 + 3 \left[\frac{\sin 2k_2 d - 2k_2 d \cos 2k_2 d}{(2k_2 d)^3} \right] + J_{1a} + J_{2a} + J_{3a} \tag{43}$$

with

$$J_{1a} = -\operatorname{Re}\frac{3}{k_2}\int_0^{k_2}\frac{\gamma_1}{\gamma_2}\cdot\frac{1}{N}\lambda^3\,d\lambda \tag{44}$$

$$J_{2a} = \operatorname{Re} 3k_2\int_0^{k_2}\frac{\gamma_1^2}{\gamma_2}\cdot\frac{1}{|N^2|}\lambda^3\,d\lambda \tag{45}$$

$$J_{3a} = -\operatorname{Re}\frac{3}{k_2}\int_0^{k_2}\frac{\gamma_1}{\gamma_2}\cdot\frac{1}{N}e^{2i\gamma_2 d}\lambda^3\,d\lambda \tag{46}$$

$$p_g = J_{1g} + J_{2g} + J_{3g} = -(J_{1a} + J_{2a}) + J_{3g} \tag{47}$$

with

$$J_{3g} = -\operatorname{Im}\frac{3}{k_2}\int_{k_2}^{\infty}\frac{\gamma_1}{\nu_2}\cdot\frac{1}{N}e^{-2\nu_2 d}\lambda^3\,d\lambda \tag{48}$$

and

$$p_t = p_a + p_g = 1 + 3\left[\frac{\sin 2k_2d - 2k_2d\cos 2k_2d}{(2k_2d)^3}\right] + J_{3a} + J_{3g} \tag{49}$$

The integrals involved in (43)–(49) are evaluated in the appendices, subject to the condition $k_1^2 \gg k_2^2$. With

$$\operatorname{Si} x = \int_0^x \frac{\sin u}{u}\,du \quad \text{and} \quad \operatorname{Cin} x = \int_0^x \frac{1-\cos u}{u}\,du$$

the results are

$$\begin{aligned}
p_a = 1 &+ 3\left[\frac{\sin 2k_2d - 2k_2d\cos 2k_2d}{(2k_2d)^3}\right] \\
&- \frac{3k_2}{k_1}\left\{\left(1 - \frac{3k_2^2}{k_1^2}\right)\ln\left(1 + \frac{k_1}{k_2}\right) - \frac{3}{2} + \frac{3k_2}{k_1}\right. \\
&+ \left(1 - \frac{k_2^2}{k_1^2}\right)\cos\frac{2k_2^2d}{k_1} \\
&\times\left[\ln\left(1 + \frac{k_1}{k_2}\right) - \operatorname{Cin} 2k_2d\left(1 + \frac{k_2}{k_1}\right) + \operatorname{Cin}\frac{2k_2^2d}{k_1}\right] \\
&+ \left(1 - \frac{k_2^2}{k_1^2}\right)\sin\frac{2k_2^2d}{k_1}\left[\operatorname{Si} 2k_2d\left(1 + \frac{k_2}{k_1}\right) - \operatorname{Si}\frac{2k_2^2d}{k_1}\right] \\
&\left.+ \frac{1 - \cos 2k_2d}{4k_2^2d^2} - \left(1 - \frac{k_2}{k_1}\right)\frac{\sin 2k_2d}{2k_2d}\right\}
\end{aligned} \tag{50}$$

$$p_g = \frac{3k_2}{k_1}\left\{\left(1-\frac{3k_2^2}{k_1^2}\right)\ln\left(1+\frac{k_1}{k_2}\right)-\frac{3}{2}+\frac{3k_2}{k_1}\right.$$
$$-\cos\frac{2k_2^2 d}{k_1}\left(\gamma+\ln\frac{2k_2^2 d}{k_1}-\operatorname{Cin}\frac{2k_2^2 d}{k_1}\right)$$
$$-\sin\frac{2k_2^2 d}{k_1}\left(\operatorname{Si}\frac{2k_2^2 d}{k_1}-\frac{\pi}{2}\right)+\left(1-\cos\frac{2k_2^2 d}{k_1}\right)E_1(2k_1 d)$$
$$\left.+\gamma+\ln 2k_1 d-1+\ln 2-\frac{\pi}{2}\int_0^{2k_1 d}[I_1(z)-L_1(z)]\frac{dz}{z}\right\}$$
$$+\frac{k_1}{k_2}\left\{1-\frac{3\pi}{4k_1 d}\left[I_2(2k_1 d)-L_2(2k_1 d)\right]\right\} \tag{51}$$

Here I is a modified Bessel function, L is a Struve function. These are defined more explicitly in the appendices.

When the dipole is on the surface in the air, $d = 0$ and the above formulas reduce to

$$p_a = 2-\frac{6k_2}{k_1}\left\{\left(1-\frac{2k_2^2}{k_1^2}\right)\ln\left(1+\frac{k_1}{k_2}\right)-1+\frac{2k_2}{k_1}\right\} \tag{52}$$

$$p_g = \frac{3k_2}{k_1}\left\{\left(1-\frac{3k_2^2}{k_1^2}\right)\ln\left(1+\frac{k_1}{k_2}\right)+2\ln\frac{k_1}{k_2}-\frac{5}{2}+\ln 2+\frac{3k_2}{k_1}\right\}$$
$$+\frac{k_1}{k_2} \tag{53}$$

Note that p_a is not zero but a finite quantity and p_g is not infinite.

The following specific values of p_a, p_g and η with $d = 0$ are of interest: (a) Region 1 is fresh water with $k_1/k_2 = 9$ and $\omega\epsilon_1 \gg \sigma_1$: $p_a = 1.021, p_g = 10.71, \eta = 0.087$. (b) Region 1 is rock with $k_1/k_2 = 3$ and $\omega\epsilon_1 \gg \sigma_1$: $p_a = 0.510, p_g = 5.314, \eta = 0.088$. (c) Region 1 is dry earth with $k_1/k_2 = 2$ and $\omega\epsilon_1 \gg \sigma_1$: $p_a = 0.352, p_g = 4.031, \eta = 0.080$. Note that in this last case the condition $k_1^2 \gg k_2^2$ is not well satisfied so that the formulas are not as good approximations as for larger values of k_1/k_2.

Complete graphs of p_a, p_g and the radiation efficiency $\eta = p_a/(p_a + p_g)$ are shown in Fig. 1.2. In all cases, the frequency is assumed to be sufficiently high $(\omega\epsilon_1 \gg \sigma_1)$ so that Region 1 is a good dielectric with negligible conductivity. These results are in good agreement with the corresponding ones of Hansen [3] except near and at $d = 0$. The ratio η remains finite for all values of k_1/k_2 and does not vanish as indicated by Hansen [3].

It appears from (51) and (53) that $p_g \to \infty$ when $k_1 \to \infty$ due to the presence of the term multiplied by k_1/k_2. Actually this is not the case because the formulas (51) and (53) are not valid when $k_1 \to \infty$. This is evident from the following asymptotic expansions. In (51) one of the crucial terms is $L_2(2k_1d) - I_2(2k_1d)$. With formula (9.6.6) on p. 375 and (12.2.6) on p. 498 of Abramowitz and Stegun [5], it follows that

$$\begin{aligned} L_2(2k_1d) - I_2(2k_1d) &\sim \frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}\Gamma(m+\frac{1}{2})}{\Gamma(\frac{5}{2}-m)(k_1d)^{2m-1}} \\ &= \frac{k_1d}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}\Gamma(m+\frac{1}{2})}{\Gamma(\frac{5}{2}-m)(k_1^2d^2)^m} \\ &= \frac{k_1d}{\pi}\left\{-\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{2})} + \sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(m+\frac{3}{2})}{\Gamma(\frac{3}{2}-m)(k_1^2d^2)^{m+1}}\right\} \end{aligned} \tag{54a}$$

With $\Gamma(\frac{1}{2})/\Gamma(\frac{5}{2}) = \sqrt{\pi}/(3\sqrt{\pi}/4) = 4/3$, it follows that

$$\begin{aligned} &\frac{k_1}{k_2}\left\{1 - \frac{3\pi}{4k_1d}[I_2(2k_1d) - L_2(2k_1d)]\right\} \\ &\quad\sim \frac{k_1}{k_2}\left\{1 - 1 + \frac{3}{4}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(m+\frac{3}{2})}{\Gamma(\frac{3}{2}-m)(k_1^2d^2)^{m+1}}\right\} \\ &\quad= \frac{3k_2}{4k_1}\cdot\frac{1}{k_2^2d^2}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(m+\frac{3}{2})}{\Gamma(\frac{3}{2}-m)(k_1^2d^2)^m} \end{aligned} \tag{54b}$$

Clearly this *decreases* with increasing k_1/k_2 for any nonzero value of k_1d.

A large-argument approximation of the second crucial term in (51), viz.,

$$\frac{\pi}{2}\int_0^{2k_1d}\left[I_1(z)-L_1(z)\right]\frac{dz}{z}$$
$$=\frac{\pi}{2}\left\{\int_0^{2k_1d}\left[I_0(z)-L_0(z)\right]dz-\left[I_1(z)-L_1(z)\right]\right\}$$

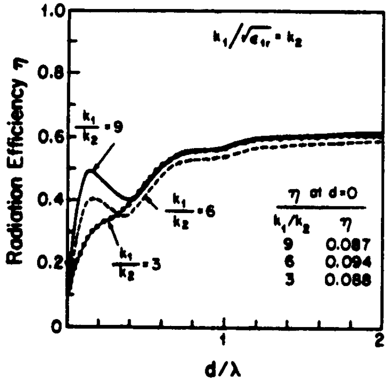
can also be obtained with formulas (12.2.6), (12.2.8) on p. 498 of Abramowitz and Stegun [5]. These give

$$\frac{\pi}{2}\int_0^{2k_1d}\left[I_1(z)-L_1(z)\right]\frac{dz}{z}$$
$$\sim \ln 4k_1d+\gamma-\sum_{m=1}^{\infty}\frac{(2m)!(2m-1)!}{(m!)^2(4k_1d)^{2m}}$$
$$+\frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}\Gamma(m+\frac{1}{2})}{\Gamma(\frac{3}{2}-m)(k_1d)^{2m}}$$
$$=\gamma+\ln 4k_1d-\sum_{m=1}^{\infty}\frac{(2m)!(2m-1)!}{(m!)^2(4k_1d)^{2m}}$$
$$-\frac{1}{2}\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})}+\frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(m+\frac{3}{2})}{\Gamma(\frac{1}{2}-m)(k_1^2d^2)^{m+1}}$$
$$=\gamma+\ln 2k_1d+\ln 2-1+\frac{1}{2k_1^2d^2}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(m+\frac{3}{2})}{\Gamma(\frac{1}{2}-m)(k_1^2d^2)^m}$$
$$-\sum_{m=1}^{\infty}\frac{(2m)!(2m-1)!}{(m!)^2(4k_1d)^{2m}} \tag{54c}$$

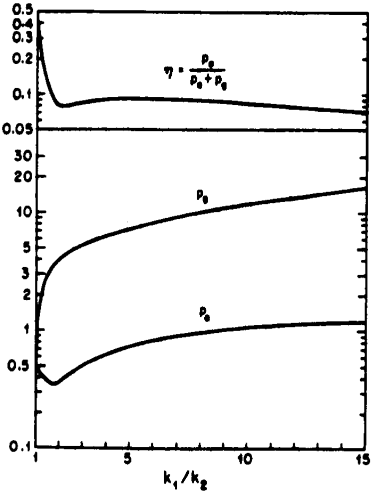
Here the only term that increases without limit as $k_1\to\infty$ is $\ln 2k_1d$. However, the entire expression is multiplied in (51) by k_2/k_1 and

$$\lim_{k_1\to\infty} 2k_2d\,\frac{\ln 2k_1d}{2k_1d}=2k_2d\lim_{x\to\infty}\frac{\ln x}{x}=2k_2d\lim_{x\to\infty}\frac{1}{x}=0 \tag{54d}$$

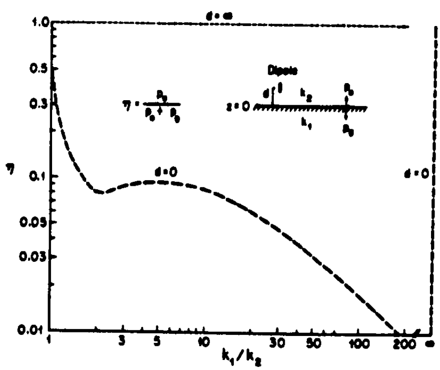
Evidently the critical quantity in the evaluation of the power into the earth is k_1d. The simultaneous limits $d\to 0$ and $k_1\to\infty$ are



(a)



(b)



(c)

Figure 1.2 (a) Radiation efficiency η of a vertical dipole at height d in air (k_2) over a dielectric half-space (k_1). (b) Normalized powers into the air (p_a) and earth (p_g) and radiation efficiency (η) of a vertical dipole on the surface of the dielectric (k_1) at $d = 0$ in the air (k_2). (c) Fraction of power η in upper half-space with dipole at $d = 0$ and $d = \infty$.

indeterminate. In (51), the limit $d \to 0$ leads to (53), but this is not valid with $k_1 \to \infty$ because the quantity $k_1 d$ has been set equal to zero.

The limit $k_1 \to \infty$ has been carried out in conjunction with (37), and the subsequent application of the limit $d \to 0$ led to (39).

The graphs in Fig. 1.2(a) show that the radiation efficiency of a vertical electric dipole, as defined in (8), drops to a small value as the height d of the dipole is reduced to zero. This means that most of the power radiated by the dipole is ultimately transferred into the earth. It will be shown that this is necessary for transmission to receivers on or near the surface of the earth and for the transmission and reception of low-angle radiation in over-the-horizon radar.

An analytical determination of the power transferred into the earth by a vertical dipole was carried out by Sommerfeld and Renner [6, eqs. (46) and (46a)] specifically for a dipole over sea water. Many approximations were made to obtain a solution of the order of k_2/k_1. When applied to a perfect dielectric, their formula is simply $p_g \sim 2\pi k_2/k_1$, which is independent of the height d of the dipole and is a completely inadequate approximation of (51) and (53). This raises serious questions about the accuracy of the formula when applied to sea water including the leading term, $1/(2k_1 d)^2$, which becomes infinite when $d \to 0$. Note that this is incompatible with the formulas (56)-(58) for the electromagnetic field in the air which are valid for complex k_1 and remain finite when $d \to 0$.

1.5 The Electromagnetic Field of the Vertical Electric Dipole in Air Over a Dielectric or Conducting Half-Space

The general integrals (1)–(3) for the three components of the electromagnetic field of a vertical electric dipole are difficult to interpret. Fortunately quite simple integrated formulas are available [1,2]. Subject to the one condition

$$|k_1| \geq 3k_2 \tag{55}$$

the following formulas have been derived [2]:

$$
\begin{aligned}
B_{2\phi}(\rho, z) = -\frac{\mu_0}{4\pi}\Bigg\{ & e^{ik_2 r_1}\left(\frac{\rho}{r_1}\right)\left(\frac{ik_2}{r_1} - \frac{1}{r_1^2}\right) \\
& + e^{ik_2 r_2}\left[\left(\frac{\rho}{r_2}\right)\left(\frac{ik_2}{r_2} - \frac{1}{r_2^2}\right)\right. \\
& \left.- \frac{2k_2^3}{k_1}\left(\frac{\pi}{k_2 r_2}\right)^{1/2} e^{-iP}\mathcal{F}(P)\right]\Bigg\}
\end{aligned}
\tag{56}
$$

$$
\begin{aligned}
E_{2\rho}(\rho, z) = -\frac{\omega\mu_0}{4\pi k_2}\Bigg\{ & e^{ik_2 r_1}\left(\frac{\rho}{r_1}\right)\left(\frac{z-d}{r_1}\right)\left(\frac{ik_2}{r_1} - \frac{3}{r_1^2} - \frac{3i}{k_2 r_1^3}\right) \\
& + e^{ik_2 r_2}\left(\frac{\rho}{r_2}\right)\left(\frac{z+d}{r_2}\right)\left(\frac{ik_2}{r_2} - \frac{3}{r_2^2} - \frac{3i}{k_2 r_2^3}\right) \\
& - \frac{2k_2}{k_1} e^{ik_2 r_2}\left[\left(\frac{\rho}{r_2}\right)\left(\frac{ik_2}{r_2} - \frac{1}{r_2^2}\right)\right. \\
& \left.- \frac{k_2^3}{k_1}\left(\frac{\pi}{k_2 r_2}\right)^{1/2} e^{-iP}\mathcal{F}(P)\right]\Bigg\}
\end{aligned}
\tag{57}
$$

$$
\begin{aligned}
E_{2z}(\rho, z) = \frac{\omega\mu_0}{4\pi k_2}\Bigg\{ & e^{ik_2 r_1}\left[\frac{ik_2}{r_1} - \frac{1}{r_1^2} - \frac{i}{k_2 r_1^3}\right. \\
& \left.- \left(\frac{z-d}{r_1}\right)^2\left(\frac{ik_2}{r_1} - \frac{3}{r_1^2} - \frac{3i}{k_2 r_1^3}\right)\right] \\
& + e^{ik_2 r_2}\left[\frac{ik_2}{r_2} - \frac{1}{r_2^2} - \frac{i}{k_2 r_2^3}\right. \\
& - \left(\frac{z+d}{r_2}\right)^2\left(\frac{ik_2}{r_2} - \frac{3}{r_2^2} - \frac{3i}{k_2 r_2^3}\right) \\
& \left.- \frac{2k_2^3}{k_1}\left(\frac{\pi}{k_2 r_2}\right)^{1/2}\left(\frac{\rho}{r_2}\right) e^{-iP}\mathcal{F}(P)\right]\Bigg\}
\end{aligned}
\tag{58}
$$

where

$$
r_1 = [\rho^2 + (z-d)^2]^{1/2}, \qquad r_2 = [\rho^2 + (z+d)^2]^{1/2} \tag{59}
$$

$$
P = \frac{k_2^3 r_2}{2k_1^2}\left[\frac{k_2 r_2 + k_1(z+d)}{k_2 \rho}\right]^2 \tag{60}
$$

$$\mathcal{F}(P) = \int_P^\infty \frac{e^{it}}{\sqrt{2\pi t}}\, dt = \frac{1}{2}(1+i) - C_2(P) - iS_2(P) \tag{61}$$

Here $C_2(P) + iS_2(P)$ is the Fresnel integral.

In (56)–(58), the part of the field with the factor $e^{ik_2r_1}$ represents the direct field, the part with the factor $e^{ik_2r_2}$ the reflected (image) field, and the part with the Fresnel integral the surface-wave field. The above formulas for the direct and image fields are exact and not subject to the condition (55); this applies only to the surface-wave terms.

Of particular interest is the field of the dipole when it is on or close to the boundary but still in the air. With $d \sim 0$, (56)–(58) become

$$B_{2\phi}(\rho, z) = -\frac{\mu_0}{2\pi} e^{ik_2r_0}\left[\left(\frac{\rho}{r_0}\right)\left(\frac{ik_2}{r_0} - \frac{1}{r_0^2}\right) - \frac{k_2^3}{k_1}\left(\frac{\pi}{k_2r_0}\right)^{1/2} e^{-iP_z}\mathcal{F}(P_z)\right] \tag{62}$$

$$E_{2\rho}(\rho, z) = -\frac{\omega\mu_0}{2\pi k_2} e^{ik_2r_0}\left\{\left(\frac{\rho z}{r_0^2}\right)\left(\frac{ik_2}{r_0} - \frac{3}{r_0^2} - \frac{3i}{k_2r_0^3}\right) - \frac{k_2}{k_1}\left[\left(\frac{\rho}{r_0}\right)\left(\frac{ik_2}{r_0} - \frac{1}{r_0^2}\right) - \frac{k_2^3}{k_1}\left(\frac{\pi}{k_2r_0}\right)^{1/2} e^{-iP_z}\mathcal{F}(P_z)\right]\right\} \tag{63}$$

$$E_{2z}(\rho, z) = \frac{\omega\mu_0}{2\pi k_2} e^{ik_2r_0}\left[\frac{ik_2}{r_0} - \frac{1}{r_0^2} - \frac{i}{k_2r_0^3} - \frac{z^2}{r_0^2}\left(\frac{ik_2}{r_0} - \frac{3}{r_0^2} - \frac{3i}{k_2r_0^3}\right) - \frac{k_2^3}{k_1}\left(\frac{\pi}{k_2r_0}\right)^{1/2}\left(\frac{\rho}{r_0}\right) e^{-iP_z}\mathcal{F}(P_z)\right] \tag{64}$$

where

$$P_z = \frac{k_2^3 r_0}{2k_1^2}\left(\frac{k_2r_0 + k_1z}{k_2\rho}\right)^2; \qquad r_0 = (\rho^2 + z^2)^{1/2} \tag{65}$$

Note that when the dipole is moved from $d \sim 0$ in the air to $d \sim 0$ in the earth, the field (62)–(64) is unchanged except that it is multiplied by the small factor k_2^2/k_1^2.

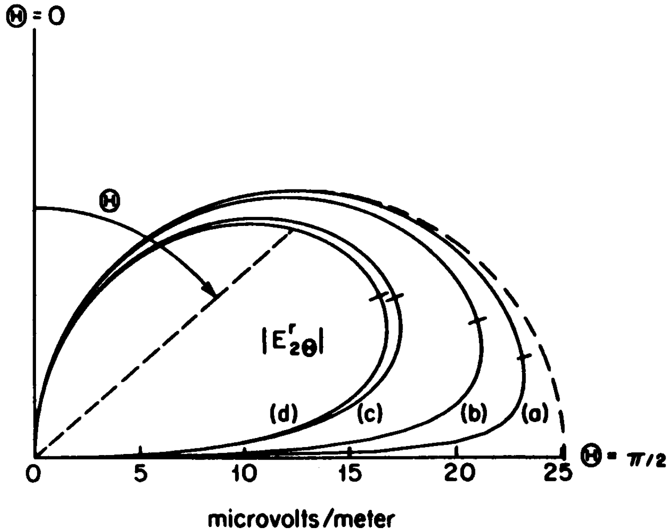


Figure 1.3 Complete field of $|E^r_{2\Theta}(r_0, \Theta)|$ for vertical dipole in air on boundary between air and (a) sea water, (b) wet earth, (c) dry earth, and (d) lake water. Frequency $f = 10$ MHz, radial distance $r_0 = 500$ km. The dashed curve is for $\sigma_1 = \infty$.

	σ_1 S/m	ϵ_{1r}	Θ_{max}	$E^r_{2\Theta}(r_0,\Theta)_{max}$ V/m	$E^r_{2\Theta}(r_0,\pi/2)$ V/m
a	4.0	80	78.°5	2.36×10^{-5}	1.73×10^{-6}
b	0.4	12	73.°0	2.20×10^{-5}	1.73×10^{-7}
c	0.04	8	66.°0	1.87×10^{-5}	1.74×10^{-8}
d	0.004	80	65.°5	1.80×10^{-5}	1.93×10^{-8}

The field in the air on the surface $z = 0$ of the earth is

$$B_{2\phi}(\rho,0) = -\frac{\mu_0}{2\pi}\, e^{ik_2\rho}\left[\frac{ik_2}{\rho} - \frac{1}{\rho^2} - \frac{k_2^3}{k_1}\left(\frac{\pi}{k_2\rho}\right)^{1/2} e^{-iR}\mathcal{F}(R)\right] \quad (66)$$

$$E_{2\rho}(\rho,0) = \frac{\omega\mu_0}{2\pi k_1}\, e^{ik_2\rho}\left[\frac{ik_2}{\rho} - \frac{1}{\rho^2} - \frac{k_2^3}{k_1}\left(\frac{\pi}{k_2\rho}\right)^{1/2} e^{-iR}\mathcal{F}(R)\right] \quad (67)$$

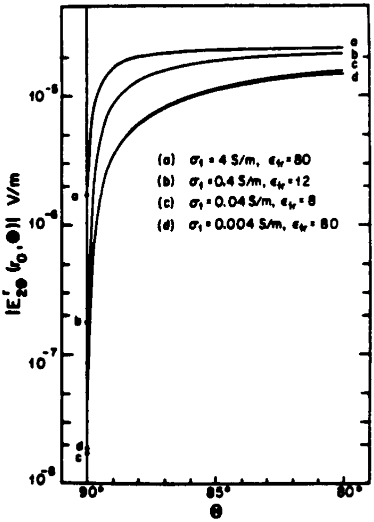


Figure 1.4 **Enlarged section of Fig. 1.3 near $\Theta = 90°$.**

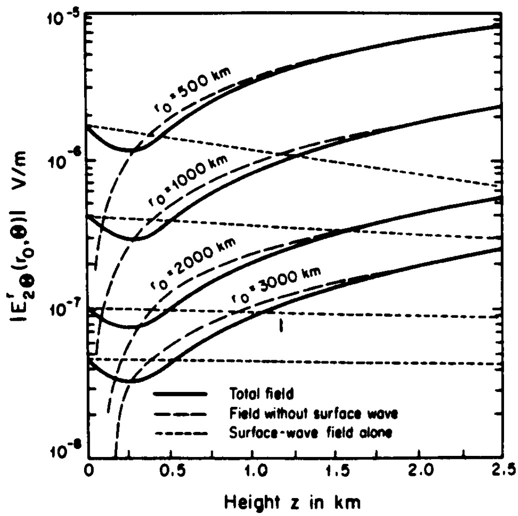


Figure 1.5 **Magnitude of $E^r_{2\Theta}(r_0, \Theta)$ at height z in air over sea water in range $z \ll r_0$. [Note that $|E^r_{2\Theta}(r_0, \Theta)| \sim 1/r_0^2$ to agree with (82).]**

$$E_{2z}(\rho,0) = \frac{\omega\mu_0}{2\pi k_2} e^{ik_2\rho} \left[\frac{ik_2}{\rho} - \frac{1}{\rho^2} - \frac{i}{k_2\rho^3} - \frac{k_2^3}{k_1}\left(\frac{\pi}{k_2\rho}\right)^{1/2} e^{-iR}\mathcal{F}(R)\right] \tag{68}$$

where $R = k_2^3\rho/2k_1^2$.

At very large distances from the source where $|P| \geq 4$, the Fresnel-integral term assumes a simple asymptotic form. This is

$$\begin{aligned} T \equiv \frac{k_2^3}{k_1}\left(\frac{\pi}{k_2 r_0}\right)^{1/2} e^{-iP_z}\mathcal{F}(P_z) &\to \frac{k_2^3}{k_1}\left(\frac{\pi}{k_2 r_0}\right)^{1/2} \frac{1}{(2\pi P_z)^{1/2}}\left(\frac{1}{2P_z} + i\right) \\ &= \frac{ik_2\rho}{r_0^2[1 + (k_1 z/k_2 r_0)]} + \frac{k_1^2\rho^3}{k_2^2 r_0^5[1 + (k_1 z/k_2 r_0)]^3} \end{aligned} \tag{69}$$

When $z = d = 0$ and $k_2\rho \geq 8|k_1^2|/k_2^2$,

$$T = \frac{ik_2}{\rho} + \frac{k_1^2}{k_2^2\rho^2} \tag{70}$$

With (70), (66)–(68) reduce to

$$\frac{k_1}{k_2} E_{2\rho}(\rho,0) = -\frac{\omega}{k_2} B_{2\phi}(\rho,0) = E_{2z}(\rho,0) = -\frac{\omega\mu_0}{2\pi k_2}\frac{k_1^2}{k_2^2}\frac{e^{ik_2\rho}}{\rho^2} \tag{71}$$

Note that the asymptotic form of the Fresnel-integral term contains a $1/\rho$ term *that cancels* the $1/\rho$ terms in (66)–(68). It also includes a $1/\rho^2$ term multiplied by the large factor k_1^2/k_2^2. This term becomes the dominant part of the far field along the boundary.

It is usually convenient to use the spherical coordinates r_0, Θ, Φ, where Θ is measured from the vertical z-axis. With $\rho = r_0 \sin\Theta$, $z = r_0\cos\Theta$, and

$$B_{2\Phi}(r_0,\Theta) = B_{2\phi}(\rho,z) \tag{72a}$$

$$E_{2r}(r_0,\Theta) = E_{2\rho}(\rho,z)\sin\Theta + E_{2z}(\rho,z)\cos\Theta \tag{72b}$$

$$E_{2\Theta}(r_0,\Theta) = E_{2\rho}(\rho,z)\cos\Theta - E_{2z}(\rho,z)\sin\Theta \tag{72c}$$

the field components (62)–(64) become

$$B_{2\Phi}(r_0,\Theta) = -\frac{\mu_0}{2\pi}\,e^{ik_2r_0}\left[\sin\Theta\left(\frac{ik_2}{r_0} - \frac{1}{r_0^2}\right) - T\right] \tag{73}$$

$$E_{2r}(r_0,\Theta) = \frac{\omega\mu_0}{2\pi k_2}\,e^{ik_2r_0}\left\{\left(\frac{2}{r_0^2} + \frac{2i}{k_2r_0^3} - T\sin\Theta\right)\cos\Theta\right.$$
$$\left. + \frac{k_2}{k_1}\left[\sin\Theta\left(\frac{ik_2}{r_0} - \frac{1}{r_0^2}\right) - T\right]\sin\Theta\right\} \tag{74}$$

$$E_{2\Theta}(r_0,\Theta) = -\frac{\omega\mu_0}{2\pi k_2}\,e^{ik_2r_0}\left\{\left(\frac{ik_2}{r_0} - \frac{1}{r_0^2} - \frac{i}{k_2r_0^3} - T\sin\Theta\right)\sin\Theta\right.$$
$$\left. - \frac{k_2}{k_1}\left[\sin\Theta\left(\frac{ik_2}{r_0} - \frac{1}{r_0^2}\right) - T\right]\cos\Theta\right\} \tag{75}$$

where

$$T \equiv \frac{k_2^3}{k_1}\left(\frac{\pi}{k_2r_0}\right)^{1/2} e^{-iP_z}\mathcal{F}(P_z) \tag{76}$$

Of primary interest is the far field defined by $k_2\rho \geq 8|k_1^2|/k_2^2$. In this range, all $1/r_0^2$ and $1/r_0^3$ terms are negligible compared to $1/r_0$ terms. Where all $1/r_0$ terms cancel, the $1/r_0^2$ terms are retained. The dominant $1/r_0^2$ terms are those in the Fresnel-integral term which are multiplied by the very large factor k_1^2/k_2^2. Since they are significant primarily near $\Theta = \pi/2$, the other $1/r_0^2$ terms in $E_r(r_0,\Theta)$ are also retained although they are relatively small. The asymptotic form of the Fresnel-integral term in (69) is

$$T^r = \frac{ik_2\sin\Theta}{r_0[1 + (k_1/k_2)\cos\Theta]} + \frac{k_1^2\sin^3\Theta}{k_2^2r_0^2[1 + (k_1/k_2)\cos\Theta]^3} \tag{77}$$

The dominant terms in the far field are

$$B_{2\Phi}^r(r_0,\Theta) = -\frac{\mu_0}{2\pi}\,e^{ik_2r_0}\left\{\frac{ik_2}{r_0}\left(\frac{k_1\sin\Theta\cos\Theta}{k_2 + k_1\cos\Theta}\right)\right.$$
$$\left. - \frac{k_1^2\sin^3\Theta}{k_2^2r_0^2[1 + (k_1/k_2)\cos\Theta]^3}\right\} \tag{78}$$

$$E_{2r}^r(r_0,\Theta) = \frac{\omega\mu_0}{2\pi k_2}\,e^{ik_2r_0}\left\{\frac{2}{r_0^2}\cos\Theta\right.$$

$$\left. - \frac{k_1 \sin^4 \Theta}{k_2 r_0^2 [1 + (k_1/k_2)\cos\Theta]^2} \right\} \tag{79}$$

$$E_{2\Theta}^r(r_0, \Theta) = -\frac{\omega\mu_0}{2\pi k_2} e^{ik_2 r_0} \left\{ \frac{ik_2}{r_0} \left(\frac{k_1 \sin\Theta \cos\Theta}{k_2 + k_1 \cos\Theta} \right) - \frac{k_1^2}{k_2^2 r_0^2} \left[\frac{\sin^3\Theta \left[\sin^2\Theta - (k_2/k_1)\cos\Theta\right]}{\left[1 + (k_1/k_2)\cos\Theta\right]^3} \right] \right\} \tag{80}$$

Note that $E_{2r}^r(r_0, \Theta)$ has no $1/r_0$ component. The dominant component of the electric field, $E_{2\Theta}^r(r_0, \Theta)$, has a large $1/r_0$ term except near $\Theta = \pi/2$ where it vanishes. At $\Theta = \pi/2$,

$$T^r = \frac{ik_2}{r_0} + \frac{k_1^2}{k_2^2 r_0^2}; \quad E_{2\Theta}^r(r_0, \pi/2) = \frac{\omega\mu_0}{2\pi k_2} \frac{k_1^2}{k_2^2} \frac{e^{ik_2 r_0}}{r_0^2}; \quad r_0 = \rho \tag{81}$$

When $|k_1 z| \ll k_2 r_0$, $\sin\Theta = \rho/r_0 \sim 1$, $\cos\Theta = z/r_0 \ll 1$, and

$$E_{2\Theta}^r(r_0, \Theta) \sim -\frac{\omega\mu_0}{2\pi k_2} \frac{k_1}{k_2} \frac{e^{ik_2 r_0}}{r_0^2} \left(ik_2 z - \frac{k_1}{k_2} \right) \tag{82}$$

The $1/r_0$ term in (80) is the far-field contribution by the plane-wave reflection coefficient; the $1/r_0^2$ term is the surface wave. Graphs of $|E_{2\Theta}^r(r_0, \Theta)|$ are in Fig. 1.3 in a conventional polar plot, and in Fig. 1.4 in a linear/logarithmic plot for the range $80° \le \Theta \le 90°$ for the low-angle radiation. Figure 1.5 shows $|E_{2\Theta}^r(r_0, \Theta)|$ when $z \ll r_0$ as a function of z specifically when Region 1 is sea water. The magnitudes of the field without the surface-wave term and of the field due to the surface-wave term alone are shown separately. Note that both terms decrease with distance as $1/r_0^2$ for any fixed z.

For engineering applications it may be useful to generalize the far-field formulas (78) and (80) so that they can be used for all values of k_1 without the restriction $|k_1| \ge 3k_2$. This is readily done for the $1/r_0$ terms simply by substituting the exact formula for the plane-wave transmission coefficient f_{er} for the approximate one that appears in (78) and (80). That is,

$$\tfrac{1}{2}(1 + f_{er}) = \frac{n^2 \cos\Theta}{n^2 \cos\Theta + (n^2 - \sin^2\Theta)^{1/2}}; \qquad n^2 = \frac{k_1^2}{k_2^2} \tag{83}$$

is substituted in the $1/r_0$ terms for

$$\tfrac{1}{2}(1 + f_{er}) \sim \frac{k_1 \cos\Theta}{k_2 + k_1 \cos\Theta} \tag{84}$$

(which is obtained from the exact formula when $\sin^2\Theta \leq 1$ is neglected compared to n^2). With this substitution, (78) and (80) have the following forms:

$$B^r_{2\Phi}(r_0,\Theta) = -\frac{\mu_0}{2\pi}\, e^{ik_2 r_0} \times \left\{ \frac{ik_2}{r_0}\left[\frac{(k_1^2/k_2^2)\sin\Theta\cos\Theta}{(k_1^2/k_2^2)\cos\Theta + [(k_1^2/k_2^2) - \sin^2\Theta]^{1/2}}\right] - \frac{k_1^2 \sin^3\Theta}{k_2^2 r_0^2 [1+(k_1/k_2)\cos\Theta]^3} \right\} \tag{85}$$

$$E^r_{2\Theta}(r_0,\Theta) = -\frac{\omega\mu_0}{2\pi k_2}\, e^{ik_2 r_0} \times \left\{ \frac{ik_2}{r_0}\left[\frac{(k_1^2/k_2^2)\sin\Theta\cos\Theta}{(k_1^2/k_2^2)\cos\Theta + [(k_1^2/k_2^2) - \sin^2\Theta]^{1/2}}\right] - \frac{k_1^2}{k_2^2 r_0^2}\left[\frac{\sin^3\Theta\,[\sin^2\Theta - (k_2/k_1)\cos\Theta]}{[1+(k_1/k_2)\cos\Theta]^3}\right] \right\} \tag{86}$$

These formulas are valid subject to the far-field condition

$$k_2 r_0 \geq |8k_1^2/k_2^2| \tag{87}$$

When $k_1 = k_2$, this condition approximates the usual far-field requirement $k_2 r_0 \gg 1$, so that the terms $1/(k_2^2 r_0^2)$ and $1/(k_2^3 r_0^3)$ can be neglected compared to $1/k_2 r_0$. When $k_1 = k_2$, the far field in (85) and (86) becomes

$$B^r_{2\Phi}(r_0,\Theta) = -\frac{\mu_0}{4\pi}\, e^{ik_2 r_0}\, \frac{ik_2}{r_0}\sin\Theta;$$

$$E^r_{2\Theta}(r_0,\Theta) = -\frac{\omega\mu_0}{4\pi k_2}\, e^{ik_2 r_0}\, \frac{ik_2}{r_0}\sin\Theta \tag{88}$$

which is the far field of the isolated dipole in air. Note that, when $k_1 = k_2$, the $1/r_0$ terms do not vanish when $\Theta = \pi/2$ but have maxima, so that *all* $1/r_0^2$ terms are negligible.

The accuracy of the surface-wave term is reduced when $|k_1/k_2|$ is smaller than 3, but it should still give the right order of magnitude at $\Theta = \pi/2$ until, as k_1 approaches k_2, it becomes negligibly small. The fact that it does not vanish when $k_1 = k_2$ is irrelevant since r_0 must be chosen large enough so that it — along with all the other $1/r_0^2$ terms — is negligible.

1.6 The Poynting Vector Near the Boundary

The Poynting vector in Region 2 (air) is defined by

$$\begin{aligned}\overline{S}_2(\rho,z) &= \hat{z}S_{2z}(\rho,z)+\hat{\rho}S_{2\rho}(\rho,z)\\ &= \frac{1}{2\mu_0}\left[\overline{E}_2(\rho,z)\times\overline{B}_2^*(\rho,z)\right]\end{aligned} \tag{89}$$

$$\begin{aligned}S_{2z}(\rho,z) &= \frac{1}{2\mu_0}\,E_{2\rho}(\rho,z)B_{2\phi}^*(\rho,z);\\ S_{2\rho}(\rho,z) &= -\frac{1}{2\mu_0}\,E_{2z}(\rho,z)B_{2\phi}^*(\rho,z)\end{aligned} \tag{90}$$

The slope of the locus of the Poynting vector is

$$\frac{dz}{d\rho}=\frac{\operatorname{Re}S_{2z}(\rho,z)}{\operatorname{Re}S_{2\rho}(\rho,z)} \tag{91}$$

The general equation of the locus obtained from (90) with the components of the field (62)–(64) is complicated and no explicit solution is available. In the near and intermediate range, an approximate formula can be obtained that indicates that the Poynting vector at a point $(\rho,0)$ on the boundary has followed a curved path from the dipole at $(0,0)$ upward into the air and back down to the earth at $(\rho,0)$. For present purposes, it is adequate to determine its slope as it reaches the surface. This is readily determined from the field in both the intermediate and far ranges. In the former, defined by $k_2\rho<|k_1^2|/k_2^2$, the Fresnel term contributes negligibly and (66)–(68) give the following leading terms:

$$S_{2z}(\rho,0)=\frac{1}{2\mu_0}\,E_{2\rho}(\rho,0)B_{2\phi}^*(\rho,0)\sim-\frac{\omega\mu_0}{8\pi^2k_1}\,\frac{k_2^2}{\rho^2} \tag{92}$$

$$S_{2\rho}(\rho,0)=-\frac{1}{2\mu_0}\,E_{2z}(\rho,0)B_{2\phi}^*(\rho,0)\sim\frac{\omega\mu_0}{8\pi^2k_2}\,\frac{k_2^2}{\rho^2} \tag{93}$$

In the far field, where $k_2\rho\geq 8|k_1^2|/k_2^2$, (71) gives

$$-\frac{k_2}{k_1}\,S_{2\rho}^r(\rho,0)=S_{2z}^r(\rho,0)=-\frac{\omega\mu_0}{8\pi^2k_1}\,\frac{(k_1k_1^*)^2}{k_2^4}\,\frac{1}{\rho^4} \tag{94}$$

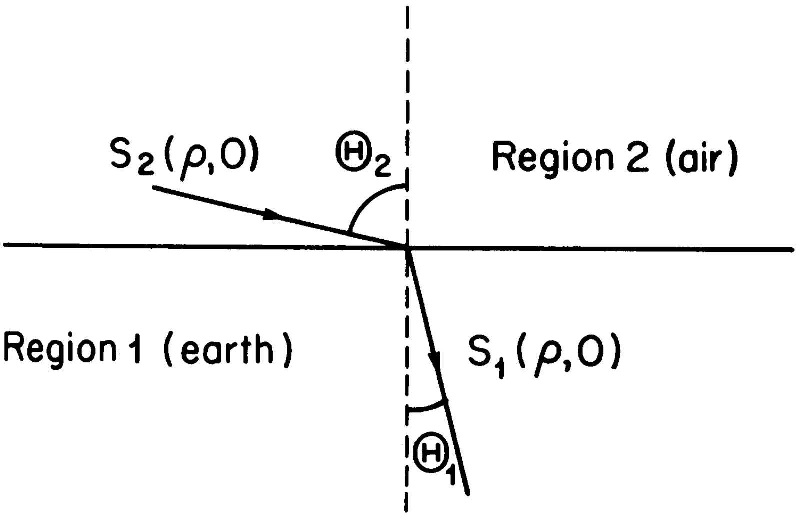


Figure 1.6 **Poynting vector at a point on the air-earth boundary in the far zone.**

In each range,

$$\frac{dz}{d\rho} = \frac{\operatorname{Re} S_{2z}(\rho,0)}{\operatorname{Re} S_{2\rho}(\rho,0)} \sim -\frac{k_2}{\beta_1} \tag{95}$$

where β_1 is the real part of $k_1 = \beta_1 + i\alpha_1$. With the boundary conditions at $z = 0$, it follows that

$$S_{1z}(\rho,0) = S_{2z}(\rho,0); \qquad S_{1\rho}(\rho,0) = \frac{k_2^2}{k_1^2} S_{2\rho}(\rho,0) \tag{96}$$

Hence, the slope of the Poynting vector in the earth just below the surface is

$$\left(\frac{dz}{d\rho}\right)_{z=0-} = -\frac{k_2}{\beta_1} \cdot \frac{\beta_1^2}{k_2^2} = -\frac{\beta_1}{k_2} \tag{97}$$

The Poynting vector in both regions at a point on their boundary in the intermediate or the far field is shown in Fig. 1.6 with $\beta_1/k_2 = 2$. This rather small value is chosen to permit a clearer diagram. Actual values of β_1/k_2 are much greater. It is seen that the Poynting vector arrives at the point of observation $(\rho, 0)$ traveling almost horizontally, i.e., with a large angle of incidence in the air, and enters the earth to travel almost vertically downward, i.e., with a small angle of refraction. Since $S_{2\rho}(\rho, 0)$ is greater than $S_{2z}(\rho, 0)$ by a factor k_1/k_2, most of the power continues to travel radially along the surface in the air. However, a small fraction is continuously transferred into the earth where it travels almost vertically downward. This is a complicated phenomenon that includes incident, reflected (reradiated) and refracted fields and associated powers. It is important to note that in the far field only the surface wave is involved in this transfer of power into the earth.

1.7 Conclusion

A study of the fraction of power that remains in the air (upper half-space) and the fraction that is transferred into the earth (lower half-space) when the radiating source is a vertical dipole at a height d in the air confirms the results obtained by Hansen [3] except that, as $d \to 0$, the power in the air is reduced to a deep minimum and not zero. A study of the properties of the electromagnetic field generated by the same dipole shows that the power transferred into the earth is associated with the surface wave that travels outward along the boundary in the air. Furthermore, when $d \sim 0$, the entire far field along that boundary is due to the surface wave. This also provides a significant part of the field in the low-angle range when $|k_1 z| \ll k_2 r_0$ that is of interest for over-the-horizon radar.

An antenna with high radiation efficiency in the sense defined by Hansen [3] necessarily must generate a very weak surface wave. On the other hand, an antenna that generates a strong low-angle field with a significant value along the boundary surface, $\Theta = 90^\circ$, necessarily generates a strong surface wave that transfers all the power associated with it into the earth. Therefore, it has a low radiation efficiency. It may be concluded that the concept of radiation efficiency as defined by (8) is not a useful figure-of-merit.

The design of an antenna that maximizes the surface wave and the entire low-angle field in the range $80^\circ \leq \Theta \leq 90^\circ$ is a problem of considerable interest. Such an antenna will have a low radiation efficiency and most of the power radiated will not remain in the upper half-space (air) but will be transferred to the lower half-space (earth).

Appendix A. Evaluation of the Integrals for the Power in the Air

The two integrals J_{1a} and J_{2a} in (44) and (45) can be evaluated together. They are

$$
\begin{aligned}
J_{1a} + J_{2a} &= \mathrm{Re}\,\frac{3}{4k_2^3}\int_0^{k_2} \gamma_2^{-1}\left[|G_{21}|^2 - 2G_{21}\right]\lambda^3\,d\lambda \\
&= \mathrm{Re}\,\frac{3}{k_2^3}\int_0^{k_2} \gamma_2^{-1}\left(\frac{k_2^4\gamma_1^2}{N^2} - \frac{k_2^2\gamma_1}{N}\right)\lambda^3\,d\lambda \\
&= \mathrm{Re}\,\frac{3}{k_2}\int_0^{k_2} \gamma_2^{-1}\left[\frac{k_2^2\gamma_1^2 - \gamma_1(k_2^2\gamma_1 + k_1^2\gamma_2)}{N^2}\right]\lambda^3\,d\lambda \\
&= -\,\mathrm{Re}\,\frac{3k_1^2}{k_2}\int_0^{k_2} \frac{\gamma_1}{(k_1^2\gamma_2 + k_2^2\gamma_1)^2}\,\lambda^3\,d\lambda
\end{aligned}
\tag{A1}
$$

With the condition $k_2^2 \ll k_1^2$ it follows that, in the range $0 \le \lambda \le k_2$, $\lambda^2 \ll k_1^2$. Hence, $\gamma_1 = (k_1^2 - \lambda^2)^{1/2} \sim k_1$. The integral therefore becomes

$$
\begin{aligned}
J_{1a} + J_{2a} &\sim -\,\mathrm{Re}\,\frac{3k_1^3}{k_2}\int_0^{k_2} \frac{\lambda^3\,d\lambda}{(k_1^2\gamma_2 + k_2^2 k_1)^2} \\
&= -\,\mathrm{Re}\,\frac{3k_1}{k_2}\int_0^{k_2} \frac{\lambda^3\,d\lambda}{\left[k_1(k_2^2 - \lambda^2)^{1/2} + k_2^2\right]^2}
\end{aligned}
\tag{A2}
$$

Let the variable be changed to

$$
\zeta = \frac{(k_2^2 - \lambda^2)^{1/2}}{k_2}, \qquad \lambda^2 = k_2^2(1 - \zeta^2), \qquad \lambda\,d\lambda = -k_2^2\zeta\,d\zeta
$$

so that

$$
\begin{aligned}
J_{1a} + J_{2a} &= -\,\mathrm{Re}\,\frac{3k_2}{k_1}\int_0^1 \frac{(1 - \zeta^2)\zeta\,d\zeta}{(\zeta + k_2 k_1^{-1})^2} \\
&= -\,\mathrm{Re}\,\frac{3k_2}{k_1}\left\{\int_0^1 \frac{\zeta\,d\zeta}{(\zeta + k_2k_1^{-1})^2} - \int_0^1 \frac{\zeta^3\,d\zeta}{(\zeta + k_2k_1^{-1})^2}\right\}
\end{aligned}
\tag{A3}
$$

These are elementary integrals with the following integrated values:

$$
J_{1a} + J_{2a} = -\frac{3k_2}{k_1}\left\{\ln\left(\zeta + \frac{k_2}{k_1}\right) + \frac{k_2}{k_1(\zeta + k_2k_1^{-1})}\right.
$$

$$
\begin{aligned}
& - \left[\frac{1}{2}\left(\zeta + \frac{k_2}{k_1}\right)^2 - \frac{3k_2}{k_1}\left(\zeta + \frac{k_2}{k_1}\right) \right. \\
& \quad + \frac{3k_2^2}{k_1^2} \ln\left(\zeta + \frac{k_2}{k_1}\right) \\
& \quad \left. \left. + \frac{k_2^3}{k_1^3 \left(\zeta + k_2 k_1^{-1}\right)} \right] \right\}_0^1 \\
= & - \frac{3k_2}{k_1} \left[\left(1 - \frac{3k_2^2}{k_1^2}\right) \ln\left(1 + \frac{k_1}{k_2}\right) - \frac{3}{2} + \frac{3k_2}{k_1} \right] \qquad \text{(A4)}
\end{aligned}
$$

The third integral (46) is

$$
\begin{aligned}
J_{3a} &= - \operatorname{Re} \frac{3}{k_2} \int_0^{k_2} \frac{\gamma_1}{\gamma_2 N} e^{2i\gamma_2 d} \lambda^3 \, d\lambda \\
&\sim - \operatorname{Re} \frac{3}{k_2} \int_0^{k_2} \frac{1}{(k_2^2 - \lambda^2)^{1/2}} \frac{e^{2id(k_2^2 - \lambda^2)^{1/2}}}{k_1 (k_2^2 - \lambda^2)^{1/2} + k_2^2} \lambda^3 \, d\lambda \qquad \text{(A5)}
\end{aligned}
$$

The approximation $\gamma_1 \sim k_1$ has been made because $\lambda^2 \ll k_1^2$ in the range $0 \le \lambda \le k_2$. With the same change of variable made in (A1), i.e., $\zeta = (k_2^2 - \lambda^2)^{1/2}/k_2$, the integral becomes

$$
J_{3a} = - \operatorname{Re} \frac{3k_2}{k_1} \int_0^1 \frac{1 - \zeta^2}{\zeta + k_2 k_1^{-1}} e^{2ik_2 d\zeta} \, d\zeta \qquad \text{(A6)}
$$

Now let $m = k_2/k_1$ and $u = \zeta + m$, $\zeta = u - m$, $1 - \zeta^2 = 1 - u^2 + 2um - m^2$. Then

$$
\begin{aligned}
J_{3a} = - \operatorname{Re} 3m \, e^{-2ik_2 dm} & \left\{ \int_m^{m+1} (1 - m^2) \frac{e^{2ik_2 du}}{u} \, du \right. \\
& - \int_m^{m+1} e^{2ik_2 du} u \, du \\
& \left. + 2m \int_m^{m+1} e^{2ik_2 du} du \right\} \qquad \text{(A7)}
\end{aligned}
$$

Next, let $v = 2k_2 du$, so that

$$
J_{3a} = - \operatorname{Re} 3m \, e^{-2ik_2 dm} \left\{ \int_{2k_2 dm}^{2k_2 d(m+1)} (1 - m^2) \frac{e^{iv}}{v} \, dv \right.
$$

$$
- \int_{2k_2dm}^{2k_2d(m+1)} \frac{e^{iv} v\, dv}{4k_2^2 d^2}
+ \frac{m}{k_2 d} \int_{2k_2dm}^{2k_2d(m+1)} e^{iv}\, dv \Bigg\} \tag{A8}
$$

These integrals can be expressed in terms of the sine and cosine integrals, viz.,

$$
\operatorname{Si} x = \int_0^x \frac{\sin u}{u}\, du; \qquad \operatorname{Cin} x = \int_0^x \frac{1 - \cos u}{u}\, du \tag{A9}
$$

Thus,

$$
\begin{aligned}
J_{3a} &= -\operatorname{Re} 3m\, e^{-2ik_2dm} \Bigg\{ (1 - m^2)\left[\ln v - \operatorname{Cin} v + i \operatorname{Si} v\right] \\
&\qquad - \frac{1}{4k_2^2 d^2} e^{iv}(-iv + 1) - \frac{mi}{k_2 d} e^{iv} \Bigg\}_{2k_2dm}^{2k_2d(1+m)} \\
&= -\operatorname{Re} 3m \Bigg\{ e^{-2ik_2dm}(1 - m^2) \left[\ln\left(\frac{1+m}{m}\right) - \operatorname{Cin} 2k_2d(1+m) \right. \\
&\qquad \left. + \operatorname{Cin} 2k_2dm + i \operatorname{Si} 2k_2d(1+m) - i \operatorname{Si} 2k_2dm \right] \\
&\qquad - \frac{1}{4k_2^2 d^2} \left\{ e^{2ik_2d}[1 - 2ik_2d(m+1)] - 1 + 2ik_2dm \right\} \\
&\qquad - i \frac{m}{k_2 d} (e^{2ik_2d} - 1) \Bigg\}
\end{aligned} \tag{A10}
$$

Rearrangement and substitution for $m = k_2/k_1$ give the final formula. It is

$$
\begin{aligned}
J_{3a} = -\frac{3k_2}{k_1} \Bigg(\left(1 - \frac{k_2^2}{k_1^2}\right) \Bigg\{ \cos \frac{2k_2^2 d}{k_1} \Bigg[\ln\left(1 + \frac{k_1}{k_2}\right) \\
- \operatorname{Cin} 2k_2 d\left(1 + \frac{k_2}{k_1}\right) + \operatorname{Cin} \frac{2k_2^2 d}{k_1} \Bigg] \\
+ \sin \frac{2k_2^2 d}{k_1} \left[\operatorname{Si} 2k_2 d\left(1 + \frac{k_2}{k_1}\right) - \operatorname{Si} \frac{2k_2^2 d}{k_1} \right] \Bigg\}
\end{aligned}
$$

$$+\frac{1-\cos 2k_2 d}{4k_2^2 d^2} - \left(1-\frac{k_2}{k_1}\right)\frac{\sin 2k_2 d}{2k_2 d}\Bigg) \tag{A11}$$

Appendix B. Evaluation of the Integral J_{3g}

Since $J_{1g} + J_{2g} = -(J_{1a} + J_{2a})$, the only remaining integral is

$$\begin{aligned} J_{3g} &= -\frac{3}{k_2}\,\mathrm{Im}\int_{k_2}^{\infty}\frac{e^{-2\nu_2 d}}{\nu_2}\,\frac{\gamma_1}{N}\,\lambda^3\,d\lambda \\ &= -\frac{3}{k_2}\,\mathrm{Im}\int_{k_2}^{\infty}\frac{(k_1^2-\lambda^2)^{1/2}\,e^{-2d(\lambda^2-k_2^2)^{1/2}}\lambda^3\,d\lambda}{(\lambda^2-k_2^2)^{1/2}[ik_1^2(\lambda^2-k_2^2)^{1/2}+k_2^2(k_1^2-\lambda^2)^{1/2}]} \end{aligned} \tag{B1}$$

In the range $\lambda > k_1$, $(k_1^2-\lambda^2)^{1/2} = i(\lambda^2-k_1^2)^{1/2} = i\nu_1$. Hence,

$$J_{3g} = -\frac{3}{k_2}\,\mathrm{Im}\left\{\int_{k_2}^{k_1}\frac{\gamma_1 e^{-2d\nu_2}\lambda^3\,d\lambda}{\nu_2(ik_1^2\nu_2+k_2^2\gamma_1)} + \int_{k_1}^{\infty}\frac{i\nu_1 e^{-2d\nu_2}\lambda^3\,d\lambda}{\nu_2(ik_1^2\nu_2+ik_2^2\nu_1)}\right\} \tag{B2}$$

Here the second integral is real and, therefore, has no imaginary part.

$$J_{3g} = -\frac{3}{k_2}\,\mathrm{Im}\int_{k_2}^{k_1}\frac{\gamma_1(ik_1^2\nu_2-k_2^2\gamma_1)}{\nu_2(-k_1^4\nu_2^2-k_2^4\gamma_1^2)}\,e^{-2d\nu_2}\lambda^3\,d\lambda \tag{B3}$$

The integral that includes the second term in the numerator is real and has no imaginary part. Thus,

$$J_{3g} = \frac{3k_1^2}{k_2}\,\mathrm{Re}\int_{k_2}^{k_1}\frac{(k_1^2-\lambda^2)^{1/2}e^{-2d(\lambda^2-k_2^2)^{1/2}}\lambda^3\,d\lambda}{k_1^4(\lambda^2-k_2^2)+k_2^4(k_1^2-\lambda^2)} \tag{B4}$$

In this integral, let $\zeta = \lambda^2/k_2^2$ or $\lambda^2 = \zeta k_2^2$, $\lambda\,d\lambda = \frac{1}{2}k_2^2\,d\zeta$. Then,

$$J_{3g} = \frac{3k_1^2}{2k_2}\,\mathrm{Re}\int_{1}^{k_1^2/k_2^2}\frac{(k_1^2-k_2^2\zeta)^{1/2}e^{-2k_2 d(\zeta-1)^{1/2}}k_2^4\zeta\,d\zeta}{k_1^4k_2^2(\zeta-1)+k_2^4(k_1^2-k_2^2\zeta)} \tag{B5}$$

Here the denominator is

$$k_2^2(k_1^4 - k_2^4)\zeta - k_1^2k_2^2(k_1^2 - k_2^2) = k_2^2(k_1^4 - k_2^4)\left(\zeta - \frac{k_1^2}{k_1^2 + k_2^2}\right)$$

so that

$$J_{3g} = \mathrm{Re}\,\frac{3k_1^2k_2}{2(k_1^4 - k_2^4)} \int_1^{k_1^2/k_2^2} \frac{(k_1^2 - k_2^2\zeta)^{1/2} e^{-2k_2d(\zeta-1)^{1/2}} \zeta\, d\zeta}{\zeta - k_1^2(k_1^2 + k_2^2)^{-1}} \tag{B6}$$

Now let $\tau^2 = \zeta - 1$ or $\zeta = \tau^2 + 1$, $d\zeta = 2\tau\, d\tau$. Also let $n^2 = (k_1^2/k_2^2) - 1$, $m^2 = k_2^2/(k_1^2 + k_2^2)$. Then,

$$J_{3g} = \mathrm{Re}\,\frac{3k_1^2k_2^2}{k_1^4 - k_2^4} \int_0^n \frac{(n^2 - \tau^2)^{1/2} e^{-2k_2d\tau}(\tau^2 + 1)\tau\, d\tau}{\tau^2 + m^2} \tag{B7}$$

Since $k_1^4 \gg k_2^4$, this reduces to the following two integrals:

$$J_{3g} = \mathrm{Re}\,\frac{3k_2^2}{k_1^2}(I_1 + I_2) \tag{B8}$$

where

$$I_1 = \mathrm{Re}\int_0^n \frac{(n^2 - \tau^2)^{1/2} e^{-2k_2d\tau}\tau\, d\tau}{\tau^2 + m^2} \tag{B9a}$$

$$I_2 = \mathrm{Re}\int_0^n \frac{(n^2 - \tau^2)^{1/2} e^{-2k_2d\tau}\tau^3\, d\tau}{\tau^2 + m^2} \tag{B9b}$$

Since $m^2 = k_2^2/(k_1^2 + k_2^2) < k_2^2/k_1^2 \ll 1$, it can be neglected in I_2 but must be retained in I_1 because the integral becomes infinite when $m = 0$.

The evaluation of I_1 is carried out in two parts obtained with $\tau = nt$. Thus, with $z = 2k_2dn$, $a = m/n \sim k_2^2/k_1^2$,

$$I_1 = n\int_0^1 \frac{e^{-zt}(1 - t^2)^{1/2}}{t^2 + a^2}\, t\, dt = I_{11} + I_{12} \tag{B10}$$

where

$$I_{11} = n\int_0^1 e^{-zt}\frac{t\, dt}{t^2 + a^2};$$

$$I_{12} = -n\int_0^1 e^{-zt}\frac{1 - (1 - t^2)^{1/2}}{t^2 + a^2}\, t\, dt \tag{B11}$$

The integral I_{11} can be evaluated with formula (5.1.44) on p. 230 of Abramowitz and Stegun [5]. With $x = -zt$ or $t = -x/z$,

$$\begin{aligned} I_{11} &= n \int_0^{-z} e^x \frac{x\,dx}{x^2 + a^2 z^2} \\ &= -\frac{n}{2} \left[e^{iaz} E_1(-x + iaz) + e^{-iaz} E_1(-x - iaz) \right] \Big|_0^{-z} \\ &= \frac{n}{2} \left[e^{iaz} E_1(iaz) + e^{-iaz} E_1(-iaz) \right. \\ &\qquad \left. - e^{iaz} E_1(z + iaz) - e^{-iaz} E_1(z - iaz) \right] \end{aligned} \tag{B12}$$

where $E_1(x)$ is defined in (5.1.1) on p. 228 of Abramowitz and Stegun [5]. This can be expressed in terms of the more familiar exponential integral $E_i(-z) = -E_1(z)$ [see (5.1.2) on p. 228]. Also with $z(1 \pm ia) = z[1 \pm i(k_2^2/k_1^2)] \sim z$ (since $k_2^2 \ll k_1^2$), the desired formula is

$$I_{11} = -\frac{n}{2} \left[e^{iaz} E_i(-iaz) + e^{-iaz} E_i(iaz) + 2E_1(z) \cos az \right] \tag{B13}$$

Note that

$$\begin{aligned} E_i(\pm iaz) &= \operatorname{Ci} az \pm \left(\operatorname{Si} az - \frac{\pi}{2} \right) \\ &= \gamma + \ln az - \operatorname{Cin} az \pm i \left(\operatorname{Si} az - \frac{\pi}{2} \right) \end{aligned} \tag{B14}$$

where $\gamma = 0.5772$ and $\operatorname{Cin} x$ and $\operatorname{Si} x$ are defined in (A9). With this notation, the final formula for I_{11} is

$$\begin{aligned} I_{11} = -\frac{n}{2} \Big\{ & e^{iaz} \left[\gamma + \ln az - \operatorname{Cin} az - i \left(\operatorname{Si} az - \frac{\pi}{2} \right) \right] \\ & + e^{-iaz} \left[\gamma + \ln az - \operatorname{Cin} az + i \left(\operatorname{Si} az - \frac{\pi}{2} \right) \right] \\ & + 2E_1(z) \cos az \Big\} \\ = -n \Big\{ & \cos az \left[\gamma + \ln az - \operatorname{Cin} az \right] + \sin az \left(\operatorname{Si} az - \frac{\pi}{2} \right) \\ & + E_1(z) \cos az \Big\} \end{aligned} \tag{B15}$$

The integral I_{12} in (B11) can be evaluated as follows:

$$I_{12} = -n \lim_{\epsilon\to 0}\left\{ \int_0^1 e^{-zt} t^{-1+\epsilon}\, dt - \int_0^1 e^{-zt} t^{-1+\epsilon}(1-t^2)^{1/2}\, dt \right\} \tag{B16}$$

Here the first integral is given by formula (1) on p. 255 of Bateman [4], Vol. 1, with $c = 1+\epsilon$, $a = \epsilon$, and $x = -z$. The second integral is given by Gradshteyn and Ryzhik [7], p. 323, formula 1, with $\mu = -z$, $\rho = \frac{3}{2}$, $\nu = \frac{1}{2}\epsilon$, and $u = 1$. The results are

$$\begin{aligned} I_{12} = -\, n \lim_{\epsilon\to 0}\Bigg\{ &\frac{\Gamma(\epsilon)}{\Gamma(\epsilon+1)}\ \Phi(\epsilon, 1+\epsilon; -z) \\ &- \frac{1}{2} B(\tfrac{1}{2}\epsilon, \tfrac{3}{2})\, {}_1F_2(\tfrac{1}{2}\epsilon;\, \tfrac{1}{2},\, \tfrac{1}{2}\epsilon+\tfrac{3}{2};\, \tfrac{1}{4}z^2) \\ &+ \frac{z}{2} B(\tfrac{1}{2}\epsilon+\tfrac{1}{2}, \tfrac{3}{2})\, {}_1F_2(\tfrac{1}{2}\epsilon+\tfrac{1}{2};\, \tfrac{3}{2},\, \tfrac{1}{2}\epsilon+2;\, \tfrac{1}{4}z^2) \Bigg\} \end{aligned} \tag{B17}$$

The function Φ is defined in (1) on p. 248 of Bateman [4], Vol. 1. It is

$$\begin{aligned} \Phi(\epsilon,\, 1+\epsilon;\, -z) &= \sum_{k=0}^{\infty} \frac{(\epsilon)_k(-z)^k}{(1+\epsilon)_k k!} = 1 + \sum_{k=1}^{\infty} \frac{\epsilon}{k+\epsilon}\, \frac{(-z)^k}{k!} \\ &\sim 1 + \epsilon \sum_{k=1}^{\infty} \frac{(-z)^k}{k(k!)} \\ &= 1 - \epsilon \left[\gamma + \ln z + E_1(z)\right] \end{aligned} \tag{B18}$$

The last step makes use of (5.1.11) on p. 229 of Abramowitz and Stegun [5]. The Beta function is defined in (6.2.2) on p. 258 of Abramowitz and Stegun [5]. It is

$$\begin{aligned} B(\tfrac{1}{2}\epsilon,\, \tfrac{3}{2}) &= \frac{\Gamma(\frac{1}{2}\epsilon)\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2}\epsilon+\frac{3}{2})} \equiv \frac{2}{\epsilon}\, \frac{\Gamma(1+\frac{1}{2}\epsilon)\frac{1}{2}\sqrt{\pi}}{\Gamma(\frac{3}{2}+\frac{1}{2}\epsilon)} \\ &= \frac{\sqrt{\pi}}{\epsilon}\, \frac{\Gamma(1)}{\Gamma(\frac{3}{2})} \left[1 + \tfrac{1}{2}\epsilon\, \psi(1) - \tfrac{1}{2}\epsilon\, \psi(\tfrac{3}{2})\right] \\ &= \frac{2}{\epsilon}\left[1 - \tfrac{1}{2}\epsilon\gamma - \tfrac{1}{2}\epsilon(-\gamma + 2 - 2\ln 2)\right] \end{aligned} \tag{B19}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ as defined in (6.3.1) on p. 258 of Abramowitz and Stegun [5], and with (6.3.2) and (6.3.4) on the same page. Thus,

$$B(\tfrac{1}{2}\epsilon,\, \tfrac{3}{2}) \sim \frac{2}{\epsilon} - 2 + 2\ln 2 \tag{B20}$$

Similarly,

$$B(\tfrac{1}{2}\epsilon + \tfrac{1}{2},\, \tfrac{3}{2}) = \frac{\Gamma(\frac{1}{2}\epsilon + \frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2}\epsilon + 2)} \tag{B21}$$

With (6.1.18) and (6.1.15) on p. 256 of Abramowitz and Stegun [5],

$$\begin{aligned}\Gamma(\tfrac{1}{2}\epsilon + \tfrac{1}{2}) &= \frac{2^{1-\epsilon}\sqrt{\pi}\,\Gamma(\epsilon)}{\Gamma(\frac{1}{2}\epsilon)} \\ &= \frac{2^{1-\epsilon}\sqrt{\pi}\,\epsilon^{-1}\Gamma(\epsilon+1)}{2\epsilon^{-1}\Gamma(\frac{1}{2}\epsilon + 1)} \\ &= \frac{\sqrt{\pi}\,\Gamma(\epsilon+1)}{2^{\epsilon}\Gamma(\frac{1}{2}\epsilon + 1)}\end{aligned} \tag{B22}$$

Also, $\Gamma(\frac{3}{2}) = \sqrt{\pi}/2$ and $\Gamma(\frac{1}{2}\epsilon + 2) = (\frac{1}{2}\epsilon + 1)\Gamma(\frac{1}{2}\epsilon + 1)$, so that

$$B(\tfrac{1}{2}\epsilon + \tfrac{1}{2},\, \tfrac{3}{2}) = \frac{\pi\Gamma(\epsilon+1)}{2^{1+\epsilon}(\frac{1}{2}\epsilon + 1)\Gamma^2(\frac{1}{2}\epsilon + 1)} \sim \frac{\pi\Gamma(1)}{2\Gamma^2(1)} \sim \frac{\pi}{2} \tag{B23}$$

The hypergeometric series ${}_1F_2$ is defined as follows:

$$\begin{aligned}{}_1F_2(\tfrac{1}{2}\epsilon;\, \tfrac{1}{2},\, \tfrac{1}{2}\epsilon + \tfrac{3}{2};\, \tfrac{1}{4}z^2) &= 1 + \sum_{k=0}^{\infty} \frac{(\frac{1}{2}\epsilon)_{k+1}(\frac{1}{4}z^2)^{k+1}}{(\frac{1}{2})_{k+1}(\frac{1}{2}\epsilon + \frac{3}{2})_{k+1}(k+1)!} \\ &= 1 + \frac{\frac{1}{2}\epsilon}{\frac{1}{2}(\frac{1}{2}\epsilon + \frac{3}{2})} \\ &\quad \times \sum_{k=0}^{\infty} \frac{(1 + \frac{1}{2}\epsilon)_k(\frac{1}{4}z^2)^{k+1}}{(\frac{3}{2})_k(\frac{1}{2}\epsilon + \frac{5}{2})_k(k+1)!}\end{aligned} \tag{B24}$$

When higher-order terms in ϵ have been neglected, this reduces to

$${}_1F_2(\tfrac{1}{2}\epsilon;\, \tfrac{1}{2},\, \tfrac{1}{2}\epsilon + \tfrac{3}{2};\, \tfrac{1}{4}z^2) \sim 1 + \frac{2\epsilon}{3} \sum_{k=0}^{\infty} \frac{k!}{(\frac{3}{2})_k(\frac{5}{2})_k} \frac{(\frac{1}{4}z^2)^{k+1}}{(k+1)!} \tag{B25}$$

The second series is

$$ {}_1F_2(\tfrac{1}{2}\epsilon+\tfrac{1}{2};\ \tfrac{3}{2},\ \tfrac{1}{2}\epsilon+2;\ \tfrac{1}{4}z^2) = \sum_{k=0}^{\infty} \frac{(\frac{1}{2}\epsilon+\frac{1}{2})_k(\frac{1}{4}z^2)^k}{(\frac{3}{2})_k(\frac{1}{2}\epsilon+2)_k k!} \sim \sum_{k=0}^{\infty} \frac{\frac{1}{2}(\frac{1}{4}z^2)^k}{(2)_k(\frac{1}{2}+k)k!} \tag{B26} $$

With these values I_{12} becomes

$$ \begin{aligned} I_{12} = -n\lim_{\epsilon\to 0}\Bigg(& \frac{1}{\epsilon}\{1-\epsilon[E_1(z)+\gamma+\ln z]\} \\ & -\frac{1}{2}\left(\frac{2}{\epsilon}-2+2\ln 2\right)\left[1+\frac{2\epsilon}{3}\sum_{k=0}^{\infty}\frac{k!(\frac{1}{4}z^2)^{k+1}}{(\frac{3}{2})_k(\frac{5}{2})_k(k+1)!}\right] \\ & +\frac{\pi z}{4}\sum_{k=0}^{\infty}\frac{(\frac{1}{4}z^2)^k}{(2)_k(1+2k)k!}\Bigg) \\ = -n[-E_1(z) & -\gamma-\ln z+1-\ln 2-P(z)] \end{aligned} \tag{B27} $$

where

$$ \begin{aligned} P(z) = & \frac{2}{3}\sum_{k=0}^{\infty}\frac{k!(\frac{1}{4}z^2)^{k+1}}{(\frac{3}{2})_k(\frac{5}{2})_k(k+1)!} \\ & -\frac{\pi z}{4}\sum_{k=0}^{\infty}\frac{(\frac{1}{4}z^2)^k}{(2)_k(1+2k)k!}; \qquad P(0)=0 \end{aligned} \tag{B28} $$

$$ \begin{aligned} P'(z) &= \frac{2}{3}\,\frac{z}{2}\sum_{k=0}^{\infty}\frac{(\frac{1}{4}z^2)^k}{(\frac{3}{2})_k(\frac{5}{2})_k} - \frac{\pi}{4}\sum_{k=0}^{\infty}\frac{(\frac{1}{4}z^2)^k}{(2)_k k!} \\ &= \frac{z}{2}\,{}_1F_2(1;\ \tfrac{3}{2},\ \tfrac{5}{2};\ \tfrac{1}{4}z^2) - \frac{\pi}{4}\sum_{k=0}^{\infty}\frac{(\frac{1}{4}z^2)^k}{(k+1)!k!} \end{aligned} \tag{B29} $$

With Bateman [4], Vol. 2, pp. 38–39, Formula (57), and p. 5, Formula (12), this becomes

$$ P'(z) = \frac{z}{3}\left[\frac{\Gamma(\frac{5}{2})L_1(z)}{2\pi^{-1/2}(\frac{1}{4}z^2)}\right] - \frac{\pi}{4}\cdot\frac{2}{z}I_1(z) \tag{B30} $$

Since $\Gamma(\frac{5}{2}) = \frac{3}{4}\Gamma(\frac{1}{2}) = \frac{3}{4}\sqrt{\pi}$,

$$ P'(z) = \frac{\pi}{2z}[L_1(z) - I_1(z)] \tag{B31} $$

and

$$P(z)=\frac{\pi}{2}\int_0^z [L_1(z)-I_1(z)]\,\frac{dz}{z} \tag{B32}$$

In these formulas, $L_1(z)$ is the modified Struve function, $I_1(z)$ the modified Bessel function of order 1.

With $z=2k_2nd\sim 2k_1d$,

$$I_{12}=\frac{k_1}{k_2}\left\{E_1(2k_1d)+\gamma+\ln 2k_1d-1+\ln 2\right.$$
$$\left.+\frac{\pi}{2}\int_0^{2k_1d}[L_1(z)-I_1(z)]\,\frac{dz}{z}\right\} \tag{B33}$$

With (B33) and (B15), (B10) becomes

$$\begin{aligned}I_1&=I_{11}+I_{12}\\&=-\frac{k_1}{k_2}\left\{\cos\frac{2k_2^2d}{k_1}\left(\gamma+\ln\frac{2k_2^2d}{k_1}-\mathrm{Cin}\,\frac{2k_2^2d}{k_1}\right)\right.\\&\quad+\sin\frac{2k_2^2d}{k_1}\left(\mathrm{Si}\,\frac{2k_2^2d}{k_1}-\frac{\pi}{2}\right)\\&\quad-\left(1-\cos\frac{2k_2^2d}{k_1}\right)E_1(2k_1d)\\&\quad-\gamma-\ln 2k_1d-\ln 2+1\\&\quad\left.+\frac{\pi}{2}\int_0^{2k_1d}[I_1(z)-L_1(z)]\,\frac{dz}{z}\right\}\end{aligned} \tag{B34}$$

The integral in (B34) can be rearranged with the recurrence relation

$$z^{-1}\left[I_1(z)-L_1(z)\right]=I_0(z)-L_0(z)-\left[I_1'(z)-L_1'(z)\right] \tag{B35}$$

It follows that

$$\begin{aligned}&\frac{\pi}{2}\int_0^{2k_1d}[I_1(z)-L_1(z)]\,\frac{dz}{z}\\&\quad=\frac{\pi}{2}\left\{\int_0^{2k_1d}[I_0(z)-L_0(z)]\,dz-[I_1(2k_1d)-L_1(2k_1d)]\right\}\\&\quad=\frac{\pi}{2}\left\{f_0(2k_1d)-[I_1(2k_1d)-L_1(2k_1d)]\right\}\end{aligned} \tag{B36}$$

The functions $f_0(2k_1d)$ and $I_1(2k_1d) - L_1(2k_1d)$ are tabulated in the range $0 \le 2k_1d \le 5$ on p. 501 of Abramowitz and Stegun [5]. For large arguments,

$$\frac{\pi}{2}\int_0^{2k_1d} [I_1(z) - L_1(z)]\,\frac{dz}{z}$$
$$= \ln 2k_1d + \frac{\pi}{2}\left\{f_2(2k_1d) - [I_1(2k_1d) - L_1(2k_1d)]\right\} \quad \text{(B37)}$$

The functions $f_2(2k_1d)$ and $I_1(2k_1d) - L_1(2k_1d)$ are tabulated on p. 502 of Abramowitz and Stegun [5] in the range $0 \le (2k_1d)^{-1} \le 0.2$ or $5 \le 2k_1d \le \infty$.

Alternatively, an asymptotic formula can be derived for the integral in (B36) when k_1d is large. This is given in (54c). With it, (B34) becomes

$$I_1 \sim -\frac{k_1}{k_2}\left\{\cos\frac{2k_2^2d}{k_1}\left(\gamma + \ln\frac{2k_2^2d}{k_1} - \operatorname{Cin}\frac{2k_2^2d}{k_1}\right)\right.$$
$$+ \sin\frac{2k_2^2d}{k_1}\left(\operatorname{Si}\frac{2k_2^2d}{k_1} - \frac{\pi}{2}\right)$$
$$- \sum_{m=1}^{\infty}\frac{(2m)!(2m-1)!}{(m!)^2(4k_1d)^{2m}}$$
$$\left. + \frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(m+\frac{3}{2})}{\Gamma(\frac{1}{2}-m)(k_1d)^{2(m+1)}}\right\} \quad \text{(B38)}$$

The integral I_2 in (B9b) is evaluated with the help of (1) on p. 323 of Gradshteyn and Ryzhik [7] with $\mu = -2k_2d$, $\nu = 1$, $\rho = \frac{3}{2}$, and $u = n$. The result, with $k_2nd \sim k_1d$, is

$$I_2 = \tfrac{1}{2}B(1, \tfrac{3}{2})n^3\,{}_1F_2(1; \tfrac{1}{2}, \tfrac{5}{2}; k_1^2d^2)$$
$$- k_1dB(\tfrac{3}{2}, \tfrac{3}{2})n^3\,{}_1F_2(\tfrac{3}{2}; \tfrac{3}{2}, 3; k_1^2d^2) \quad \text{(B39)}$$

Here

$$B(1, \tfrac{3}{2}) = \frac{\Gamma(1)\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2})} = \frac{2}{3}; \qquad B(\tfrac{3}{2}, \tfrac{3}{2}) = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{\Gamma(3)} = \frac{\pi}{8} \quad \text{(B40)}$$

Hence,

$$I_2 = n^3\left[\frac{1}{3}\,{}_1F_2(1; \tfrac{1}{2}, \tfrac{5}{2}; k_1^2d^2) - \frac{\pi}{8}\,k_1d\,{}_1F_2(\tfrac{3}{2}; \tfrac{3}{2}, 3; k_1^2d^2)\right] \quad \text{(B41)}$$

The hypergeometric series can be expanded in terms of the modified Struve function L and Bessel function I. Thus, with $(1)_k = k!$,

$$
\begin{aligned}
{}_1F_2(1;\, \tfrac{1}{2},\, \tfrac{5}{2};\, k_1^2 d^2) &= 1 + \sum_{k=1}^{\infty} \frac{(k_1^2 d^2)^k}{(\frac{1}{2})_k (\frac{5}{2})_k} \\
&= 1 + \sum_{k=0}^{\infty} \frac{(k_1^2 d^2)^{k+1}}{\frac{1}{2}(\frac{3}{2})_k \times \frac{5}{2}(\frac{7}{2})_k} \\
&= 1 + \frac{4}{5}\, k_1^2 d^2 \sum_{k=0}^{\infty} \frac{(k_1^2 d^2)^k}{(\frac{3}{2})_k (\frac{7}{2})_k} \\
&= 1 + \frac{4}{5}\, k_1^2 d^2\, {}_1F_2(1;\, \tfrac{7}{2},\, \tfrac{3}{2};\, k_1^2 d^2)
\end{aligned}
\tag{B42}
$$

It follows with the first line on p. 39 of Bateman [4], Vol. 2, that

$$
\begin{aligned}
{}_1F_2(1;\, \tfrac{1}{2},\, \tfrac{5}{2};\, k_1^2 d^2) &= 1 + \frac{4}{5}\, k_1^2 d^2\, \frac{L_2(2k_1 d)\Gamma(\frac{7}{2})}{2\pi^{-1/2} k_1^3 d^3} \\
&= 1 + \frac{3\pi}{4k_1 d}\, L_2(2k_1 d)
\end{aligned}
\tag{B43}
$$

Also, with formula (12) on p. 5 of Bateman [4], Vol. 2,

$$
\begin{aligned}
{}_1F_2(\tfrac{3}{2};\, \tfrac{3}{2},\, 3;\, k_1^2 d^2) &= \sum_{k=0}^{\infty} \frac{(k_1^2 d^2)^k}{(3)_k k!} \\
&= 2 \sum_{k=0}^{\infty} \frac{(k_1^2 d^2)^k}{k!(k+2)!} \\
&= \frac{2}{k_1^2 d^2}\, I_2(2k_1 d)
\end{aligned}
\tag{B44}
$$

It follows that

$$
I_2 = \frac{k_1^3}{k_2^3} \left\{ \frac{1}{3} - \frac{\pi}{4k_1 d} \left[I_2(2k_1 d) - L_2(2k_1 d) \right] \right\}
\tag{B45}
$$

This can be evaluated with the recurrence relations (12.2.4) on p. 498 and (9.6.26) on p. 376 of Abramowitz and Stegun [5]:

$$
L_2(z) = L_0(z) - \frac{2}{z}\, L_1(z) - \frac{\frac{1}{2} z}{\sqrt{\pi}\, \Gamma(\frac{5}{2})}
\tag{B46}
$$

and

$$I_2(z) = I_0(z) - \frac{2}{z} I_1(z) \tag{B47}$$

With these formulas,

$$\begin{aligned} I_2(2k_1d) - L_2(2k_1d) = I_0(2k_1d) - L_0(2k_1d) \\ - \frac{1}{k_1d}[I_1(2k_1d) - L_1(2k_1d)] \\ + \frac{k_1d}{\sqrt{\pi}\,\Gamma(\frac{5}{2})} \end{aligned} \tag{B48}$$

The functions $I_0(2k_1d) - L_0(2k_1d)$ and $I_1(2k_1d) - L_1(2k_1d)$ are tabulated on p. 501 of Abramowitz and Stegun [5] in the range $0 \le 2k_1d \le 5$ and on p. 502 in the range $5 \le 2k_1d < \infty$.

Alternatively, use can be made of the power-series expansions of $L_2(2k_1d)$ and $I_2(2k_1d)$. These are convergent for all values of the argument. With $z = 2k_1d$, they are

$$\begin{aligned} L_2(z) &= \left(\frac{z}{2}\right)^3 \sum_{k=0}^{\infty} \frac{(\frac{1}{2}z)^{2k}}{\Gamma(k+\frac{3}{2})\Gamma(k+\frac{7}{2})} \\ I_2(z) &= \left(\frac{z}{2}\right)^2 \sum_{k=0}^{\infty} \frac{(\frac{1}{2}z)^{2k}}{k!\,\Gamma(k+3)} \end{aligned} \tag{B49}$$

When (B34) and (B45) are combined in (B8), the result is

$$\begin{aligned} J_{3g} &= \frac{3k_2^2}{k_1^2}(I_1 + I_2) \\ &= -\frac{3k_2}{k_1}\left\{\cos\frac{2k_2^2d}{k_1}\left(\gamma + \ln\frac{2k_2^2d}{k_1} - \operatorname{Cin}\frac{2k_2^2d}{k_1}\right)\right. \\ &\qquad + \sin\frac{2k_2^2d}{k_1}\left(\operatorname{Si}\frac{2k_2^2d}{k_1} - \frac{\pi}{2}\right) \\ &\qquad - \left(1 - \cos\frac{2k_2^2d}{k_1}\right)E_1(2k_1d) \\ &\qquad - \gamma - \ln 2k_1d + 1 - \ln 2 \\ &\qquad \left. + \frac{\pi}{2}\int_0^{2k_1d} [I_1(z) - L_1(z)]\,\frac{dz}{z}\right\} \\ &\quad + \frac{k_1}{k_2}\left\{1 + \frac{3\pi}{4k_1d}\,[L_2(2k_1d) - I_2(2k_1d)]\right\} \end{aligned} \tag{B50}$$

When $d = 0$,

$$J_{3g} = -\frac{3k_2}{k_1}\left(1 - \ln 2 - 2\ln\frac{k_1}{k_2}\right) + \frac{k_1}{k_2} \tag{B51}$$

For large values of $k_1 d$, asymptotic formulas for the integral in (B50) and for the term $L_2(2k_1d) - I_2(2k_1d)$ are, respectively, in (54c) and (54a,b). With these,

$$\begin{aligned} J_{3g} = -\frac{3k_2}{k_1}\Bigg\{ & \cos\frac{2k_2^2 d}{k_1}\left(\gamma + \ln\frac{2k_2^2 d}{k_1} - \operatorname{Cin}\frac{2k_2^2 d}{k_1}\right) \\ & + \sin\frac{2k_2^2 d}{k_1}\left(\operatorname{Si}\frac{2k_2^2 d}{k_1} - \frac{\pi}{2}\right) \\ & - \sum_{m=1}^{\infty}\frac{(2m)!(2m-1)!}{(m!)^2(4k_1 d)^{2m}} \\ & + \frac{1}{2}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(m+\frac{3}{2})}{\Gamma(\frac{1}{2}-m)(k_1 d)^{2(m+1)}} \\ & - \frac{1}{4k_2^2 d^2}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(\frac{3}{2}+m)}{\Gamma(\frac{3}{2}-m)\left(k_1^2 d^2\right)^m}\Bigg\} \end{aligned} \tag{B52}$$

Note that when $k_1 \to \infty$, $J_{3g} \to 0$.

Acknowledgments

The author is indebted to T. T. Wu and M. M. Weiner for many helpful discussions, to B. H. Sandler for carrying out the calculations, and to M. Owens for help in correcting the manuscript and preparing it for publication. This research was supported in part by the Joint Services Electronics Program under Grant N00014-89-J-1023 with Harvard University and in part by Project No. 91260 of the MITRE Corp.

References

[1] King, R. W. P., "New formulas for the electromagnetic field of a vertical electric dipole in a dielectric or conducting half-space near its horizontal interface," *J. Appl. Phys.*, **53**, 8476–8482, 1982; Erratum, **56**, 3366, 1984.

[2] King, R. W. P., "Electromagnetic field of a vertical dipole over an imperfectly conducting half-space," *Radio Science*, **25**, 149–160, 1990.

[3] Hansen, P., "The radiation efficiency of a dipole antenna located above an imperfectly conducting ground," *IEEE Trans. Antennas Propagat.*, **AP-20**, 766–769, 1972.

[4] Bateman Manuscript Project, *Higher Transcendental Functions*, Vols. 1 and 2, edited by A. Erdélyi, McGraw–Hill Book Co., New York, 1953.

[5] Abramowitz, M., and I. A. Stegun, Eds., *Handbook of Mathematical Functions*, Dover Publications, New York, 1972.

[6] Sommerfeld, A., and F. Renner, "Strahlungsenergie und Erdabsorption bei Dipolantennen," *Ann. der Physik*, Series 5, **41**, 1–36, 1942.

[7] Gradshteyn, I. S., and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York, 1980.