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THE PERMITTIVITY AND LOSS OF FIBER-LOADED MATERIALS

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1. Background to the Present Study

1.1 The Clausius-Mossotti Formula.

There are currently numerous mixture formulas in the literature, catering for various circumstances of loading, size and shape of loading materials, wavelength, randomization of particles, etc. Almost all these formulas reduce to a version of the Clausius-Mossotti equation which gives the effective, or bulk, permittivity ϵ_m of the mixture in terms of that of the matrix material ϵ_1 by an equation of the form

$$\epsilon = \epsilon_m/\epsilon_1 = 1 + M \tag{1.1}$$

where the quantity M depends on the loading-particle size, electrical properties and the volumetric loading factor v.

1.2 Loading Strengths

A number of regions of different loading strengths can be identified. The following can be conveniently characterized, although the nomenclature used here is not an official one:

1.2.1 Very weak

In this region the loading particles are so far apart that multiple scattering between them is negligible. Under these circumstances the quantity M in (1.1) can be written in the form

$$M = \alpha N \tag{1.2}$$

where N is the number of particles per unit volume, and α, called the *polarizability,* is a measure of the particle scattering. The particle scatters *as if* it were a point dipole, with no size or structure, though its size and shape are used in calculating α.

As an example, for a very weak loading of small spherical metallic particles, it is found that

$$M = 3v \tag{1.3}$$

where v is the geometric volumetric loading. For spheres of radius r in a cubic lattice of side D,

$$v = (4/3)\pi r^3/D^3 \tag{1.4}$$

with a maximum value of $\pi/6$ when $2r = D$. However, (1.3) breaks down well before a 50% loading occurs.

1.2.2 Weak

The transition from very weak to weak loading does not occur at a precisely defined point. When interparticle dipole-dipole scattering becomes significant, (1.2) no longer describes the situation. If M becomes "appreciable" with respect to unity, for example $M = 1/2$, multiple scattering between the loading particles leads to a modified version of (1.2);

$$M = \alpha N/(1 - A'\alpha N) \tag{1.5}$$

where A', called the *array factor*, depends on the mutual disposition of the particles. It has the value 1/3 for particles in a cubic array, and (1.3) for small metallic spheres becomes

$$M = 3v/(1 - v) \tag{1.6}$$

For example, $v = 1/7$ for $M = 1/2$; and $v = 1/4$ for $M = 1$.

1.2.3 Strong

Beyond a certain point, depending also on the polarizability, (1.5) and (1.6) fail because the particles are so close that dipolar scattering is insufficient to account for their mutual interaction. Quadrupole, octopole and higher-order terms must be included, and terms that depend on higher powers of N or v enter the denominators. Thus (1.6) becomes augmented by an octopole term (in the cubic lattice case the quadrupole term vanishes by symmetry) to give [1]

$$M = 3v/(1 - v - 1.65v^{10/3}) \tag{1.7}$$

This formula gives $M = 4.46$ for $v = 1/2$, when the spheres are very nearly touching. This region may be referred to as one of *strong* loading; and its upper limit is reached when the spherical surfaces, known as exclusion spheres, which can be thought of as just circumscribing the particles, are touching. In the case of a spherical particle the actual surface and the circumscribing spherical surface coincide. Otherwise, the particle resides within the circumscribing sphere. Clearly, the greatest volumetric loading (for a cubic lattice of spheres) is $v = \pi/6 = 0.523$, at which value, of course, still more terms would be needed in the denominator of (1.7).

1.2.4 Dense

Although the circumscribing spheres cannot get any closer than just touching, the individual particles can, provided they are not spherical. In particular, for elongated aligned particles in a regular lattice structure, much closer spacings are, in principle, possible; although physical touching can occur if the alignment is disturbed. When such touching, *on average*, is limited to perhaps one or two close neighbors, at most, the loading can be called *dense*.

1.2.5 Very dense

If the particles are so close that, on randomizing their orientations, a particle can typically touch several others, a new phenomenon called *reticulation* occurs. The particles begin to form a coherent three-dimensional network of contacting particles, and this marks a sort of phase change between a particle-loaded matrix, and a new type of material in which the role of matrix and particle material is reversed. The particles now form the body of the mixture, in which small pockets of the original matrix material are embedded; much like the gas bubbles in an expanded or foamed plastic material. The loading in such a material may be characterized as *very dense*, and is not the subject of the present study; though the approximate point of transition from dense to very dense will be discussed briefly in a later section.

It is clear that if the particles are highly elongated, the start of the dense region, as a function of volume loading, can occur at an extremely low value of *geometric* loading. For example, if the particles are cylindrical fibers with an aspect ratio of 100, the transition from strong to dense loading occurs when $v = (\pi/4)10^{-4}$, less than a hundredth of a percent—this would normally be thought of as being in the very weak category! Clearly, particle shape as well as the actual volume loading must both be considered in determining the loading region of concern. The present study is primarily concerned with the region here classified as *dense*, i.e., interpenetrating exclusion spheres.

1.3 The Exclusion Sphere

The exclusion sphere concept plays a crucial role in developing mixture formulas. As long as one has not crossed the threshold from strong to dense, i.e. just-touching exclusion spheres, the loading particles cannot touch, whatever their shape or orientation. Thus a random orientation can be considered by averaging particle orientations without consideration of particle contact, which otherwise *enormously* complicates calculations of this character. The exclusion sphere, however, comprises a much more basic consideration than that of merely preventing particle contact. To see this we must introduce the T-matrix, due to Waterman [2], which is essentially a matrix whose elements give the scattering into one order of spherical harmonic when the particle is excited by another. Most modern mixture calculations depend on this feature, which is a very powerful one, permitting the mixture analysis to be separated from the analysis of the particle properties *per se*. These are embedded in the elements of the T-matrix, and depend on the possibility of the particle field, under various conditions of excitation, being expressible in terms of out-going waves of various orders of spherical harmonics. (The different orders of spherical harmonics replace the higher multi-pole radiations of earlier analyses.) Such an expansion in out-going waves is possible outside the exclusion sphere, but *not* inside it. Thus the limitation to non-penetrating exclusion spheres is *quite essential* to the use of the T-matrix formulation; not just to prevent particle contact on angular averaging, but, very basically, just to permit the T-matrix formulation to be valid. This *electromagnetic requirement* seems not to have been remarked upon before; and attempts to apply formulas to densities in excess of those thus permitted have been made [3] without consideration of the invalidity of the application.

One of the paradoxes encountered in this may arise for certain rectangular lattices, for which the array factor A' in (1.5) can be large enough to permit the denominator to go through zero and become negative. The resulting non-physical value of ϵ clearly indicates that something is amiss.

It follows that, to be able to make any sort of valid analysis *at all* in the dense region, the T-matrix concept has to be abandoned. Angular averaging will be a *very difficult operation*, and even an analysis of aligned particles in a regular array will call for a new approach.

An initial attempt at the latter is examined in the remainder of this study. As will be seen, not only is the T-matrix concept abandoned, but also that of separation of excitation and the resulting scattering, an essential feature of the T-matrix element determination.

2. Aligned Fibers in a Rectangular Array

2.1 Approximations Used

Approximations of various sorts need to be made along the way. In the present study we are concerned with loading particles in the form of finitely conducting cylinders, of diameter $2a$ and length $2L$, in which the aspect ratio L/a is large, of the order of 100. The conductivity of the fibers, σ, may be large, but the radius is sufficiently small for the fields to substantially penetrate the body of the fiber. The fibers are small compared to wavelength, so terms of order L/λ_0 are small and usually may be neglected; here, λ_0 is the free space wavelength, typically in the microwave band.

2.2 Geometric Arrangement

The fibers are considered arranged in a rectangular array, in which the co-linear spacing, g, is necessarily greater than $2L$; but in the examples given will not be much greater than this. The equatorially adjacent fibers are separated by f, with $f \ll g$ for loadings of the order of a percent. In fact, if we take $g = 2L$, then $f = 0.18L$ for a 1 percent loading when $L/a = 100$, giving $f/g = 0.09$.

The geometric arrangement is shown in Figure 1 in which a z-polarized wave propagates along the x-axis. Particles are located at $x = mf,\ 0 \leq m < \infty;\ y = nf,\ -\infty < n < \infty;\ z = pg, -\infty < p < \infty$. A typical particle could be located anywhere in the y-z plane, but because the distribution is to $\pm\infty$ in both dimensions, all such particles are equivalent, so for convenience we take the y and z coordinates of a representative particle as both zero. Not so, however, for the x coordinate, since the front surface of the mixture is at $x = 0$; and $x = m_0 f$ will be taken as the x coordinate, where we take $m_0 \gg 1$ so as to be dealing with the bulk properties, rather than the special properties of a thin "skin" near the interface at $x = 0$. This localized region is of no interest in the present study.

2.3 The Fiber Current

We treat the fibers somewhat like short antennas, each carrying a current $I(\zeta/L)$, where ζ is an axial coordinate along the fiber, measured from the fiber center. A common process used in antenna theory is to consider the radiation as due to a hypothetical current filament of strength I located along the antenna axis. For this to be valid at the antenna surface the field calculated there needs to be the same as that due to the actual current, which is usually distributed uniformly round the antenna surface. It can be shown that this is then indeed so; but in the present investigation the current is assumed to substantially penetrate the body of the fiber, and it is not immediately obvious that the assertion still holds. The matter is analyzed in Appendix A, "Effective Wire Radius", where it is shown that, irrespective of the cross-sectional current distribution, so long as it is azimuthally uniform, the effective radius is always a: the current can validly be considered as concentrated along the axis, easing many subsequent calculations.

All fibers in a y-z plane (x = constant) are equivalent, but as x increases, the fields at the fibers, and therefore their currents, progress with the amplitude and phase of the mixture wave whose propagation constant is $k\epsilon^{1/2}$, where $k = k_0\epsilon_1^{1/2}$ is the propagation constant of the matrix. Apart from this, all fiber currents are equivalent, and we can write, for the currents in the plane $x = mf$

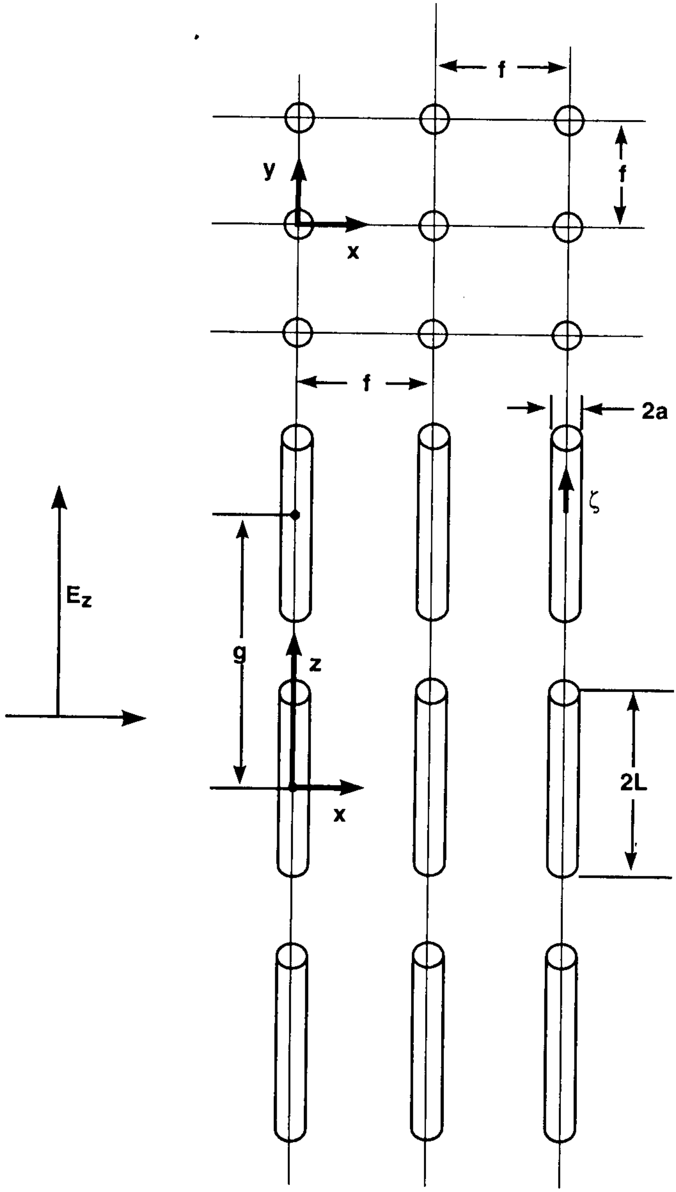

Figure 1. Cylinders in a regular rectangular lattice.

$$I_{x=mf} = I(\zeta/L)\exp[-jk\epsilon^{1/2} fm] \tag{2.1}$$

where ζ is considered measured from the center of any particle, irrespective of its placement.

On this basis, the distance between a point ζ on a particle at (mf, nf, pg) to a point on the surface of a representative particle at $(m_0f, 0, 0)$ is

$$R_{mnp} = [(m - m_0)^2 f^2 + n^2 f^2 + (pg + \zeta - z)^2 + a^2]^{1/2} \tag{2.2}$$

In this expression the very small quantity a^2 can be ignored unless $m = m_0$, $n = p = 0$; i.e. for the field produced by the representative particle on its own surface.

The Hertzian vector from a particle at (mf, nf, pg) to the representative particle at $(m_0f, 0, 0)$ is given in Appendix E. It has a single component Π_z, and in equation $(E2)$ the current is normalized by defining

$$i(\zeta/L) = -(j30/\epsilon_1 k_0) I(\zeta/L) \tag{2.3}$$

(where ϵ_1 is the relative permittivity of the matrix material) From here on, only this normalized current is used. In terms of it, the boundary condition for the tangential component of electric field E at the fiber surface is given by $(E3)$

$$E = jri \tag{2.4}$$

where $(E4)$,

$$r = \epsilon_1 / 15a^2 \sigma \lambda_0 \tag{2.5}$$

is a convenient measure, for the purposes of this study, of the fiber resistivity.

2.4 An Integral Equation

The electric field at any point is the sum of the incident field E_0e^{-jkx} and a triple radiation sum taken over all particles. To get the field from the Hertzian vector requires the operation (grad div + k^2), which reduces to $(\partial^2/\partial z^2 + k^2)$ for the z component. If we specialize the calculation to a point on the surface of the representative particle, and use the boundary condition (2.4), we get an integral equation for the fiber current:

$$jri(z/L)e^{-jk\epsilon^{1/2}m_0 f} = E_0 e^{-jkm_0 f} + (\partial^2/\partial z^2 + k^2)\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty}$$
$$\int_{-L}^{L} i(\zeta/L)\frac{e^{-jkR_{mnp}}}{R_{mnp}}e^{-jk\epsilon^{1/2}mf}d\zeta \quad ; \quad -L \le z \le L \tag{2.6}$$

3. A Variational Expression for ϵ

3.1 Approximations for the Kernel

The difficulty with (2.6) is the intractable nature of the triple sum; even with a much simpler kernel the equation has all the difficulties associated with determining an antenna current. Fortunately, in the present instance, the form of the antenna current is not itself the object of study, and providing the mixture permittivity can be suitably extracted, a variational form can be sought in which the exact form of the current does not need to be known. Since ϵ occurs primarily in the triple sum, this must first be investigated; two useful instruments for this are i) approximating a sum by an integral; and ii) the use of Poisson's theorem, which replaces a sum by a related sum of Fourier transforms. These matters are discussed in detail in Appendices B and C; the main features and conclusions are given in the ensuing sections.

3.2 The Improved Sum Formula

For smoothly varying functions the well-known Newton's formula relates a sum to an integral, with an approximate correction equal to half the function at the end points. A much improved version, the Euler-Maclaurin sum formula, gives the correction terms as a sequence of derivatives at the end points. A variant of the latter, in which the first correction term is the second-order derivative, comes from approximating the area under a curve between the limits $m-1/2$ to $m+1/2$ by the unit-width rectangle at m:

$$f(m)=\int_{m-1/2}^{m+1/2} f(x)dx + O(f''(m)) \tag{3.1}$$

This leads to the moderately accurate approximation

$$\sum_{m=M_1}^{M_2} f(m)=\int_{M_1-1/2}^{M_2+1/2} f(x)dx \tag{3.2}$$

In Appendix B a substantial improvement, which appears to be novel, is found in which (3.1) is replaced by

$$f(m)=\int_{m-1/2}^{m+1/2} \mathrm{Re}\, f(x+i\delta/2)dx + O(f^{iv}(m)) \tag{3.3}$$

where $\delta = 1/3^{1/2}$. Instead of (3.2) one finds

$$\sum_{m=M_1}^{M_2} f(m)\approx \mathrm{Re}\int_{M_1-1+(1+i\delta)/2}^{M_2+(1+i\delta)/2} f(x)dx \tag{3.4}$$

This is a considerable improvement over (3.2) and is used in Appendix C for handling sums with $f(m)=(m^2+\alpha^2)^{-1/2}$, with α a parameter determined by the details of the application. Equation (3.4) is exact for cubic curves, and its application amounts to approximating a function piece-wise by cubic segments at unit intervals.

3.3 The Triple Sum

Poisson's theorem states that

$$\sum_{n=-\infty}^{\infty} f(\alpha n)=\frac{1}{\alpha}\sum_{q=-\infty}^{\infty} F(2\pi q/\alpha) \tag{3.5}$$

where

$$F(\omega)=\int_{-\infty}^{\infty} e^{i\omega z} f(z)dz \tag{3.6}$$

This replaces one sum by another; however one may be much more slowly convergent than the other, leading to very good approximations in some cases by taking only a very few terms of the more convergent series. This happens to be the case if

$$f(z) = e^{-ik(r^2+z^2)^{1/2}}/(r^2+z^2)^{1/2} \, ,$$

for which $F(\omega) = 2K_0[r(\omega^2 - k^2)^{1/2}]$.
This is a highly attenuating function when the argument is large, leading to $(C6)$ in which the K_0 functions are further approximated by a sum handled by the method of Appendix B. There are many more such operations needed before the triple sum can be adequately approximated and these are detailed in Appendix C, leading eventually to $(C25)$:

$$\sum_{m=0}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} \frac{e^{-jkR_{mnp}}}{R_{mnp}} e^{-jk\epsilon^{1/2}fm} = \frac{-2\pi e^{-jkfm_0}}{k^2 f^2 g(\epsilon^{1/2}-1)}$$
$$+ e^{-jk\epsilon^{1/2}fm_0}\left\{\frac{4\ \pi}{k^2 f^2 g(\epsilon-1)} + \frac{2\pi g}{f^2}\left[\frac{1}{6}+\frac{\phi^2}{g^2}-\frac{|\phi|}{g}\right]\right.$$
$$\left. +(a^2+\phi^2)^{-1/2} + \frac{2}{f}\mathrm{Re}\log\left[\frac{(a^2+\phi^2)^{1/2}}{\beta+(\phi^2+\beta^2+a^2)^{1/2}}\right]\right\} \qquad (3.7)$$

where $\phi = (z-\zeta)$; $\beta = f(1+i\delta)/2$; $\delta = 1/\sqrt{3}$.

3.4 A Variational Expression

The above expression can at once be seen to consist of two distinct parts, one varying as $\exp(-jkx)$ and the other as $\exp(-jk\epsilon^{1/2}x)$, where $x = m_0 f$ is the coordinate of the representative particle. Since the *net* field in the mixture varies only as $\exp(-jk\epsilon^{1/2}x)$ the term in $\exp(-jkx)$ must somehow be removed. As is apparent from (2.6) the excitation field $E_0\exp(-jkx)$ also occurs in that equation. The *extinction* theorem requires that it be cancelled out, leading to the relation

$$0 = E_0 + \int_{-L}^{L} i(\zeta/L)d\zeta\left[\frac{-2\pi}{f^2 g(\epsilon^{1/2}-1)}\right] \qquad (3.8)$$

This equation determines the current moment in terms of the incident field, although it happens that the subsequent analysis is homogeneous in the current, so that (3.8) does not need to be used here. In what remains of (2.6) the factor $\exp(-jk\epsilon^{1/2}fm_0)$ cancels throughout, leading to

$$jri(z/L) = (\partial^2/\partial z^2 + k^2)\int_{-L}^{L} i(\zeta/L)\left\{\frac{4\pi}{k^2 f^2 g(\epsilon-1)} + F(z-\zeta)\right\} d\zeta \tag{3.9}$$

where $F(\phi) = F_1(\phi) + F_2(\phi) + F_3(\phi)$ and

$$F_1(\phi) = (a^2+\phi^2)^{-1/2} \tag{3.10}$$

$$F_2(\phi) = (1/f)\{\log(a^2+\phi^2) - 2\text{Re}\log[\beta + (\phi^2+\beta^2+a^2)^{1/2}]\} \tag{3.11}$$

$$F_3(\phi) = \frac{2\pi g}{f^2}\left[\frac{1}{6} + \frac{\phi^2}{g^2} - \frac{|\phi|}{g}\right] \tag{3.12}$$

Now the first term in the integral is independent of z, so $(\partial^2/\partial z^2 + k^2)$ reduces to just k^2. In the term in F the representative lengths are of order L or less, so $\partial^2/\partial z^2 = O(1/L^2)$, in relation to which the k^2 term is negligible. Hence (3.9) becomes, on re-arrangement,

$$\frac{4\pi}{f^2 g(\epsilon-1)} = \int_{-L}^{L} i(\zeta/L)d\zeta = jri(z/L) - \partial^2/\partial z^2 \int_{-L}^{L} i(\zeta/L)F(z-\zeta)d\zeta \tag{3.13}$$

This equation can be put in variational form by multiplying both sides by $i(z/L)$ and integrating over the fiber length:

$$\frac{4\pi}{f^2 g(\epsilon-1)} = \frac{jr\int_{-L}^{L} i^2(z/L)dz + \int_{-L}^{L}\int_{-L}^{L} i'(z/L)i'(\zeta/L)F(z-\zeta)dz d\zeta/L^2}{\left[\int_{-L}^{L} i(z/L)dz\right]^2} \tag{3.14}$$

Use has been made of an integration by parts on z and ζ, as discussed in the first section of Appendix D. And as shown in Appendix F, this form is a variational structure in which small departures of any assumed form for the current from its correct form lead to second-order errors in ϵ. Moreover, since the integral equation (3.13) is embedded in (3.14), further improvements in the value of ϵ from an assumed current form can be found. The method is explained in Appendix F.

Two further changes are made to (3.14). All lengths are normalized to L from here on, so that, for example, a now means a/L, f means f/L, etc; and use is made of the integration-variable change of Appendix D. This leads to

$$\frac{8\pi}{f^2 g(\epsilon-1)} = \frac{jrL^2\int_0^1 i^2(z)dz + \int_0^2 F(\lambda)\left[\int_0^{2-\lambda} i'\left(\frac{\mu-\lambda}{2}\right) i'\left(\frac{\mu+\lambda}{2}\right) d\mu\right] d\lambda}{\left[\int_0^1 i(z)dz\right]^2} \tag{3.15}$$

The symmetry of $i(z)$ around $z=0$, plus the fact that $i(\pm 1) = 0$, the vanishing of the current at the fiber ends, has been utilized in arriving at (3.15). The purpose of the variable change of Appendix D is to separate the double integration of z and ζ into an integration over a variable μ involving *only* the currents, and a variable λ, the original ($z-\zeta$), in F. The first integration can thus be carried out, for an assumed form of current, irrespective of the function F of (3.10) to (3.12).

Manipulations of (3.15) form the bulk of the remainder of the analysis leading to an expression for ϵ.

4. A Variational Solution for ϵ

4.1 Current Power-Series Expansion

In the quasi-static limit for a linear antenna excited by a uniform tangential field, the current produced is proportional to

$$i_{qs} = \cos kz - \cos kL \tag{4.1}$$

For kz and kL both $\ll 1$, (4.1) can be approximated by expanding the cosines to the first two terms. Apart from an irrelevant factor $L^2k^2/2$, this approximation is $i_{qs} = (L^2 - z^2)/L^2$, or if lengths are normalized to L,

$$i_{qs} = 1 - z^2 \tag{4.2}$$

This exhibits the necessary symmetry around $z = 0$, and vanishes as required at $z = \pm 1$. It is therefore the simplest and most obvious form for insertion into (3.15). In fact, if the fibers were far enough apart for their net effect to appear as a constant field, (4.2) is all that would be needed. But their close proximity ensures that the field incident on a fiber is far from uniform, and, in the earlier type of analysis this would call forth the existence of higher multipole components; or, in the case of the T-matrix, higher-order spherical harmonics. What is the equivalent of these for the present analysis? One could contemplate, for instance, an expansion in terms of a set of suitable functions $f_n(z)$, of the form

$$i(z) = (1 - z^2)\left[\sum_{n=0}^{\infty} A_n f_n(z)\right] \tag{4.3}$$

where the outside factor ensures the vanishing of $i(z)$ at $z = \pm 1$. But (3.13) gives no obvious clue to the best choice of $f_n(z)$. Perhaps the simplest, in the absence of any other guidance, is to assume a power series expansion, with the powers even, to ensure symmetry. This form will be pursued in this section, with the expansion limited here to a single additional term Az^2:

$$i(z) = (1 - z^2)[1 + Az^2] \tag{4.4}$$

However, the method could be extended to more terms, though the resulting analysis would be quite lengthy.

4.2 Current Integrals

The first step is to evaluate the μ-integration of (3.15). This is straightforward since only polynomials are involved. The integration is given in ($D8$); the resulting functiôn is denoted by $I(\lambda)$, and it can be shown that

$$\begin{aligned} I(\lambda) = (2/3)(4 - 6\lambda + \lambda^3) + 4A(4/15 - 2\lambda + 4\lambda^2 - 7\lambda^3/3 + \lambda^5/5) \\ + 2A^2(44/105 - 2\lambda + 8\lambda^2/5 + \lambda^3/3 - 2\lambda^5/5 + 2\lambda^7/35) \end{aligned} \tag{4.5}$$

Two other integrals occurring in (3.15) are

$$\left[\int_0^1 i(z)dz\right]^2 = (4/9)(1 + A/5)^2 \tag{4.6}$$

and

$$\int_0^1 i^2(z)dz = (8/15)[1 + 2A/7 + A^2/21] \tag{4.7}$$

The integration of $I(\lambda)$ with $F(\lambda)$ is more involved; it is given in Appendix G, and use is made of some appropriate approximations:

$$\begin{aligned}\int_0^2 I(\lambda)F_1(\lambda)d\lambda =& \frac{8}{3}\left[\log\frac{4}{a} - \frac{7}{3}\right] + \frac{16A}{15}\left[\log\frac{4}{a} - \frac{53}{15}\right] \\ &+ \frac{88A^2}{105}\left[\log\frac{4}{a} - \frac{3931}{1155}\right]\end{aligned} \tag{4.8}$$

$$\begin{aligned}\int_0^2 I(\lambda)F_2(\lambda)d\lambda =& \frac{-8}{3}\left[\log\frac{4\cdot 3^{1/2}}{f} - \frac{4}{3} + \frac{\pi}{6\cdot 3^{1/2}} + \frac{f}{4}\right] \\ &- \frac{16A}{15}\left[\log\frac{4\cdot 3^{1/2}}{f} - \frac{38}{15} + \frac{\pi}{6\cdot 3^{1/2}} + \frac{5f}{4}\right] \\ &- \frac{88A^2}{105}\left[\log\frac{4\cdot 3^{1/2}}{f} - \frac{2776}{1155} + \frac{\pi}{6\cdot 3^{1/2}} + \frac{35f}{44}\right]\end{aligned} \tag{4.9}$$

$$\int_0^2 I(\lambda)F_3(\lambda)d\lambda = \frac{32\pi}{45f^2g}[(3g-5) + 2(3g-7)A/7 + (5g-7)A^2/35] \tag{4.10}$$

4.3 Angular Averaging

Before inserting these results into (3.15) it is necessary to consider an aspect of angular averaging which impinges on these calculations. Since the particles are all aligned along the z-axis, no such averaging is apparently required. However, a practical material consists of fibers in random orientation. The present analysis has not been set up to take this into account; and in fact it is not currently known how to do this. One approach is therefore to ignore it completely and proceed with the aligned-fiber calculations. An alternative is to note that, when the fibers are well-separated, the impact of random averaging on an equation like (2.6) is to reduce the contribution of all the terms in the triple summation except that of the representative particle. If another fiber makes an angle θ with respect to the representative particle its contribution to the field at the particle is reduced by a factor $\cos^2\theta$, and this averaged over a sphere has the value 1/3. Thus the triple summation, apart from the term in $(m_0, 0, 0)$ is reduced by a third; or, what amount to the same thing, the other terms are to be multiplied by 3. This affects E_0 in (3.8), and r and F_1 in (3.9) and (3.10).

It is not claimed that doing this will account for the randomizing of the fiber directions. Rather, that this feature would be one component of such a calculation, and correctly gives the very weak, or perhaps weak, limit. By including it, one has a partial comparison with other formulas which include it. Although it is somewhat optional whether to do this, it will be done here. If results, absent this feature, are desired they can be found by the method of this section by omitting this factor 3 in r and F_1.

Because of a similarity of the integrals involving F_1 and F_2 it is desirable to combine them. When the factor 3 is included with F_1 the combination is written in the form $2F_1 + (F_1 + F_2)$. The term coming from $2F_1$ is dropped if the randomizing option is not followed.

4.4 Variational Solution

Adding (4.8) to (4.9), and restoring now the original meanings of a, f, etc.,i.e. writing a/L, f/L for a and f in these equations, we get

$$\int_0^2 I(\lambda)[F_1(\lambda) + F_2(\lambda)]d\lambda = \frac{8}{3}\left[\log\frac{f}{a\cdot 3^{1/2}} - 1 - \frac{\pi}{6\cdot 3^{1/2}} - \frac{f}{4L}\right]$$
$$+ \frac{16A}{15}\left[\log\frac{f}{a\cdot 3^{1/2}} - 1 - \frac{\pi}{6\cdot 3^{1/2}} - \frac{5f}{4L}\right]$$
$$+ \frac{88A^2}{105}\left[\log\frac{f}{a\cdot 3^{1/2}} - 1 - \frac{\pi}{6\cdot 3^{1/2}} - \frac{35f}{44L}\right] = H_2 \qquad (4.11)$$

The term $1/f^2g$ in (4.10) and (3.15) reverts to L^3/f^2g. Since the volumetric loading is $\pi a^2 \cdot 2L/f^2 g$ we can write

$$\frac{8\pi L^3}{f^2 g} = v(2L/a)^2 \tag{4.12}$$

Solving (3.15) for $\epsilon = \epsilon_m/\epsilon_1$ gives

$$\epsilon_m = \epsilon_1[1 + v(2L/a)^2/G] \tag{4.13}$$

where

$$G = \frac{9/4}{(1 + A/5)^2}\{jrL^2(8/5)[1 + 2A/7 + A^2/21] + H\} \tag{4.14}$$

where $H = H_1 + H_2 + H_3$ and

$$\begin{aligned} H_1 =& 2\left\{\frac{8}{3}\left[\log\frac{4L}{a} - \frac{7}{3}\right] + \frac{16A}{15}\left[\log\frac{4L}{a} - \frac{53}{15}\right]\right. \\ &\left. + \frac{88A^2}{105}\left[\log\frac{4L}{a} - \frac{3931}{1155}\right]\right\} \end{aligned} \tag{4.15}$$

H_2 is given in (4.11)

$$H_3 = (4/45)v(2L/a)^2[(3g/L - 5) + 2(3g/L - 7)A/7 + (5g/L - 7)A^2/35] \tag{4.16}$$

(The term H_1 is dropped, and r in (4.14) is divided by 3 if the angular averaging is omitted).

G is seen to be the ratio of two quadratics in A. The variational solution requires that one form the equation

$$\partial G/\partial A = 0 \tag{4.17}$$

solve for A, and insert the solution back into (4.14) and then (4.13). The resulting expression is the variational solution for ϵ_m.

4.5 Correction Term Evaluation

We can write G in the form

$$G = (P + QA + RA^2)/(1 + CA)^2 \tag{4.18}$$

where $C = 1/5$ and

$$P = jrL^2\frac{18}{5} + 12\left[\log\frac{4L}{a} - \frac{7}{3}\right] + 6\left[\log\frac{f}{a\cdot 3^{1/2}} - 1 - \frac{\pi}{6\cdot 3^{1/2}} - \frac{f}{4L}\right] + \frac{v(2L/a)^2}{5}\left(\frac{3g}{L} - 5\right) \tag{4.19}$$

$$Q = jrL^2\frac{36}{35} + \frac{24}{5}\left[\log\frac{4L}{a} - \frac{53}{15}\right] + \frac{12}{5}\left[\log\frac{f}{a\cdot 3^{1/2}} - 1 - \frac{\pi}{6\cdot 3^{1/2}} - \frac{5f}{4L}\right] + \frac{2v(2L/a)^2}{35}\left(\frac{3g}{L} - 7\right) \tag{4.20}$$

$$\begin{aligned} R = {} & jrL^2\frac{6}{35} + \frac{132}{35}\left[\log\frac{4L}{a} - \frac{3931}{1155}\right] \\ & + \frac{66}{35}\left[\log\frac{f}{a\cdot 3^{1/2}} - 1 - \frac{\pi}{6\cdot 3^{1/2}} - \frac{35f}{44L}\right] \\ & + \frac{v(2L/a)^2}{175}\left(\frac{5g}{L} - 7\right) \end{aligned} \tag{4.21}$$

$\partial G/\partial A = 0$ gives $A = (2CP - Q)/(2R - CQ)$; whence, after some elementary reduction, it is found that

$$1/G = \frac{1}{P}\left[1 + \frac{(2CP - Q)^2}{4PR - Q^2}\right] \tag{4.22}$$

This form is interesting in itself, because if the term in A had not been included the result would have been simply $G = P$. Hence the term $(2CP - Q)^2/(4PR - Q^2)$, compared to unity, shows the relative importance of the term in Az^2 in (4.4), and hence the likelihood of (4.22) being an accurate form. It happens that the expression $(2CP - Q)$ is especially simple, and is given by

$$(2CP - Q) = \frac{12}{5}\left[jrL^2\frac{6}{35} + \frac{12}{5} + \frac{f}{L} + v(2L/a)^2\frac{g}{35L}\right] \tag{4.23}$$

But $4PR - Q^2$ does not reduce to anything compact, and hence is left to be determined numerically from (4.19) to (4.21).

5. Spatial Averaging

5.1 Lattice Displacements

The results of the preceding sections contain no machinery for spatial averaging. A realistic material will not only have fibers at all angles, but they will spatially overlap as well. The arrangement considered so far, essentially parallel layers of aligned particles, has no overlap at all. It is possible to allow for overlap by interleaving the lattice with another one spaced $g/2$ in the z-direction, but this would involve a non-physical interpenetration unless the second lattice were also displaced laterally: for example a $g/2$ z-displacement *plus* an $f/2$ y-displacement as in Figure 2. The antenna currents would be altered, but would still be equivalent, by symmetry, in the two sub-lattices. However, the double lattice would now be more sparse in the x-direction than in the y-direction. One can overcome this by having also a similar $g/2$ z-displaced lattice displaced $f/2$ in the x-direction, but now the currents in the sub-lattices would no longer be equivalent. The balance can be restored by considering the original lattice, with no z-displacement, but with $f/2$ displacements in both the y and x directions. Figure 3 attempts to show this by giving a cross-section in a plane z constant. The shaded cross-sections are z-displaced by $g/2$ and the unshaded ones are not. All antenna currents are equivalent, since they have correspondingly arranged neighbors.

5.2 Modified Formulas

The effects of the displaced lattices are discussed in Appendix H. Essentially, there is no net term corresponding to $F_1(\phi)$ of (3.10); and the term corresponding to (3.11) is negligible, so there is no $F_2(\phi)$ term either. The term on the left of (3.14) containing ϵ is multiplied by 4, one for each of the four sub-lattices. But since the volume loading, for the same f and g, also increases by 4 the term in v in (4.13) stays the same, though (4.12) is replaced by

$$8\pi L^3/f^2 g = v(L/a)^2 \tag{5.1}$$

The big difference comes in the $F_3(\phi)$ of (3.12); for the original lattice and the x-y displaced lattice it is the same, but for the two z-displaced lattices it takes the form

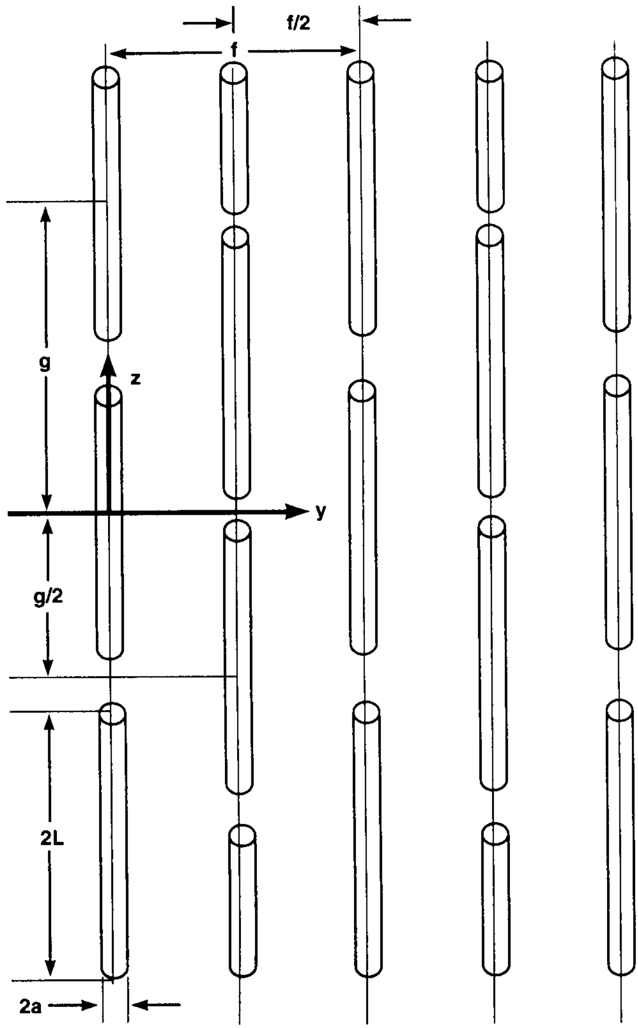

Figure 2. Two double-displaced lattices in $y - z$-plane.

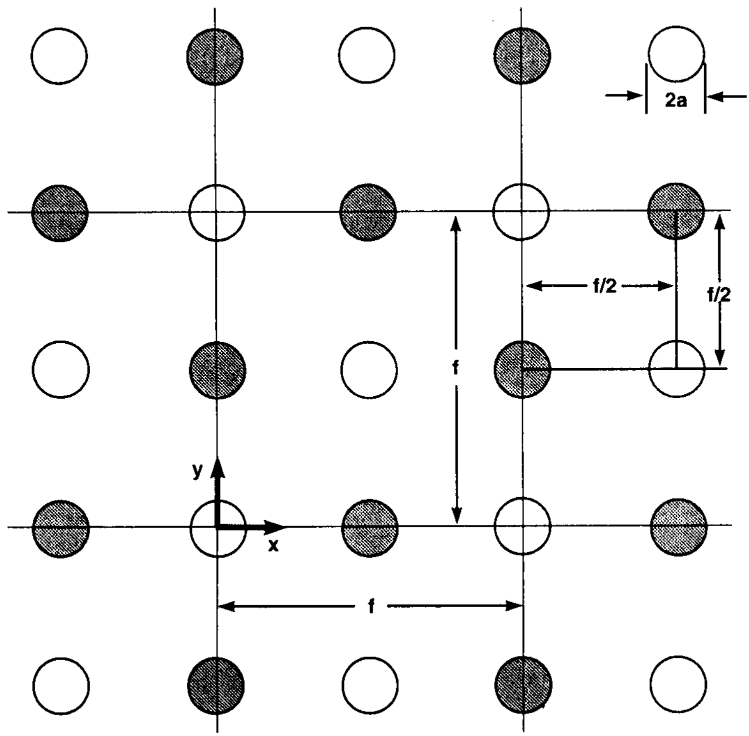

Figure 3. Nest of four lattices; shaded are z−displaced.

$$\overline{F}_3(\phi) = \frac{2\pi g}{f^2}\left[\frac{1}{6} + \frac{\phi_m^2}{g^2} - \frac{|\phi_m|}{g}\right] \tag{5.2}$$

where $\phi_m = (z - \zeta \pm g/2)$, with the $\pm$ sign chosen so that $|\phi_m| \leq g$. This affects (4.10) which becomes considerably more complicated in form. However, near $g = 2$, which is the only region of interest here, it takes the approximation

$$\int_0^2 I(\lambda)\overline{F}_3(\lambda)d\lambda$$
$$= -\frac{4\pi}{45f^2g}\left[7\left(\frac{3g}{L} - 5\right) + 26\left(\frac{3g}{L} - 7\right)A/7 + 4\left(\frac{5g}{L} - 7\right)A^2/35\right] \tag{5.3}$$

The initial factor $1/f^2g$, when put in terms of the volume, is reduced by 4 because of (5.1). Hence the term corresponding to H_3 of (4.16) is just the average for the four lattices, and takes the form

$$\overline{H}_3 = (1/180)v(2L/a)^2$$
$$\left[\left(\frac{3g}{L} - 5\right) - 10\left(\frac{3g}{L} - 7\right)A/7 + 6\left(\frac{5g}{L} - 7\right)A^2/35\right] \tag{5.4}$$

The only terms affected in P, Q, and R of (4.19) to (4.21) are the final ones in v, which take respectively, additional factors

$$1/16, \ -5/16, \ 6/16 \tag{5.5}$$

Correspondingly, the term in v in (4.23) for $2CP - Q$ has the factor $g/35L$ replaced by

$$(g/35L - 1/16) \tag{5.6}$$

(This goes through zero and changes sign when $g = 35/16 = 2.19$; but for practical purposes it will not be significantly less than zero.)

6. Results and Discussion

6.1 Formula Structures

It is apparent from the later equations of section 4 that the formulas are too complicated to be easily read. Nevertheless some general conclusions can be made. The *structure* of (3.15) is essentially of the Claussius-Mossotti type; this is also apparent from (4.13) and (4.14). From discussions in a previous report [4], agreement with measurements cannot be obtained with this structure, no matter what values may be given to the various parameters. In fact, the new formulas, despite the quite extensive work put into their generation, are remarkably similar to the earlier ones. Thus, from reference 4, equation (10), we have, (using the present notation)

$$\epsilon_m/\epsilon_1 - 1 = \alpha' v/(1 - A'\alpha' v) \tag{6.1}$$

were A' is the array factor and

$$\alpha' = (2L^2/9a^2)/\{[\log(4L/a) - 7/3] + j\epsilon_1 L^2/(a^2 60\lambda_0\sigma)\} \tag{6.2}$$

The differences between this and (4.13) boil down to:
i) A factor 4/5 in the resistive term; this is due to using the average of the square of the current rather than simply the current, as in the earlier formulation. The effect is minor, but reduces the already too low imaginary part of ϵ_m by 20%.
ii) The presence in (4.14) of the term in A coming from the assumed current form $(1 - z^2)(1 + Az^2)$, whereas (6.2) has effectively ignored this improvement. It will be discussed in more detail later.
iii) A determination of the array factor, which is implicit in (4.16). It too will be discussed in more detail later.
iv) The polarizability term $[\log(4L/a) - 7/3]$, which appertains to an isolated cylinder, is replaced by the more complicated terms in H_1 and H_2 of (4.11) and (4.15). These terms include both proximity effects from neighboring fibers, and also the terms in A coming from the assumed form of fiber current.

Because of the complicated form of the equations, much of the remaining discussion will be based on numerical computations.

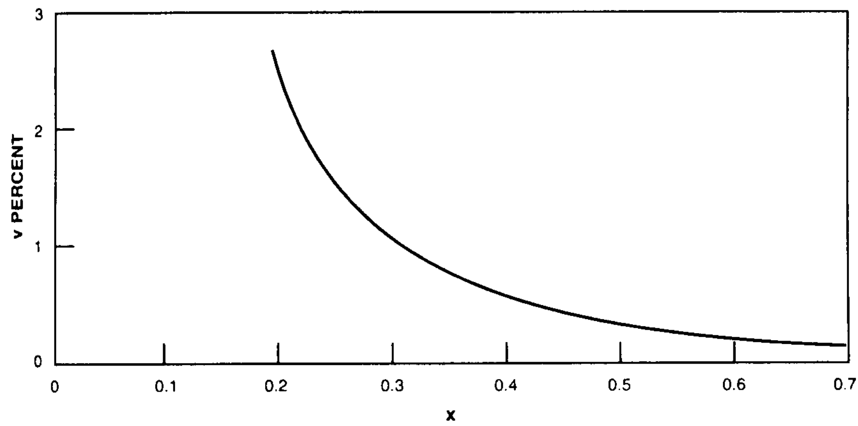

Figure 4. Plot of v versus $x = f/L; L/a = 100$; 4 lattices.

6.2 Volumetric Loading

Taking the four-lattice structures of section 5 leads to (5.1) for the volumetric loading v in terms of other parameters, principally the lattice spacings f and g. These are independent parameters, and in order to provide a dense structure, g should be taken as close as practical to its minimum value $2L$. However, from approximations used earlier, the spacing between the tips of adjacent co-linear fibers should be of the same order as their equatorial spacing, so we take $g = 2L+2f$ as a relation between these quantities. If we write $f/L = x$ then (5.1) becomes

$$v = 4\pi(a/L)^2/x^2(1+x) \tag{6.3}$$

A plot of this relation between v and x is shown in Figure 4 for the case $L/a = 100$, a value used throughout this study. Two particular values selected for further calculations are $x = 0.3$, $v = 1.07$ percent, and $x = 0.5$, $v = 0.335$ percent.

6.3 The Case $f/L = 0.3$, $v = 1.07$ *Percent*

The value of P, bearing in mind the changes due to the four-lattice structure given by (5.5), is calculated as

$$P = 67.75 + j0.82 \tag{6.4}$$

In calculating the imaginary part, the values $\epsilon_1 = 2.05$, $\lambda_0 = 7.5$ cm and $\sigma = 8 \cdot 10^4$ mho/m have been used. The value of ϵ_m comes out as

$$\epsilon_m = 15.05 - j0.161 \tag{6.5}$$

This ignores the current term in A. If this is included we find, from (4.22), an additional factor ($1.073 - j2.5 \cdot 10^{-4}$), to give

$$\epsilon_m = 16.0 - j0.172 \tag{6.6}$$

The corresponding value of A (neglecting the minute imaginary part) is

$$A = 0.55 \tag{6.7}$$

This value of A is far from negligible, but it produces only a 7 percent increase in the calculation of ϵ_m; apart from this the increase in the imaginary part is almost negligible, affecting only the third significant figure. This latter is hardly surprising since, physically, the fiber resistance would not be expected to have much influence on the *form* of the fiber current.

The value of ϵ_m from (6.1), neglecting for the moment the array factor A', would have been $\epsilon_m = 15.4 - j0.209$, not too different from (6.5) or (6.6). However, the value of $\alpha' v$ is 6.52, and if the term $A'v$ in (6.1) is included, with $A' = 1/3$ for a cubic lattice (this seems, from symmetry, the only appropriate form to choose) then the denominator in (6.1) goes negative and the ensuing result is non-physical. The question of lattice array factor will be further discussed later. Chalupa [5] suggests an empirical value of $1/15$, which keeps the denominator in (6.1) positive for this value of v, but which would eventually lead to non-physical effects at higher loading. However, none of these features helps with the very small imaginary part, which needs to be compared to measured values of about

$$\epsilon_m(\text{measured}) = 22 - j5 \tag{6.8}$$

At this concentration an additional factor of about 30 is needed for the imaginary term in (4.14).

6.4 The Case $f/L = 0.5, v = 0.335$ Percent

The value of P, modified by (5.5), is

$$P = 62.21 + j0.82 \tag{6.9}$$

The additional factor from (4.22) is $1.057 - j1.1 \times 10^{-4}$, not too different from the case $f/L = 0.3$, and the resulting ϵ_m is

$$\epsilon_m = 6.72 - j0.062 \tag{6.10}$$

to be compared with

$$\epsilon_m(\text{measured}) = 7.42 - j0.38 \tag{6.11}$$

(obtained by quadratic interpolation from neighboring measured values, given in reference 4) .

The value of the current term A is

$$A = 0.51 \tag{6.12}$$

not very different from the case $f/L = 0.3$.

To account for the resistive term an additional factor in r of about 6 is needed. For what it may be worth, such an empirical factor to resolve these discrepancies could be constructed in the form

$$1 + 2.5(L/a)^2 v^{3/2} \tag{6.13}$$

This reduces to unity for weak concentrations, but otherwise has no obvious theoretical basis.

6.5 The Current Correction Term

One of the features of the last five sections is the value of the current term A, a little over 0.5 in both cases, dropping slowly for smaller volumetric concentrations. The question can be raised as to whether this value is much influenced by the dense concentrations (recalling that the "dense" region starts near $v = 10^{-4}$), or whether it is a carryover from the current form in the weak region. The latter cannot be found from the formulas of this study because approximations pertaining to the dense region are involved. But the calculations of the appendix of reference 4 give the charge density on an isolated cylinder in the form $\tau(z) = z(1 - \overline{A}z^2/L^2 + \overline{B}z^4/L^4)$. (Note that the notation of the reference has been altered slightly to prevent confusion). The coefficients $\overline{A}$ and $\overline{B}$ are given in the reference, and were there calculated by comparing coefficients of power expansions near the origin. Although this is a different procedure from the variational one used in section 3, it should be expected to give comparable results.

The current on the fiber is obtained by integrating the charge density, with the integration constant determined by the vanishing of the current at the fiber ends. The details are given in Appendix I, where the value of A corresponding to this integration is given by

$$A = \frac{\log(2L/a) - 253/120}{3\log^2(2L/a) - (227/20)\log(2L/a) + 56/5} \tag{6.14}$$

When $L/a = 100$ this gives $A = 0.142$, about a quarter of the value in the dense region. The balance must therefore be presumed due to the dense nature of the structure.

6.6 The Array Factor

An important consideration is the array factor, which has the value 1/3 for a cubic lattice. A genuine cubic lattice is not possible for aligned cylinders because of interpenetration. Probably the nearest that could be constructed would be a 'nest' of sub-lattices with the colinear displacement of neighboring sub-lattices equal to the equatorial spacing—this is clearly not the case for the structure analyzed here.

In reference 3 the lattice array factor for a rectangular lattice of aligned cylinders is given, and it is seen that, as the lattice becomes more compressed laterally the array factor drops, goes through zero and becomes negative. Thus the non-physical change of sign of the denominator of the Clausius-Mossotti type of formula is obviated in this region. This appears to be the sort of result generated here where the lattice is also of the laterally compressed character. But, as discussed in reference 4, the array factor calculation of reference 3 is formally invalid at these high concentrations because of the exclusion sphere interpenetration that would be involved. The calculations involve dipolar terms used way outside their legitimate range. Nevertheless, the calculations seem to be giving usable results in a region where their use is technically invalid. Why should this be so? The reason seems to be connected with the process whereby the multipolar (or spherical harmonic) analysis fails inside the surface of the exclusion sphere. If *all* the relevant harmonics are retained, then the resulting series of terms becomes divergent at and inside the surface. But if only the first few are used, and, in particular, if the dipolar term predominates, there may be no *apparent* divergence. Now there is known to exist a large class of divergent asymptotic expansions in which the early terms decrease, and only eventually do they increase to give a divergent result. The error resulting from stopping the series at a particular early term, while the terms are still decreasing, can be shown to be less than that term. If the situation should be at all similar with the spherical harmonic expansion inside the exclusion sphere, then the dipole term alone may be a much better representation of the field, even inside the exclusion sphere, than one has a right to expect. This sort of conclusion may not be relevant when one is dealing with a slowly divergent series of decreasing terms like, say, the harmonic series $1/n$; but at this time nothing is known about the divergence of the spherical harmonic series inside the exclusion sphere. All that is known is that the Clausius-Mossotti formula seems to hold, well above $v = 10^{-4}$ (for $L/a = 100$), although the dipolar calculations are not properly usable there. The above discussion may partially explain why one can seem to be getting away with it well into the dense region, at least for the real part of ϵ_m.

6.7 Dielectric Loss

The loss term, giving the imaginary part of ϵ_m, is clearly very poorly handled by these calculations, even at the 0.1 percent level. The reason seems to be due to the interparticle contact that can occur, even at $v = 10^{-4}$. Chalupa [6] has shown that at about $v = a/L$, or 1 percent in the present instance, there is so much contact that can occur, on averaging the particle orientations, that it heralds the onset of reticulation. Measurements show that somewhere between 1 and 2 percent the DC resistance of a sample of the material drops suddenly from many megohms to kilohms, indicating the presence of a continuous, even though tenuous, web of conducting material through the mixture.

The present analysis, designed to provide an intial attempt at investigating a mixture in the dense region, was not intended to handle non-aligned orientations; and, in fact, it is not currently known how to tackle this problem. The most that can be concluded at this time is that inter-particle contact, which starts to occur as soon as the dense region is entered, is primarily responsible for the anomalously high losses. The use of dipolar and similar expansion terms inside the exclusion sphere, where their use is nominally invalid, would appear not to be the prime cause.

6.8 Non-Random Alignment

Although the present analysis cannot handle the randomly aligned fibers, it will be recalled that a primitive attempt at averaging, leading to a factor of 1/3, was introduced in section 4.3. It is not difficult to negate this process. For $f/L = 0.3$, ($v = 1.07$ percent) the calculation then gives $\epsilon_m = 38.8 - j0.42$, to be compared to the measured value $\epsilon_m = 22 - j5$. The formula now grossly overestimates the real part, but the major discrepancy between the imaginary parts remains. It is concluded that this partial averaging, though weakly justified, is nevertheless warranted; though it is quite unable to take inter-particle contact and the onset of reticulation into account.

7. Conclusions

From the results and discussion of Section 6 the following tentative conclusions have been drawn.

I) Prior calculations using the T-matrix or multipole methods may be usefully extended into the dense region, even though they are technically invalid there because of interpenetration of exclusion spheres, upon which the calculations are based.

II) When the particles can touch, which will always be so in the dense region, the loss term will be in excess of any of these calculations, which do not take inter-particle contact into account. When the amount of contact is excessive a phase change occurs in which a three-dimensional web of conducting material forms in the mixture.

III) The loss term cannot be assessed without taking inter-particle

contact adequately into account.
IV) There is a substantial change in the current form due to proximity effects, but the effect of this on the variational calculation is minor, even in the dense region.
V) A negative array factor can be properly generated, which prevents the non-physical negative permittivity that can be found with the Clausius-Mossotti formula.
VI) The methods of this study, although providing a valid solution in the dense region, necessarily produces a formula of the Clausius-Mossotti character, which is known to be unable to explain the measured results. The missing feature is, of course, the inter-particle contact on randomizing particle orientations. To make further progress, this is the feature that will have to be studied further.

Appendix A: Effective Wire Radius

Because most germane calculations in the literature pertain to current distributions on the surface of cylinders, it is necessary to evaluate the effective wire radius for current density distributions that are uniform in angle, but may vary with radial depth.

It is only the near-field that is of concern, and this is obtainable from a potential that varies as $1/R$ near the origin, where

$$R = (z^2 + r^2 + a^2 - 2ar\cos\theta)^{1/2} \qquad (A1)$$

and

r refers to a point at radius r

a refers to a point on the cylinder surface, where $r = a$

z is an axial coordinate

θ is an azimuthal angle

In all subsequent usages there is an integration with respect to z over a region, taken here from $-L$ to $+L$, where $L \gg a$. The current density variation as a function of z is not important unless it varies rapidly with z over a region of size of the order of a. In practice this only happens close to the cylinder end, where the current is zero in any case. Thus, for all practical purposes, we are interested in an integral of the form

$$I = \int_{-L}^{L} \int_{0}^{2\pi} \int_{0}^{a} \frac{f(r/a) dz d\theta r dr}{(z^2 + r^2 + a^2 - 2ar\cos\theta)^{1/2}} \tag{A2}$$

where $f(r/a)$ represents the current density variation with radius. Put $r = au$ as change of radial variable, and integrate with respect to z, using $L \gg a$:

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^1 d\theta a^2 f(u) u \left[2\log\frac{2L}{a} - \log(1 - 2u\cos\theta + u^2) \right] du \\ &= 4\pi a^2 \log\frac{2L}{a} \int_0^1 u f(u) du + I_1 \end{aligned} \tag{A3}$$

where

$$\begin{aligned} I_1 &= -a^2 \int_0^{2\pi} \int_0^1 u f(u) \log\left[\left(1 - ue^{i\theta}\right) \left(1 - ue^{-i\theta}\right) \right] d\theta du \\ &= a^2 \int_0^{2\pi} \int_0^1 u f(u) \sum_{n=1}^{\infty} \frac{2u^n \cos n\theta}{n} d\theta du \\ &= 0 \end{aligned}$$

on carrying out the θ integration.

Hence $I = C \log\frac{2L}{a}$ where $C = 4\pi a^2 \int_0^1 u f(u) du$ is proportional to the total current. The radius a enters the formula only in the term $\log(2L/a)$, which is the form taken also when the current flows only on the cylinder surface (for which $f(u) = \delta(1-u)$). *Hence, irrespective of the form of the radial current density variation*, the effective radius of the cylinder is the actual geometric radius.

Appendix B: An Improved Sum Formula

The Euler-Maclaurin sum formula replaces a sum by an integral plus correction terms involving multiple derivatives. A somewhat improved version, which absorbs derivatives of order less than the second, comes from the approximation

$$f(m) = \int_{m-1/2}^{m+1/2} f(x) dx + O(f''(m)) \tag{B1}$$

An improved formula, believed to be new, similarly absorbs derivatives less than the fourth order, and gives excellent approximations provided the function $f(x)$ does not vary too rapidly within the range of concern.

Let $C, \alpha, \beta,$ and θ be parameters, and define

$$g(\theta) = \int_{m-\theta}^{m+\theta} [Cf(x+\alpha) + (1-C)f(x+\beta)]dx \qquad (B2)$$

Differentiating gives

$$g'(\theta) = C[f(m+\theta+\alpha)+f(m-\theta+\alpha)]+(1-C)[f(m+\theta+\beta)+f(m-\theta+\beta)] \qquad (B3)$$

Expanding this in a Taylor series around m gives, after some simplification,

$$\begin{aligned} g'(\theta) = {} & C[2f(m) + 2\alpha f'(m) + (\theta^2+\alpha^2)f''(m) + \alpha(\theta^2+\alpha^2/3)f'''(m)] \\ & + (1-C)[2f(m) + 2\beta f'(m) + (\theta^2+\beta^2)f''(m) + \\ & + \beta(\theta^2+\beta^2/3)f'''(m)] + \text{terms of order } f^{iv}(m) \end{aligned} \qquad (B4)$$

Integrate $(B4)$ with respect to θ and then take $\theta = 1/2$. The reason for choosing $\theta = 1/2$ is to produce integrals of the form $(B1)$ which, when added for an integer sequence for m, combine to give an integral over an extended range without breaks, since the lower limit of one integral equals the upper limit of the preceding one. The θ integration thus gives, at $\theta = 1/2$,

$$\begin{aligned} g(1/2) = {} & \\ & C\left[f(m) + \alpha f'(m) + \frac{1}{24}(1+12\alpha^2)f''(m) + \frac{\alpha}{24}(1+4\alpha^2)f'''(m)\right] \\ & + (1-C)[f(m) + \beta f'(m) + \frac{1}{24}(1+12\beta^2)f''(m) \\ & + \frac{\beta}{24}(1+4\beta^2)f'''(m)] + O(f^{iv}(m)) \end{aligned} \qquad (B5)$$

$$\begin{aligned} & = f(m) + f'(m)\,[C\alpha + (1-C)\beta] \\ & + \frac{f''(m)}{24}\,[C(1+12\alpha^2) + (1-C)(1+12\beta^2)] \\ & + \frac{f'''(m)}{24}\,[\alpha C(1+4\alpha^2) + \beta(1-C)(1+4\beta^2)] + O(f^{iv}(m)) \end{aligned} \qquad (B6)$$

The object is to choose α, β and C to remove the lower-order derivatives. Setting to zero the coefficients of $f'(m)$, $f''(m)$, and $f'''(m)$ gives

i)

$$C\alpha + \beta - C\beta = 0 \text{ or } C = \beta/(\beta - \alpha),\ 1 - C = -\alpha/(\beta - \alpha) \tag{B7}$$

ii)

$$C\alpha^2 + (1 - C)\beta^2 + 1/12 = 0.$$

On elimination of C from ($B7$), this reduces to

$$\alpha\beta = 1/12 \tag{B8}$$

iii)

$$\alpha C(1 + 4\alpha^2) + \beta(1 - C)(1 + 4\beta^2) = 0.$$

On using (B7) this reduces to

$$-4\alpha\beta(\alpha + \beta) = 0, \text{ or } \alpha = -\beta \tag{B9}$$

in view of ($B8$). Combining ($B8$) and ($B9$) gives

$$\alpha = -\beta = i(1/12)^{1/2};\ C = 1/2 \tag{B10}$$

Define

$$\delta' = (1/12)^{1/2} \tag{B11}$$

Then $g(1/2) = \frac{1}{2}\int_{m-1/2}^{m+1/2}[f(x+i\delta')+f(x-i\delta')]dx = f(m)+O(f^{iv}(m))$. To the extent that terms of order $f^{iv}(m)$ can be neglected, this result can be written in the form

$$f(m) = \frac{1}{2}\int_{m-1/2}^{m+1/2}[f(x+i\delta')+f(x-i\delta')]dx \text{ or } Re\int_{m-1/2}^{m+1/2} f(x+i\delta')dx \tag{B12}$$

This is the improved version of ($B1$) sought. In particular,

$$\sum_{m=1}^{M} f(m) = \mathrm{Re}\int_{1/2}^{M+1/2} f(x + i\delta')dx \tag{B13}$$

With a slight change of variable this gives the preferred form

$$\sum_{m=1}^{M} f(m) = \mathrm{Re} \int_{(1+i\delta)/2}^{M+(1+i\delta)/2} f(x)dx \qquad (B14)$$

where

$$\delta = (1/3)^{1/2}; (1+i\delta)/2 = e^{i\pi/6}/(3)^{1/2} \qquad (B15)$$

A severe test of $(B14)$ is the harmonic series $f(x) = 1/x$ for which $\lim_{M\to\infty} \sum_{m=1}^{M} \frac{1}{m} = \log M + \gamma$, with $\gamma = 0.5772$. Equation $(B14)$ gives

$$\begin{aligned} \mathrm{Re}\ \log(x)\Big|_{(1+i\delta)/2}^{M+(1+i\delta)/2} &= \mathrm{Re}[\log M - \log \frac{1+i\delta}{2} + O(1/M)] \\ &= \log M - \frac{1}{2}\log\frac{1+\delta^2}{4} \\ &= \log M + \frac{1}{2}\log 3 \text{ as } M \to \infty. \end{aligned}$$

Hence the comparison is between $\gamma = 0.5772$ and $1/2 \log 3 = 0.5493$, an error of about 4.5 percent. In contrast, $(B1)$ would have given $\log 2 = 0.6931$, a 20 percent error.

Note: The operation Re in $(B14)$ is independent of any other use of complex quantities, such as j (used for harmonic time variation). Similarly for multiple summations, leading to the corresponding multiple integrals. One can use a system of multi-complex numbers with imaginaries i_n such that $i_n^2 = -1$ but $i_n i_p \neq -1$ if $n \neq p$; or else take the real part at each summation before proceeding to the next one. If i and j both occur in a formula, note that $ij \neq -1$.

Appendix C: The Lattice Sum

C_1: *Triple Summation*

The required lattice sum is

$$S = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \frac{e^{-jkR_{mnp}}}{R_{mnp}} e^{-jk\epsilon^{1/2}mf} \qquad (C1)$$

where

$$R_{mnp} = [(m - m_0)^2 f^2 + n^2 f^2 + (pg + \zeta - z)^2 + a^2]^{1/2} \tag{C2}$$

We seek a good approximation to S contingent on the inequalities $a \ll f$, $f \ll g$, $kg \ll 1$. Use is made of Poisson's theorem

$$\sum_{n=-\infty}^{\infty} f(\alpha n) = \frac{1}{\alpha} \sum_{q=-\infty}^{\infty} F\left(\frac{2\pi q}{\alpha}\right) \tag{C3}$$

where

$$F(\omega) = \int_{-\infty}^{\infty} e^{j\omega z} f(z) dz \tag{C4}$$

In particular, if $f(z) = e^{-jk(z^2+r^2)^{1/2}}/(z^2 + r^2)^{1/2}$ then

$$\begin{aligned} F(\omega) &= -j\pi H_0^{(2)}[r(k^2 - \omega^2)^{1/2}]; \omega^2 < k^2 \\ &= 2K_0[r(\omega^2 - k^2)^{1/2}]; \omega^2 > k^2 \end{aligned} \tag{C5}$$

Using $(C3)$ and $(C5)$, with i replaced by j, gives the initial sum

$$\alpha \sum_{n=-\infty}^{\infty} \frac{e^{-jk(r^2+\alpha^2 n^2)^{1/2}}}{(r^2 + \alpha^2 n^2)^{1/2}} = -j\pi H_0^{(2)}(kr) + 4 \sum_{q=1}^{\infty} K_0[r(4\pi^2 q^2/\alpha^2 - k^2)^{1/2}] \tag{C6}$$

We shall later be taking $\alpha = f$, and since $kf \ll 1$, k is negligible in the K_0 series in $(C6)$. This series can therefore be found by taking the limit as $k \to 0$ using $H_0^{(2)}(kr) \approx 1 - j\frac{2}{\pi} \log\left(\frac{kre^\gamma}{2}\right)$. After a little manipulation this permits $(C6)$ to be re-written as

$$\begin{aligned} \alpha \sum_{n=-\infty}^{\infty} \frac{e^{-jk\sqrt{r^2+\alpha^2 n^2}}}{\sqrt{r^2 + \alpha^2 n^2}} &\approx -j\pi H_0^{(2)}(kr) + \frac{\alpha}{r} + 2\log\frac{re^\gamma}{2\alpha} \\ &+ 2\sum_{n=1}^{\infty} \left(\frac{\alpha}{\sqrt{r^2 + \alpha^2 n^2}} - \frac{1}{n}\right); \; k\alpha \ll 1 \end{aligned} \tag{C7}$$

The sum on the right of $(C7)$ can be approximated using the method of Appendix B, leading to

$$\alpha \sum_{n=-\infty}^{\infty} \frac{e^{-jk\sqrt{r^2+\alpha^2 n^2}}}{\sqrt{r^2+\alpha^2 n^2}} \approx -j\pi H_0^{(2)}(kr) + \frac{\alpha}{r} + 2\mathrm{Re}\,\log\left[\frac{r}{\beta + (r^2+\beta^2)^{1/2}}\right] \tag{C8}$$

where $\beta = \alpha(1+i\delta)/2; \delta = 1/3^{1/2}; r > 0; k\alpha \ll 1$.

C_2 : Hankel Function Summation

A related Poisson sum for the Hankel function can be put in the form

$$\sum_{n=-\infty}^{\infty} H_0^{(2)}[k(z^2+(ng-y)^2)^{1/2}]$$
$$= \frac{2}{kg} e^{-jk|z|} + 2j \sum_{m=1}^{\infty} \cos\left(\frac{2m\pi y}{g}\right) \frac{e^{-|z|\left(\frac{4m^2\pi^2}{g^2}-k^2\right)^{1/2}}}{(m^2\pi^2 - k^2 g^2/4)^{1/2}} \tag{C9}$$

With $kg \ll 1$ the right-hand side of $(C9)$ can be approximated by

$$\frac{2}{kg} e^{-jk|z|} + \frac{2j}{\pi} \mathrm{Re} \sum_{m=1}^{\infty} \frac{e^{-2m\pi(|z|+iy)/g}}{m},$$

where Re refers to the field of i, and $|z|$ is used to mean the positive value of $(z^2)^{1/2}$. On summing the series it is found that

$$\sum_{n=-\infty}^{\infty} H_0^{(2)}[k(z^2+(ng-y)^2)^{1/2}] \approx \frac{2}{kg} e^{-jk|z|} - \frac{2j}{\pi} \mathrm{Re}\,\log[1-e^{-\frac{2\pi}{g}(|z|+iy)}] \tag{C10}$$

subject to $kg \ll 1$.

C_3 : Double Summation

Returning to $(C6)$, take $\alpha = f$, $r = [\theta^2 + (pg-\phi)^2]^{1/2}$, $\theta = [a^2 + (m-m_0)^2 f^2]^{1/2}$, $\phi = z - \zeta$, and sum over p, $(-\infty, \infty)$ to get

$$\sum_{p=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{e^{-jkR_{mnp}}}{R_{mnp}}=\sum_{p=-\infty}^{\infty}\left\{\frac{-j\pi}{f}H_0^{(2)}[k(\theta^2+(pg-\phi)^2)^{1/2}]\right.$$
$$\left.+\frac{4}{f}\sum_{q=1}^{\infty}K_0\left[(4q^2\pi^2/f^2-k^2)^{1/2}(\theta^2+(pg-\phi)^2)^{1/2}\right]\right\}\tag{C11}$$

Now $4q^2\pi^2/f^2 \gg k^2$ for $q \geq 1$ and $(\theta^2+(pg-\phi)^2)^{1/2} \gg f$ except in an extremely limited region around $\phi = 0$ at $m = m_0$ and $p = 0$. Hence the argument of the K_0 functions in $(C11)$ are always much greater than 2π when $m \neq m_0$, and we can write

$$\sum_{p=-\infty}^{\infty}\sum_{q=1}^{\infty}K_0[\,] \approx \delta(m-m_0)\sum_{q=1}^{\infty}K_0[\frac{2\pi q}{f}(a^2+\phi^2)^{1/2}]\tag{C12}$$

where $\delta(m-m_0)=0$ unless $m=m_0$, when it equals 1; and $[\,]$ refers to the K_0 argument in $(C11)$.

There is one very slight improvement of this result that may be necessary in the neighbourhood of $z=L$, $\zeta=-L$ or $z=-L$, $\zeta=L$, when ϕ is close to $\pm 2L$. If g is also close to $2L$ then $g\pm\phi$ may get small enough for the terms in $p=\pm 1$ in $(C11)$ to be noticeable. In this case the term $K_0[\frac{2\pi q}{f}(a^2+\phi^2)^{1/2}]$ is augmented by $K_0[\frac{2\pi q}{f}(a^2+(g-\phi)^2)^{1/2}]+K_0[\frac{2\pi q}{f}(a^2+(g+\phi)^2)^{1/2}]$, although these terms are negligible except very close to $\phi=\pm 2L$. As will be shown later, the subsequent integrations on z and ζ produce an almost negligible net contribution, even when $g=2L$. For simplicity $(C12)$ is retained unamended. Using the results that led to $(C7)$ and $(C8)$ gives, for the K_0 sum in $(C12)$,

$$4\sum_{p=-\infty}^{\infty}\sum_{q=1}^{\infty}K_0[\,]$$
$$\approx \delta(m-m_0)\cdot\left\{f(a^2+\phi^2)^{-1/2}-2\mathrm{Re}\log\left[\frac{\beta+(\phi^2+\beta^2+a^2)^{1/2}}{(\phi^2+a^2)^{1/2}}\right]\right\}\tag{C13}$$

where now $\beta = f(1+i\delta)/2$. If $g-2L < f$, corresponding terms in $(\phi \pm g)$ are to be added to $(C13)$ on the right-hand side.

C_4 : *Double Summation Approximation*

The Hankel function sum in $(C11)$ is found from $(C10)$ by taking $y=\phi$, $|z|=\theta \approx |m-m_0|f$ and replacing n by p, to get

$$\sum_{p=-\infty}^{\infty} H_0^{(2)}\left[k(\theta^2+(pg-\phi)^2)^{1/2}\right]$$
$$\approx \frac{2}{kg}e^{-jkf|m-m_0|} - \frac{2j}{\pi}\mathrm{Re}\left\{\log\left[1-e^{-\frac{2\pi}{g}(f|m-m_0|+i\phi)}\right]\right\} \qquad (C14)$$

Note that f in $(C13)$ can also be written as $2\mathrm{Re}\,\beta$, leading to a slight simplification. Putting $(C13)$ and $(C14)$ into $(C11)$ gives

$$\sum_{p=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{e^{-jkR_{mnp}}}{R_{mnp}}$$
$$=\frac{2}{f}\left\{\frac{-j\pi}{kg}e^{-jkf|m-m_0|} - \mathrm{Re}\log\left[1-e^{-\frac{2\pi}{g}(|m-m_0|f+i\phi)}\right]\right.$$
$$\left.+\delta(m-m_0)\mathrm{Re}\left[\beta(a^2+\phi^2)^{-1/2} - \log\frac{\beta+(\phi^2+\beta^2+a^2)^{1/2}}{(\phi^2+a^2)^{1/2}}\right]\right\} \qquad (C15)$$

C_5 : *Geometric Summation*

To calculate S in $(C1)$, $(C15)$ has to be multiplied by $e^{-jk\epsilon^{1/2}fm}$ and summed over m from 0 to ∞. The term in $\delta(m-m_0)$ gives merely a multiplier $e^{-jk\epsilon^{1/2}fm_0}$. The other terms can be dealt with by replacing the sums by integrals, though the first series is reducible to a geometric series and can be summed exactly.

Writing, temporarily, $b = k\epsilon^{1/2}f$ and $c = kf$ the first sum is

$$\begin{aligned} S_0 &= \sum_{m=0}^{\infty} e^{-jbm} e^{-jc|m-m_0|} \\ &= \sum_{m=0}^{m_0-1} e^{-j(bm+cm_0-cm)} + \sum_{m=m_0}^{\infty} e^{-j(bm+cm-cm_0)} \\ &= e^{-jcm_0} \frac{1-e^{-j(b-c)m_0}}{1-e^{-j(b-c)}} + e^{-jbm_0} \frac{1}{1-e^{-j(b+c)}} \end{aligned} \tag{C16}$$

So far this result is exact, but now we use the fact that $b \pm c \ll 1$ to expand the denominators. With $x = j(b \pm c)$, we have the expansion $1/(1-e^{-x}) = 1/x + 1/2 + x/12 + O(x^3)$. Hence $(C16)$ becomes

$$S_0 = \frac{e^{-jcm_0}}{j(b-c)} \left[1 + j\frac{(b-c)}{2} - \frac{(b-c)^2}{12}\right] + e^{-jbm_0} j2c \left[\frac{1}{b^2-c^2} + \frac{1}{12}\right] \tag{C17}$$

Reverting to the original notation and retaining only the leading terms (it can be shown that the higher-order ones give a negligible net contribution), one finds

$$S_0 \approx \frac{e^{-jkfm_0}}{jkf(\epsilon^{1/2}-1)} + e^{-jkf\epsilon^{1/2}m_0} \left[\frac{j2}{kf(\epsilon-1)}\right] \tag{C18}$$

C_6 : *Logarithmic Sum*

The remaining series is

$$\begin{aligned} S_1 &= \sum_{m=0}^{\infty} \text{Re} \log \left[1 - e^{-2\pi f|m-m_0|/g} e^{-i2\pi\phi/g}\right] e^{-jkf\epsilon^{1/2}m} \\ &= e^{-jkf\epsilon^{1/2}m_0} \sum_{m=-m_0}^{\infty} \text{Re} \log \left[1 - e^{-2\pi f|m|/g} e^{-i2\pi\phi/g}\right] e^{-jkf\epsilon^{1/2}m} \end{aligned}$$

(where Re refers only to i) $(C19)$

Now except, for a particle very near the front face, at m_0 small, the term $e^{-2\pi f|m|/g}$ is minute for $m < -m_0$. The excepted region forms a very thin "skin" at the surface, and is of no interest for bulk material properties. Hence there is negligible error in replacing $-m_0$ by $-\infty$ in the summation. The sum can now be replaced by an integral by taking $2\pi fm/g$ as integration variable u, to give

$$S_1 = e^{-jkf\epsilon^{1/2}m_0}\mathrm{Re}\int_{-\infty}^{\infty} \log\left[1 - e^{-|u|}e^{-i\psi}\right] e^{-jk\epsilon^{1/2}gu/2\pi}(g/2\pi f)du \tag{C20}$$

where $\psi = 2\pi\phi/g$.
Hence $S_1 = (g/2\pi f)e^{-jkf\epsilon^{1/2}m_0}I$, where

$$\begin{aligned} I =&\mathrm{Re}\int_{-\infty}^{\infty} \log\left[1 - e^{-|u|}e^{-i\psi}\right] e^{-jk\epsilon^{1/2}gu/2\pi}du \\ =&2\mathrm{Re}\int_{0}^{\infty} \log\left[1 - e^{-u}e^{-i\psi}\right] \cos(\eta u)du \end{aligned} \tag{C21}$$

where $\eta = k\epsilon^{1/2}g/2\pi \ll 1$. Hence the angle in the cosine is very small until u gets large, by which time $e^{-u} \ll 1$ and the integrand is negligible. Hence the cosine can be replaced by 1, and

$$I = 2\mathrm{Re}\int_{0}^{\infty} \log\left[1 - e^{-(u+i\psi)}\right] du \tag{C22}$$

Change variable by taking $v = e^{-(u+i\psi)}$, to get

$$\begin{aligned} I =&2\mathrm{Re}\int_{0}^{e^{-i\psi}} \frac{\log(1-v)}{v}dv \\ =&-2\mathrm{Re}Li_2(e^{-i\psi}) = -2\left[\frac{\pi^2}{6} + \frac{\psi^2 - 2\pi|\psi|}{4}\right] \end{aligned} \tag{C23}$$

This result requires that $|\psi| \le 2\pi$, which is satisfied, since $\psi = 2\pi\phi/g$, and $|\phi| \le 2L$, and $g \ge 2L$.
Hence

$$S_1 = (g\pi/f)e^{-jkf\epsilon^{1/2}m_0}\left[\frac{1}{6} + \frac{\phi^2}{g^2} - \frac{|\phi|}{g}\right] \tag{C24}$$

C_7 : Approximation for the Triple Sum

Collecting these results together gives

$$\sum_{m=0}^{\infty}\sum_{p=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{e^{-jkR_{mnp}}}{R_{mnp}}e^{-jk\epsilon^{1/2}fm}$$
$$\approx \frac{-2\pi}{k^2f^2g(\epsilon^{1/2}-1)}e^{-jkfm_0}$$
$$+e^{-jk\epsilon^{1/2}fm_0}\left\{\frac{4\pi}{k^2f^2g(\epsilon-1)}+\frac{2g\pi}{f^2}\left[\frac{1}{6}+\frac{\phi^2}{g^2}-\frac{|\phi|}{g}\right]\right.$$
$$\left.+\frac{1}{(a^2+\phi^2)^{1/2}}+\frac{2}{f}\mathrm{Re}\log\left[\frac{(a^2+\phi^2)^{1/2}}{\beta+(\phi^2+\beta^2+a^2)^{1/2}}\right]\right\} \quad (C25)$$

with $\phi = z-\zeta$; $\beta = f(1+i\delta)/2$; $\delta = g/3^{1/2}$.

Appendix D: Evaluation of a Double Integral

This appendix is concerned with the evaluation of

$$I=\int_{-1}^{1}\int_{-1}^{1} i(z)i(\zeta)\frac{\partial^2}{\partial z^2}F(z-\zeta)dzd\zeta \quad (D1)$$

where $i(\pm1)=0$, and both i and F are even.
Using $\frac{\partial^2}{\partial z^2}F(z-\zeta)=\frac{-\partial^2}{\partial z\partial\zeta}F(z-\zeta)$, an integration by parts with respect to both z and ζ can be performed, the integrated part being zero since $i(\pm1)=0$. Hence

$$I=-\int_{-1}^{1}\int_{-1}^{1} i'(z)i'(\zeta)F(z-\zeta)dzd\zeta \quad (D2)$$

The integration is over the square region shown in Fig. D1, which also shows axes μ and λ rotated by 45°.

We have

$$z=(\mu-\lambda)/2^{1/2};\quad \zeta=(\mu+\lambda)/2^{1/2} \quad (D3)$$
$$\mu=(z+\zeta)/2^{1/2};\quad \lambda=(\zeta-z)/2^{1/2} \quad (D4)$$
$$dzd\zeta=d\lambda d\mu \quad (D5)$$

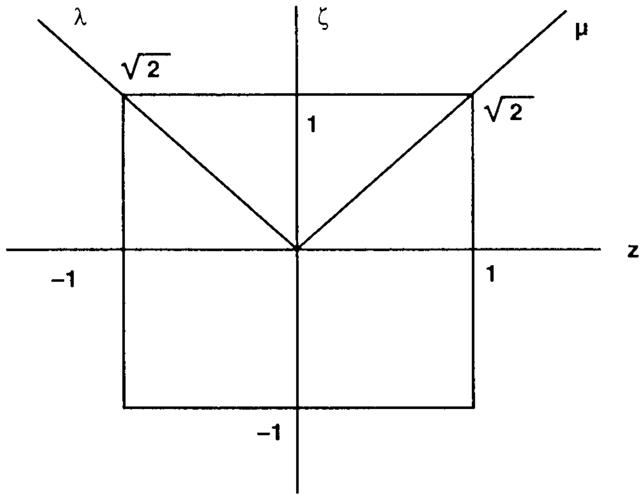

Figure D1. Integration range for z and ζ.

The integration range for λ and μ is shown in the 45° rotated square in Fig. D2.

The integration limits depend on whether the integration is first with respect to μ or to λ. In the present instance it is more convenient to choose μ first, since it does not appear in F. Then for

$$0 \leq \lambda \leq 2^{1/2}, \mu \text{ goes from } -(2^{1/2}-\lambda) \text{ to } +(2^{1/2}-\lambda)$$
$$-2^{1/2} \leq \lambda \leq 0, \mu \text{ goes from } -(2^{1/2}+\lambda) \text{ to } +(2^{1/2}+\lambda)$$

Hence

$$I = -\int_0^{2^{1/2}} F(\lambda 2^{1/2}) \left[\int_{-(2^{1/2}-\lambda)}^{(2^{1/2}-\lambda)} i'\left(\frac{\mu-\lambda}{2^{1/2}}\right) i'\left(\frac{\mu+\lambda}{2^{1/2}}\right) d\mu\right] d\lambda$$
$$-\int_{-2^{1/2}}^{0} F(\lambda 2^{1/2}) \left[\int_{-(2^{1/2}+\lambda)}^{(2^{1/2}+\lambda)} i'\left(\frac{\mu-\lambda}{2^{1/2}}\right) i'\left(\frac{\mu+\lambda}{2^{1/2}}\right) d\mu\right] d\lambda \qquad (D6)$$

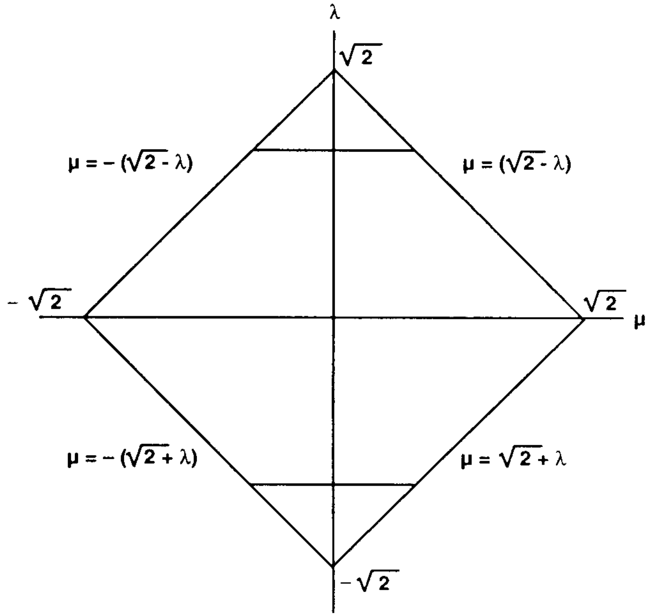

Figure D2. Integration range for λ and μ.

If in the second integral we write $-\lambda$ for λ it becomes the first. In the first, split the range of μ from 0 to $(2^{1/2} - \lambda)$ and $-(2^{1/2} - \lambda)$ to 0, and write $-\mu$ for μ in the second range. On using the fact that i is even, so that i' is odd, it is seen that the integral in the second range repeats that in the first. Finally, making a change of scale so that λ, μ go over into $\lambda/2^{1/2}$ and $\mu/2^{1/2}$, $(D6)$ becomes

$$I = -2 \int_0^2 F(\lambda) \left[\int_0^{2-\lambda} i' \left(\frac{\mu - \lambda}{2} \right) i' \left(\frac{\mu + \lambda}{2} \right) d\mu \right] d\lambda \qquad (D7)$$

This result has the merit of separating the current integration out. For example, if $i(z) = (1 - z^2)(1 + Az^2)$ the μ-integration gives

$$\int_0^{2-\lambda} i' \left(\frac{\mu - \lambda}{2} \right) i' \left(\frac{\mu + \lambda}{2} \right) d\mu$$

$$= \frac{2}{3}(4 - 6\lambda + \lambda^3) + 4A\left(\frac{4}{15} - 2\lambda + 4\lambda^2 - \frac{7}{3}\lambda^3 + \frac{\lambda^5}{5}\right)$$

$$+ 2A^2\left(\frac{44}{105} - 2\lambda + \frac{8}{5}\lambda^2 + \frac{\lambda^3}{3} - \frac{2\lambda^5}{5} + \frac{2\lambda^7}{35}\right) \qquad (D8)$$

Denote this function as $I(\lambda)$; or $I(\lambda, A)$ if the A-dependence is of concern.

Appendix E: Hertzian Vector and Fiber Loss

The Hertzian vector of a fiber located at (mf, nf, pg), oriented along the z-axis, and carrying a current $I(\zeta/L)$ at a point $z = pg + \zeta$ on the fiber, observed at a point z on a fiber located at $(m_0 f, 0, 0)$ is

$$\Pi_z = \frac{-30j}{\epsilon_1 k_0} \int_{-L}^{L} I(\zeta/L) \frac{e^{-jkR_{mnp}}}{R_{mnp}} e^{-jk\sqrt{\epsilon} fm} d\zeta \qquad (E1)$$

where $k_0 = 2\pi/\lambda_0$

$\epsilon_1 =$ matrix permittivity

$k = 2\pi\epsilon_1^{1/2}/\lambda_0 = 2\pi/\lambda$

$\epsilon = \epsilon_m/\epsilon_1$, where ϵ_m is the mixture bulk permittivity

$R_{mnp} = [(m - m_0)^2 f^2 + n^2 f^2 + (pg + \zeta - z)^2 + a^2]^{1/2}$

$a =$ fiber radius, with $a/L \ll 1$,

$2L =$ fiber length

$$\text{Write } i(\zeta/L) = -(30j/\epsilon_1 k_0) I(\zeta/L) \qquad (E2)$$

For complete penetration I is constant over the fiber cross-section, so the current density is $I/\pi a^2$. If E is the longitudinal electric field at the fiber surface then the boundary condition is $I/\pi a^2 = \sigma E$ where σ is the fiber conductivity. Using $(E2)$ this can be put in the form

$$E = jri \qquad (E3)$$

where

$$r = \epsilon_1/15a^2\sigma\lambda_0 \qquad (E4)$$

is a measure of the fiber resistivity for the purposes of this study.

Appendix F: A Variational form for the Permittivity

The structure of the equation to determine the current $i(\zeta)$ is of the form

$$K\int i(\zeta)d\zeta = \int i(\zeta)G(\zeta - z)d\zeta; \quad -1 < z < 1 \qquad (F1)$$

in which all integrations of the (normalized) variables are between the limits ± 1, and both i and G are even functions of their arguments. The constant K contains the permittivity ϵ of the mixture, which occurs nowhere else in this equation.

Ideally, $(F1)$ would be solved for $i(\zeta)$, and $(F1)$ would then give K, and hence ϵ. Let the exact solution of $(F1)$ be $i(\zeta) = i_0(\zeta)$, where $i_0(\zeta)$ satisfies $(F1)$, i.e.

$$K_0\int i_0(\zeta)d\zeta = \int i_0(\zeta)G(\zeta - z)d\zeta; \quad -1 < z < 1 \qquad (F2)$$

In the absence of knowing the exact function $i_0(\zeta)$ we can proceed as follows: multiply $(F1)$ by $i(z)$ and integrate from -1 to $+1$ with respect to z to get

$$K = \frac{\iint i(z)i(\zeta)G(\zeta - z)d\zeta dz}{\int i(\zeta)d\zeta \int i(z)dz} \qquad (F3)$$

This is, of course, exact if $i(\zeta) = i_0(\zeta)$. But if $i(\zeta)$ differs from $i_0(\zeta)$ by a small quantity $\epsilon(\zeta)i_0(\zeta)$ then, as will be shown, $(F3)$ will give a value of K differing from its correct value K_0 by an amount proportional to ϵ^2; in other words, $(F3)$ is in variational form. Not only does this enable one to calculate a reasonable value of K for a fair guess at the form of i, but the fact that $(F3)$ is variational means that it contains the integral equation $(F1)$ implicitly embedded within, so improvements of a trial function can be made directly from $(F3)$ without recourse to $(F1)$.

For convenience write $M = \int i(\zeta)d\zeta$, so that $(F3)$ can be written

$$K = M^{-2}\iint i(z)i(\zeta)G(\zeta - z)d\zeta dz \qquad (F4)$$

and put $i(\zeta) = i_0(\zeta)[1+\epsilon(\zeta)]$; or $i = i_0(1+\epsilon)$ for short. Then, if $K = K_0 + \delta K$, to first order in ϵ it is seen that

$$\delta K = M^{-2} \iint i_0(z) i_0(\zeta)[\epsilon(\zeta)+\epsilon(z)] G(\zeta - z) d\zeta dz$$
$$- 2M^{-3} [\int i_0(z)\epsilon(z)dz] \iint i_0(z) i_0(\zeta) G(\zeta - z) d\zeta dz \qquad (F5)$$

Multiply $(F2)$ by $i_0(z)\epsilon(z)dz$ and integrate:

$$K_0 M \int i_0(z)\epsilon(z)dz = \iint i_0(z)\epsilon(z) i_0(\zeta) G(\zeta - z) d\zeta dz \qquad (F6)$$

There is a similar relation obtainable by interchanging z and ζ on the righthand side of $(F6)$.

Inserting this into $(F5)$ and using $(F4)$, with K and i replaced by K_0 and i_0 it is seen that $(F5)$ is zero to first order in ϵ; the variational structure is confirmed around $i = i_0$.

Let $I(\zeta)$ be a given trial function and $I(\zeta)[1+\epsilon(\zeta)]$ be a better one, in which the form but not the amplitude of $\epsilon(\zeta)$ is set. This will not, in general, equal $i_0(\zeta)$ and the two will differ by some additional small amount $\Delta(\zeta)$, so that we can write

$$i_0 = I(1+\epsilon+\Delta) \qquad (F7)$$

Now

$$K \equiv K(i_0) = K[I(1+\epsilon+\Delta)]$$
$$= K[I(1+\epsilon)] + \Delta \frac{\partial}{\partial \Delta} K[I(1+\epsilon+\Delta)]_{\Delta=0} + O(\Delta^2) \qquad (F8)$$

But the variational character makes $\frac{\partial}{\partial\Delta} K[I(1+\epsilon+\Delta)]_{\Delta=0}$ zero. Since the latter is a function of $(\epsilon+\Delta)$ we have $\partial/\partial\Delta|_{\Delta=0} \equiv \partial/\partial\epsilon$. Hence

$$\frac{\partial}{\partial \epsilon} K[I(1+\epsilon)] = 0 \qquad (F9)$$

determines the amplitude of ϵ for optimum value of K for a given variation of $\epsilon(\zeta)$.

For example the simplest trial function, which must be even and zero at the fiber ends, is $I(\zeta) = (1-\zeta^2)$. An improved trial function would multiply this by a power series in ζ^2, with the coefficients determined by differentiation and equating to zero as required by $(F9)$. A first-order improvement would come from

$$I(\zeta) = (1-\zeta^2)(1+A\zeta^2) \tag{F10}$$

with A determined by

$$\frac{\partial}{\partial A} K[(1-\zeta^2)(1+A\zeta^2)] = 0 \tag{F11}$$

in which $K(i)$ is the form $(F3)$. This determination of A (or, more generally, the coefficients in any correction function) depends only on the right-hand side of $(F3)$, hence is not affected by the actual value of K. In other words, the form of the fiber current does not depend in any way on the value of the mixture permittivity, at least to the extent that the approximations in Appendix C give rise to a function $G(\zeta - z)$ independent of ϵ.

Appendix G: Current-Lattice Integrals

The integral needed involves $I(\lambda)$ of $(D8)$ and the functions in $(C25)$. Since the process is different for the different parts, define

$$F_1(\lambda) = (a^2+\lambda^2)^{-1/2} \tag{G1}$$

$$F_2(\lambda) = (1/f)\left\{\log(a^2+\lambda^2) - 2\mathrm{Re}\,\log\left[\beta + (\lambda^2+\beta^2+a^2)^{1/2}\right]\right\} \tag{G2}$$

$$F_3(\lambda) = \frac{2g\pi}{f^2}\left[\frac{1}{6}+\frac{\lambda^2}{g^2}-\frac{\lambda}{g}\right] \tag{G3}$$

where $\beta = f(1+i\delta)/2$; $\delta = 1/3^{1/2}$.

In this formulation all lengths have been normalized to L; i.e., a, here, means a/L, etc.

Although all the needed integrals can be evaluated exactly by elementary means, the result is long and complicated. If we make use of the inequalities $a^2 \ll 1$ and $\beta^2 \ll 1$ a lot of simplification is possible. As far as the fiber radius is concerned, a is so small that only the lowest power, which appears in the logarithm, is needed. The same is almost true for β too, but since the outside multiplier in $(G2)$ is $1/(2\mathrm{Re}\beta)$ powers up to β^2 can be considered; and in any case

$$\mathrm{Re}(\beta^3) = \mathrm{Re}[(f/3^{1/2})^3 e^{i\pi/2}] = 0.$$

Retain the form $(a^2+\lambda^2)^{-1/2}$ for integrations with the constant terms in $I(\lambda)$, and replace it by $1/\lambda$ in the remainder. Thus the integration involving the power of A^0 in $I(\lambda)$ becomes

$$\int_0^2 \left[\frac{4}{(a^2+\lambda^2)^{1/2}} - \frac{6\lambda}{\lambda} + \frac{\lambda^3}{\lambda}\right] d\lambda = 4[\log(4/a) - 7/3] \qquad (G4)$$

where the terms in square brackets can be recognized as part of the polarizability expression for a cylinder. Equation $(G4)$ is correct to $O(a)$. The net result of this process is

$$\int_0^2 I(\lambda)F_1(\lambda)d\lambda = \frac{8}{3}\left[\log\frac{4}{a} - \frac{7}{3}\right] + \frac{16A}{15}\left[\log\frac{4}{a} - \frac{53}{15}\right] + \frac{88A^2}{105}\left[\log\frac{4}{a} - \frac{3931}{1155}\right] \qquad (G5)$$

The process starts with an integration by parts. Put $\overline{I}(\lambda) = \int_0^\lambda I(\lambda)d\lambda$, to give

$$\overline{I}(\lambda) = \frac{2}{3}(4\lambda - 3\lambda^2 + \lambda^4/4) + 4A(4\lambda/15 - \lambda^2 + 4\lambda^3/3 - 7\lambda^4/12 + \lambda^6/30) + 2A^2(44\lambda/105 - \lambda^2 + 8\lambda^3/15 + \lambda^4/12 - \lambda^6/15 + \lambda^8/140) \qquad (G6)$$

Clearly, $\overline{I}(0) = 0$ and it can be readily shown that $\overline{I}(2)$ is also zero, hence the integrated part vanishes, and

$$\int_0^2 I(\lambda)F_2(\lambda)d\lambda = -\int_0^2 \overline{I}(\lambda)F_2'(\lambda)d\lambda \qquad (G7)$$

where

$$F_2'(\lambda) = \frac{2}{f}\mathrm{Re}\frac{\beta\lambda}{(\lambda^2+a^2)(\lambda^2+\beta^2+a^2)^{1/2}} \qquad (G8)$$

Near $\lambda = 0$, $I(\lambda)F_2(\lambda)$ varies as $\lambda^2/(\lambda^2 + a^2)$, which stays finite as $a \to 0$. Hence a good approximation for small a consists of ignoring a to give the approximation

$$\lambda F_2'(\lambda) \approx \frac{2}{f}\mathrm{Re}\frac{\beta}{(\lambda^2 + \beta^2)^{1/2}} \tag{G9}$$

The integration process now proceeds as in $(G1)$, except that, since β is not so very small, we also use

$$\beta \int_0^2 \frac{\lambda d\lambda}{(\beta^2 + \lambda^2)^{1/2}} = \beta(\beta^2 + \lambda^2)^{1/2}|_0^2 \approx \beta(2 - \beta) \tag{G10}$$

with the term in β not quite negligible compared to 2. We also have
$\mathrm{Re}(\beta \log \beta) = (f/2)[\log(f/3^{1/2}) - \pi/6 \cdot 3^{1/2}]$ and $\mathrm{Re}(\beta^2) = f^2/6$
to give

$$\int_0^2 I(\lambda)F_2(\lambda)d\lambda = -\left\{\frac{8}{3}\left[\log\frac{4.3^{1/2}}{f} - \frac{4}{3} + \frac{\pi}{6 \cdot 3^{1/2}} + \frac{f}{4}\right] + \frac{16A}{15}\left[\log\frac{4.3^{1/2}}{f} - \frac{38}{15} + \frac{\pi}{6 \cdot 3^{1/2}} + \frac{5f}{4}\right] + \frac{88A^2}{105}\left[\log\frac{4 \cdot 3^{1/2}}{f} - \frac{2776}{1155} + \frac{\pi}{6.3^{1/2}} + \frac{35f}{44}\right]\right\} \tag{G11}$$

The integration $I(\lambda)F_3(\lambda)$ is straightforward, since all terms are polynomials in λ. We can re-write $F_3(\lambda)$ as $(2\pi/f^2g)[g^2/6 + \lambda^2 - \lambda g]$, where the term in $g^2/6$ integrates to zero since $\overline{I}(\lambda)$, at $\lambda = 2$, is zero. Hence, it is found that

$$\int_0^2 I(\lambda)F_3(\lambda)d\lambda = \frac{32\pi}{45f^2g}[(3g-5) + 2(3g-7)A/7 + (5g-7)A^2/35] \tag{G12}$$

This is a very important term in the final formula. Since the smallest value of g possible is $g = 2$, the quantity in square brackets then becomes $1 - 2A/7 + 3A^2/35 = (1 - A/7)^2 + 16A^2/225$. It is therefore always positive, an important feature discussed in the main text.

It might be added that since $F_3(\lambda)$ is an approximation, an improved form might affect $(G12)$ significantly. It turns out that the next approximation involves a constant-term correction, in β^2, which is not only very small but also integrates out to zero. The next term involves $\mathrm{Re}(\beta^3)$ which is also zero. So $(G12)$ should be considered a quite accurate result.

Appendix H: Displaced Lattice Expressions

Determination of H_1.

The material is based on expressions in Appendix C. The z-displacement is handled by writing $pg+g/2+\zeta-z$ for $pg+\zeta-z$; or, equivalently, $g/2+\zeta-z$ for $\zeta-z$. It will be recalled that the function F in Appendix D has to be even on $\zeta-z$, and this is no longer so if it is merely a function of $g/2+\zeta-z$. However, the p-summation includes $p=-1$, for which the form $pg+g/2+\zeta-z$ becomes $-g/2+\zeta-z$; the sum of the two restores the even characteristic, so by the time the formulas of Appendix D are applied, the argument $(g/2+\lambda)$ suffices. For $(C23)$ the angle ψ has to be reduced by 2π if it exceeds that value. The result is that $(C24)$ applies with ϕ replaced by ϕ_m where

$$\phi_m = \pm g/2 + z - \zeta \qquad (H1)$$

where the $+$ or $-$ sign is chosen to keep $|\phi_m| \leq g$. See Section H_3 for more details.

Determination of H_2.

The summation on n in $(C8)$ becomes a summation with n replaced by $n+1/2$ to account for the y-displacement of $f/2$. Now $f(n+1/2)$ can be written as $(2n+1)f/2$, so the series becomes a summation over odd integers only, with f replaced by $f/2$. Now it is easily shown that if $f(n)$ is any function for which

$$\sum_{n=-\infty}^{\infty} f(\alpha n) = F(\alpha) \qquad (H2)$$

then by separating the even and odd terms one can deduce

$$\sum_{n=-\infty}^{\infty} f[\alpha(2n+1)/2] = F(\alpha/2) - F(\alpha) \qquad (H3)$$

When this is applied to $(C13)$ the terms in $f(a^2+\phi^2)^{-1/2}$ cancel out, and the logarithmic term becomes

$$\frac{1}{2}\mathrm{Re}\log\frac{[\beta+(\beta^2+\psi^2+f^2/4)^{1/2}][\beta+(\beta^2+(\psi-g)^2+f^2/4)^{1/2}]}{(\psi^2+f^2/4)^{1/2}[(\psi-g)^2+f^2/4]^{1/2}}$$
$$-\mathrm{Re}\log\left\{\frac{[\beta/2+(\beta^2/4+\psi^2+f^2/4)^{1/2}]}{(\psi^2+f^2/4)^{1/2}}\right.$$
$$\left.\frac{[\beta/2+(\beta^2/4+(\psi-g)^2+f^2/4)^{1/2}]}{[(\psi-g)^2+f^2/4]^{1/2}}\right\} \qquad (H4)$$

In this expression, $\psi = z-\zeta+g/2$ and $\beta = f(1+i\delta)/2$. Both β and $f/2$ are small compared to ψ or $\psi-g$, except in a very limited region where these may be small; but even so, $(\psi^2+f^2/4)^{1/2}$ will then be $O(f/2)$ or more. Hence the expressions in $(H4)$ can be expanded in powers of β. Now a term like $\log[\beta+(\beta^2+y^2)^{1/2}]$ comes from integrating $(\beta^2+y^2)^{-1/2}$ with respect to β. Provided $y^2>\beta^2$, as it is here with $y=(\psi^2+f^2/4)^{1/2}$, an expansion can be made in the form $1/|y|-\beta^2/2|y|^3+\cdots$ which integrates to $\beta/|y|-\beta^3/6|y|^3+\cdots$. As can be verified, the terms in β cancel in $(H4)$, while $\mathrm{Re}(\beta^3)=0$; so to a high order $(H4)$ is negligible.

Determination of H_3.

A similar expression to $(C15)$ is produced except that the term in $\delta(m-m_0)$ is absent, and $f(m-m_0)$ becomes replaced by $(2m+1-2m_0)f/2$ for the x-displaced lattices. This makes virtually no difference after the m-summation, and the remainder of the expression follows the same route as that leading to $(C22)$. The only difference of significance comes in $\mathrm{Re}Li_2(e^{-i\psi})$ since now $|\psi|$ can exceed 2π. For $(C23)$ to be valid the angle has to be between -2π and $+2\pi$, and since $Li_2(e^{-i\psi})$ is periodic in ψ with period 2π, one arrives at the specification of $(H1)$. With $z-\zeta=\lambda$ the integration corresponding to $(G12)$ becomes, apart from the factor $2\pi g/f^2$,

$$\int_0^2 I(\lambda)\overline{F}_3(\lambda)d\lambda = \int_0^{\frac{1}{2}g} I(\lambda)\left[\frac{1}{6}+\frac{(\lambda+\frac{1}{2}g)^2}{g^2}-\frac{(\lambda+\frac{1}{2}g)}{g}\right]d\lambda$$

$$+\int_{\frac{1}{2}g}^{2} I(\lambda)\left[\frac{1}{6}+\frac{(\lambda-\frac{1}{2}g)^2}{g^2}-\frac{(\lambda-\frac{1}{2}g)}{g}\right]d\lambda \quad (H5)$$

(valid for $2 \le g \le 4$)

Although there is nothing intrinsically difficult in this expression, since all terms are polynomials in λ, the actual integration is rather messy. Since the main interest is in the neighborhood of $g = 2$, the integrals were evaluated there, with results as reported in equation (5.3).

The best way to handle $(H5)$ is to add and subtract an integration of the second term from 0 to $g/2$. Recalling that the integration (from zero) of $I(\lambda)$ vanishes at $\lambda = 2$, the term in $1/6$ does not contribute, and $(H5)$ reduces to

$$\int_0^2 I(\lambda)\overline{F}_3(\lambda)d\lambda$$

$$= (1/g^2)\int_0^2 I(\lambda)(\lambda^2 - 2\lambda g)d\lambda - (1/g)\int_0^{g/2} I(\lambda)(g-2\lambda)d\lambda \quad (H6)$$

If we write $I_1(\lambda) = \int_0^\lambda I(\lambda)d\lambda$, (with $I_1(2) = 0$), and $I_n(\lambda) = \int_0^\lambda I_{n-1}(\lambda)d\lambda$, $n > 1$, then $(H6)$ can, by repeated integrations by parts, be put in the form

$$g^2\int_0^2 I(\lambda)\overline{F}_3(\lambda)d\lambda = 2I_3(2) + 2(g-2)I_2(2) - 2gI_2(g/2) \quad (H7)$$

The $I_n(\lambda)$ are obtained by repeated integration of $(D8)$. For the record, $g^2\int_0^2 I(\lambda)\overline{F}_3(\lambda)d\lambda = 2I_3(2) + (g-4)I_2(2)$, again without the initial factor $2\pi g/f^2$.

Appendix I: Isolated Fiber Current

From reference 4, $(A8)$ and $(A9)$ one finds

$$\overline{A} = [-\log(2L/a) - 38/15]/[2\log^2(2L/a) - \log(2L/a)247/30 + 1597/180] \qquad (I1)$$

$$\overline{B} = [\log(2L/a) - 5/6]/[4\log^2(2L/a) - \log(2L/a)247/15 + 1597/90] \qquad (I2)$$

From $(A3)$, the charge density is, (normalizing z with respect to L),

$$\tau(z)/EC = z(1 - \overline{A}z^2 + \overline{B}z^4) \qquad (I3)$$

The current is proportional to the integral of the charge density. The proportionality constant is not important here. Noting that $i(\pm 1) = 0$, the integration of (I3) gives

$$\begin{aligned} i(z) &= (1 - z^2) - (1 - z^4)\overline{A}/2 + (1 - z^6)\overline{B}/3 \\ &= (1 - z^2)[1 - (1 + z^2)\overline{A}/2 + (1 + z^2 + z^4)\overline{B}/3] \end{aligned} \qquad (I4)$$

Keeping terms up to z^2 in the square brackets gives a factor $1 - \overline{A}/2 + \overline{B}/3 + z^2(-\overline{A}/2 + \overline{B}/3]$ Comparing this to $(1 + Az^2)$ in (4.4) gives

$$A = (-\overline{A}/2 + \overline{B}/3)/(1 - \overline{A}/2 + \overline{B}/3) \qquad (I5)$$

From $(I1)$ and $(I2)$, this gives the formula quoted in (6.14),

$$A = \frac{\log(2L/a) - 253/120}{3\log^2(2L/a) - (227/20)\log(2L/a) + 56/5} \qquad (I6)$$

Summary

This study deals with dielectric mixtures in the "dense" region - defined here by the interpenetration of the exclusion spheres circumscribing the loading particles. Previous studies based on dipolar and multipolar expansions, or on Waterman's T-matrix using spherical harmonic expansions, are all invalid inside the exclusion sphere because of their electromagnetic requirement of out-going waves, which does not hold inside the exclusion sphere. Non-interpenetrating exclusion spheres also ensures the non-contacting of particles when averaging over particle orientations and positions, a very convenient practical consideration.

When the particles are elongated, typically cylindrical with an aspect ratio which may be 100 or more, the dense region starts at a very low geometrical volume loading, such as 10^{-4} or less. Even so, classical formulas of the Clausius-Mossotti form seem to work reasonably, well into the dense region, though they are much poorer for the loss factor. Eventually they "blow up" and give non-physical negative permittivities if the loading becomes too great.

Although a useful engineering formula is the hoped-for outcome of a scientific study such as this, the initial goal adopted for the present research was to come up with an analysis of a mixture configuration which would be fully valid in the dense region. Of the two features relating to interpenetrating exclusion spheres, the invalidity of the existing methods, for electromagnetic reasons, seemed more fundamental than the potential for particle contact on averaging, important though this must be for an eventual practical formula representing randomized loading particles in a matrix. This problem of randomizing particle positions and orientations seems to be an *extremely* difficult one when particles may contact each other (though not interpenetrate); and in the absence of a T-matrix formulation which can handle this feature well under non-contacting conditions, even the electromagnetic aspects of mutual interaction between particles under interpenetrating exclusion sphere conditions is a difficult one. The configuration that lends itself well to an initial investigation consists of *aligned* particles in a regular rectangular lattice, and this is the prime object investigated in this study. By nesting four sublattices together a reasonable amount of spatial overlap is obtained; and a very primitive orientation averaging pertinent primarily to weak mixtures, gives a simple factor, occurring in prior calculations, which goes part way to accounting for the

randomizing of the loading particles and providing comparisons with earlier formulas.

The analysis abandons the excitation/scattering concept of earlier formulations, and instead produces an integral equation for the fiber current, with an extremely complicated kernel that contains the unknown mixture permittivity embedded therein. Several mathematical appendices reduce this kernel to a manageable form, using certain approximations valid in the dense region. Some of these appendices stand alone: for example Appendix B gives a new sum formula, Appendix C handles the triple lattice sum, and Appendix A deals with effective fiber diameter under current-penetration conditions. Eventually a variational form for the mixture permittivity is obtained and suitable forms for the fiber current are used to obtain a result that is recognized to be of Clausius-Mossotti form.

Numerical comparisons with both measured results and with earlier formulations show that the new formula is no better in predicting the loss factor, though it has a negative array factor which prevents the expression from blowing up and giving non-physical negative values of permittivity.

In the discussion it is shown why some prior formulas, though technically invalid in the dense region, nevertheless may be able to give usable results. The poor prediction of the loss factor, however, is essentially due to ignoring inter-particle contact. At about 1 or 2 percent volume loading, the material undergoes a sort of phase change, and a three-dimensional web of conducting material is produced throughout the sample giving rise to, among other things, a measurable DC conductivity.

It is concluded that a valid way of randomizing particle orientations and positions is essential to an improved calculation of the mixture loss-factor. This is both a difficult geometric problem and, in the absence of a valid equivalent of the T-matrix, also a difficult electromagnetic problem.

Although a spherical harmonic expansion (or, equivalently, a multipole expansion) is not valid, for reasons already discussed, the analysis did encounter something that could be considered its equivalent: the expansion of the fiber current in terms of suitable basis functions. In the absence of any guidance as to the form these could best take, a simple power expansion was used. It turned out that the second term, which had a relative value of about 1/2, had very little effect on the outcome; leaving the dominant effect to the first term which, outside the exclusion sphere, behaved like the classical dipole term. This is possibly the reason that prior formulas, based only on this term, do better than expected into the dense region, at least as far as the permittivity is concerned.

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